Lifshitz as a deformation of AdS

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Gravity Theories and their Avatars
Crete, 18 July 2012

Reference

- Based on
  Yegor Korovin, KS, Marika Taylor,
  Lifshitz as a deformation of AdS
  to appear

Introduction

- Gauge/gravity dualities have been an important new tool in extracting strong coupling physics.
- In particular, there has been a lot of activity recently aiming at using holography in order to model strongly interacting condensed matter systems.
- More precisely, several interesting condensed matter systems exhibit strongly interacting non-relativistic scale invariant fixed points and one may hope to use gauge/gravity duality to study them.
- To this end supergravity solutions with Schrödinger and Lifshitz isometries were constructed and studied.

Introduction

- The idea here is that the holographic models may allows to uncover new universality classes, not easily accessible with conventional perturbative methods.
- One should emphasize however that there is very little *a priori* evidence that the holographic models actually describe physics appropriate for the condensed matter systems.

Introduction

- The predominant approach has been to proceed phenomenologically and probe the relevance of these models by computing observables holographically and comparing to experimental results.
- Our goal is to understand better the dual theory from first principles.

Introduction: main idea

- The main problems with getting a handle on the dual theory are:
  - These models have been constructed in a bottom-up approach so we can't directly use string theory to get clues about the dual theory.
  - The geometries are not asymptotically AdS so the usual holographic dictionary and intuition does not directly apply.
- Our main idea is try to tune the parameters of the gravitational solutions so that they become deformations of asymptotically AdS solutions.
  - Then, at least for such parameters, one can use standard AdS/CFT to understand the dual theory.

Holography for Schrödinger

This approached was exploited in [Guica, KS, Taylor, van Rees (2010)] for the Schrödinger spacetimes

$$ds^2 = -\frac{b^2 du^2}{r^{2z}} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- For $z = 2$, this metric realizes geometrically the Schrödinger group in $(d-1)$ dimensions [Son (2008)], [K. Balasubramanian, McGreevy (2008)].
- For $d = 2$ this is the "null warped AdS_3" solution of topologically massive gravity with $\mu = 3$ [Anninos et al (2008)].

Schrödinger as a deformation of AdS

$$ds^2 = -\frac{b^2 du^2}{r^4} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- This metric is not asymptotically AdS.
- The metric becomes the AdS metric when b=0: Schrödinger is a deformation of AdS
- ➠ Considering the small b limit the geometry is a small perturbation of AdS and standard AdS/CFT applies.

The QFT dual to Schrödinger

- The dual QFT is a deformation of a relativistic CFT:

$$S_{CFT} \to S_{CFT} + \int d^d x \, bX_i$$

- → X_i is an irrelevant operator from the perspective of the original CFT.
- → X_i is exactly marginal operator from the perspective of the Schrödinger symmetry [Guica, KS, Taylor, van Rees (2010)] [Kraus, Perlmutter (2011)] .
- Apart from elucidating the duality and irrespectively of it, this shows that there is a new general class of theories with Schödinger symmetry, obtained from deformations of relativistic conformal theories [for a recent application see [Hofman, Strominger (2011)].

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Lifshitz spacetimes

In this work we aim to reach a similar understanding for Lifshitz spacetimes [Kachru, Liu Mulligan (2008)].

$$ds^2 = dr^2 - e^{2zr/l}dt^2 + e^{2r/l}dx^a dx_a$$

- The Lifshitz symmetry is realized by the following isometry,

$$t \to e^{z\lambda}t, \qquad x^a \to e^{\lambda}x^a, \qquad r \to r - \lambda l.$$

- This solution in not asymptotically AdS.
- The only parameter that appears in the solution is the dynamical exponent z, so one may try to tune it in order the solution to become a deformation of AdS.

Lifshitz as a deformation of AdS

There are three known ways to view Lifshitz as a deformation of AdS.

1. In the limit $z \to \infty$ the solution becomes $AdS_2 \times R^{d-2}$.

➠ This limit is not very useful because there is no clean holographic dictionary for this solution (because of the subtleties with AdS_2 holography and the presence of the R^{d-2} factor).

Lifshitz as deformation of AdS

2. The $z = 2$ case can be oxidized to a $z = 0$ Schrödinger spacetime in one dimension higher [Costa, Taylor (2011)] [Narayan (2011)] and we just argued that Schrödinger is a deformation of AdS. This was recently pursued in [Chemissany etal (2012)].

➠ The problem with this is that the reduction circle becomes null at infinity. This implies that the dual QFT would have the interpretation of a DLCQ of a deformed CFT ...

3. For $z \approx 1$ the solution is a deformation of AdS.

➠ There are no subtleties with this case: one can use standard AdS/CFT in order to study the dual QFT.

Lifshitz as deformation of AdS

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Condensed matter systems with $z \approx 1$

A number of theoretical models with dynamical exponents close to one have appeared in the condensed matter literature.

- Quantum spin systems with quenched disorder.
- Quantum Hall systems.
- Graphene.
- Spin liquids in the presence of non-magnetic disorder.
- Quantum transitions to and from the superconducting state in high T_c superconductors.

$z \approx 1$ systems are also relevant for the study of the IR limit of Hořava-Lifshitz gravity.

Experimental evidence for $z \approx 1$ systems

There is further experimental evidence for quantum critical behavior with $z \approx 1$ in:

- the transition from the insulator to superconductor in the underdoped region of certain high T_c superconductors [Zuev et al PRL(2005)], [Matthey etal PRL(2007)] [Broun etal PRL (2007)].
- the transition from the superconductor to metal in the overdoped region of certain high T_c superconductors [Lemberger etal PLB (2011)].

Outline

Outline

The holographic model

- We will use the formulation in terms gravity coupled to a massive vector [Taylor (2008)]

$$S = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{G}\left[R + d(d-1) - \frac{1}{4}F^2 - \frac{1}{2}M^2A^2\right]$$

  Relative to prior literature we rescaled the fields such that the AdS critical point has AdS radius $l^2 = 1$.
- This model admits a Lifshitz solution

$$\begin{aligned} ds^2 &= dr^2 - e^{2zr/l}dt^2 + e^{2r/l}dx^a dx_a; \\ A &= \mathcal{A}e^{zr/l}dt, \qquad \mathcal{A}^2 = \frac{2(z-1)}{z}, \end{aligned}$$

  provided the mass is given by

$$M^2 = \frac{zd(d-1)^2}{z^2 + z(d-2) + (d-1)^2}$$

The holographic model

- Standard AdS/CFT duality implies that this Lagrangian, expanded around the AdS critical point, describes a relativistic CFT which has a vector primary operator J_i of dimension

$$\Delta = \frac{d}{2} + \sqrt{(\frac{d}{2} - 1)^2 + \frac{zd(d-1)^2}{z^2 + z(d-2) + (d-1)^2}}$$

- The same theory admits a Lifshitz critical point with dynamic critical exponent z, if

$$M^2 = \frac{zd(d-1)^2}{z^2 + z(d-2) + (d-1)^2}$$

viewed an equation for z has real solutions with $z > 1$.

Top-down models?

Thus a necessary condition for obtaining such a Lifshitz theory with critical exponent z_+ or z_- is that the spectrum of the *AdS critical point* contains vector modes with mass within the following range

	z_+	z_-
$d = 2$	$0 < M^2 \leq 1$	$M^2 = 1$
$d = 3$	$0 < M^2 \leq 2.4$	$2.33 < M^2 \leq 2.4$
$d = 4$	$0 < M^2 \leq 9/2$	$4.29 < M^2 \leq 9/2$

This is only a necessary condition because one needs to additionally check that it is consistent to retain only the massive vector mode.

Top-down models?

The simplest AdS compactifications do contain modes in the allowed range:

- The $AdS_3 \times S^3$ spectrum contains vector modes with $M^2 = 1$, leading to $z = 1$.
- The $AdS_4 \times S^7$ spectrum contains two such massive vectors:
  - $M^2 = 3/4$, leading to $z \approx 14.72$.
  - $M^2 = 1$, leading to either $z = 4$ or $z = 1$.
- The $AdS_5 \times S^5$ spectrum contains one massive vector in the allowed range, $M^2 = 3$, leading to either $z = 9$ or $z = 1$.

Consistent truncation to only these modes is a non-trivial condition that still needs to be checked (in progress).

The $z \approx 1$ case

Leaving aside the top-down models, we now focus on the case $z = 1 + \epsilon^2$, with $\epsilon \ll 1$.

- This can be achieved by taking

$$M^2 = (d-1) + (d-2)\epsilon^2 + \frac{1+d-d^2}{d(d-1)}\epsilon^4 + \cdots$$

- The Lifshitz solution now becomes Asymptotically AdS and its dual interpretation can be obtained using the standard AdS/CFT dictionary.

The QFT dual to Lifshitz with $z = 1 + \epsilon^2$

The dual theory is a deformation of a relativistic CFT by the time component of a vector primary J^μ of dimension d,

$$S_{CFT} \to S_{CFT} + \sqrt{2} \int d^d x \epsilon J^t.$$

Outline

Holographic dictionary

We are now going set up the holographic dictionary working perturbatively in ϵ.

- Holographic renormalization for general z was studied by a number of authors [Ross (2011)], [Baggio et al (2011)] [Griffin et al (2011)].
- When $\epsilon \sim 0$, the solution is Asymptotically AdS, so well established techniques can be used [de Haro, KS, Solodukhin (2000)].
- The results obtained in this fashion are more easily comparable with those of the dual QFT.

The d=2 case: General asymptotic solution

- At order ϵ^0 we just have pure gravity

$$ds^2 = dr^2 + e^{2r} g_{ij} dx^i dx^j,$$
$$g_{ij} = (\mathbf{g_{[0](0)ij}}(\mathbf{x}) + e^{-2r} g_{[0](2)ij}(x) + \ldots)$$

- This is the most general asymptotic solution with Dirichlet data $\mathbf{g_{[0](0)ij}}(\mathbf{x})$.
- The coefficient $g_{[0](2)ij}(x)$ is only partially determined by the asymptotic analysis and is related with the dual stress energy tensor $\langle \mathcal{T}_{ij} \rangle_{[0]}$.

The d=2 case: General asymptotic solution

- At order ϵ^1 the vector is turned on

$$ds^2 = dr^2 + e^{2r} g_{ij} dx^i dx^j,$$
$$g_{ij} = (\mathbf{g_{[0](0)ij}(x)} + e^{-2r} g_{[0](2)ij}(x) + \ldots)$$
$$A_i = (\epsilon e^r \mathbf{A_{(0)i}(x)} + e^{-2r}(\mathcal{A}_{(2)i}(x) + \mathbf{r}\tilde{\mathcal{A}}_{(2)i}(x)) \ldots),$$
$$A_r = e^{-r}(A_{(0)r}(x) + e^{-2r}(A_{(2)r}(x) + \mathbf{r} a_{(2)r}(x)) \ldots).$$

- $\mathbf{A_{(0)i}(x)}$ is the new Dirichlet data.
- The coefficient $\mathcal{A}_{(2)i}(x)$ is undetermined by the asymptotic analysis.
- The remaining coefficients are locally determined by the Dirichlet data $\mathbf{g_{[0](0)ij}(x)}$, $\mathbf{A_{(0)i}(x)}$ and $g_{[0](2)ij}(x)$.

The d=2 case: General asymptotic solution

- At order ϵ^2 the vector back-reacts to the metric

$$
\begin{aligned}
ds^2 &= dr^2 + e^{2r} g_{ij} dx^i dx^j, \\
g_{ij} &= (\mathbf{g}_{[0](0)ij}(\mathbf{x}) + e^{-2r} g_{[0](2)ij}(x) + \ldots) \\
&\quad + \epsilon^2 (\mathbf{r} h_{[2](0)}(x) + e^{-2r} g_{[\mathbf{2}](2)}(x) + \mathbf{r} e^{-2r} h_{[2](2)}(x) + \ldots) + \ldots \\
A_i &= (\epsilon e^r \mathbf{A}_{(\mathbf{0})\mathbf{i}}(\mathbf{x}) + e^{-2r}(\mathcal{A}_{(2)i}(x) + \mathbf{r}\tilde{\mathcal{A}}_{(2)i}(x)) \ldots), \\
A_r &= e^{-r}(A_{(0)r}(x) + e^{-2r}(A_{(2)r}(x) + \mathbf{r} a_{(2)r}(x)) \ldots).
\end{aligned}
$$

- The coefficient $g_{[\mathbf{2}](2)}(x)$ is only partially determined by the asymptotic analysis.
- The remaining coefficients are locally determined by the Dirichlet data $\mathbf{g}_{[\mathbf{0}](\mathbf{0})\mathbf{ij}}(\mathbf{x})$, $\mathbf{A}_{(\mathbf{0})\mathbf{i}}(\mathbf{x})$ and $g_{[0](2)ij}(x)$.

Renormalization of the boundary metric

- In particular, $h_{[2](0)}$ may be viewed as a renormalization of the boundary metric:

$$g_{[0](0)ij} \to g_{[0](0)ij} + r\epsilon^2 h_{[2](0)}$$

where

$$h_{[2](0)ij} = -A_{(0)i}A_{(0)j} + \frac{1}{2(d-1)} A_{(0)k}A^{k}_{(0)} g_{[0](0)ij}.$$

- Such renormalization is only possible for the specific case we discuss: $g_{[0](0)ij}$ and $A_{(0)i}A_{(0)j}$ transform the same way under Weyl transformations.

Counterterms and renormalization

- Using the asymptotic solution one can compute the most general infinities and then work out the counterterm action needed to remove them,

$$
\begin{aligned}
S_{\text{ct}} = & -\frac{1}{8\pi G_3}\int d^2x\sqrt{\gamma}\left(1+\frac{1}{2}R\mathbf{r_0}\right) \\
& +\frac{1}{32\pi G_3}\int d^2x\sqrt{\gamma}\left(\gamma^{ij}A_iA_j-\mathbf{r_0}\Big((\nabla_iA^i)^2-\frac{1}{2}F_{ij}F^{ij}\Big)\right).
\end{aligned}
$$

- The renormalized action is

$$
S_{\text{ren}} = \lim_{r_0\to\infty}(S_{\text{reg}}+S_{\text{ct}})
$$

One-point functions

$$\langle \mathcal{T}_{ij} \rangle = \langle T_{ij} \rangle_{[0]} + \epsilon^2 \langle T_{ij} \rangle_{[2]}$$

$$\begin{aligned}
\langle T_{ij} \rangle_{[2]} &= -\frac{1}{2} A^k_{(0)} \langle T_{ij} \rangle_{[0]} A^j_{(0)} g_{[0](0)ij} \\
&+ \frac{1}{8\pi G_3} \Big[- g_{[2](2)ij} + A_{(0)i} A_{(2)j} + \frac{1}{2} A_{(0)k} A^k_{(2)} g_{[0](0)ij} \\
&- \frac{1}{4} (A_{(0)i} a_{(2)j} + A_{(0)j} a_{(2)i}) + \frac{R}{16} A_{(0)k} A^k_{(0)} g_{[0](0)ij} \\
&- \frac{5}{8} A^k_{(0)} \nabla_k \nabla_l A^l_{(0)} g_{[0](0)ij} - \frac{1}{4} (\nabla_k A^k_{(0)})^2 g_{[0](0)ij} + \frac{1}{2} A_{(0)j} \nabla_i \nabla_k A^k_{(0)} \\
&+ \frac{1}{4} \left(A_{(0)i} \nabla^k F_{(0)kj} + A_{(0)j} \nabla^k F_{(0)ki} \right) - \frac{3}{8} A_{(0)l} \nabla_k F^{kl}_{(0)} \Big].
\end{aligned}$$

$$\langle \mathcal{J}^i \rangle = -\frac{2\epsilon}{16\pi G_3} A^i_{(2)} + \frac{1}{2} \langle T_{ij} \rangle_{[0]} (\epsilon A_{(0)j}).$$

Ward identities

The leading order terms satisfy the correct Ward identities as expected. The Ward identities for the subleading terms are:

- Diffeomorphism Ward identity

$$\nabla^j \langle T_{ij} \rangle_{[2]} = A_{(0)i} \nabla_j \langle J^j \rangle - \langle J^j \rangle F_{(0)ij}.$$

  This is indeed the correct Ward identity.
- Trace Ward identity

$$\langle T_i^i \rangle_{[2]} = A_{(0)i} \langle J^i \rangle + \mathcal{A},$$

  This is the expected trace anomaly with a trace anomaly $\mathcal{A}$.

The conformal anomaly

$$\mathcal{A} = \frac{1}{4} F_{(0)ij} F^{ij}_{(0)} - \frac{1}{2} (\nabla_i A^i_{(0)})^2 - \frac{1}{2} A^i_{(0)} \langle T_{ij} \rangle_{[0]} A^j_{(0)} + \frac{R}{4} A^i_{(0)} A_{(0)i}$$

- As usual, this expression should be Weyl invariant.
- Remarkably, a related Weyl invariant action for $d > 2$ has appeared in [Deser, Nepomechie (1984)]

$$\mathcal{L} = \frac{1}{4} F_{ij} F^{ij} + \frac{d-4}{2d} (\nabla_i A^i)^2 - \frac{d-4}{2} S_{ij} A^i A^j + \frac{d-4}{8(d-1)} R A_i A^i,$$

  where $S_{ij} = (R_{ij} - R/(2(d-1)) g_{ij})/(d-2)$.
- It turns out that $\langle T_{ij} \rangle_{[0]}$ has the same Weyl transformation property as S_{ij} and this action with $S_{ij} \to \langle T_{ij} \rangle_{[0]}$, $d \to 2$ is equal to $\mathcal{A}$!

The d=3 case

- The analysis and the results for $d = 3$ are exactly analogous, so we skip all details.
- We only report the following two results:

$$\left\langle \mathcal{J}^i \right\rangle = -\frac{3\epsilon}{16\pi G_4} A^i_{(3)} + \frac{1}{3} \langle T_{ij} \rangle_{[0]} (\epsilon A_{(0)j}).$$

$$\left\langle T^i_i \right\rangle_{[2]} = A_{(0)i} \left\langle J^i \right\rangle + \frac{1}{2} A^i_{(0)} \langle T_{ij} \rangle_{[0]} A^j_{(0)}.$$

Remarks

- These results hold for any solution of the bulk equations, for example, the AdS critical point and perturbation around it or the Lifshitz critical point and perturbations around it.
- By functionally differentiating wrt sources we can compute higher point functions, both for the CFT and the Lifshitz theory.
- The "cross-terms" in the 1-point functions encode OPE coefficients.

Recovering Lifshitz invariance

Let us now evaluate the trace Ward identity at the Lifshitz critical point.

- In this case,

$$A_{(0)t} = \sqrt{2}, \qquad g_{(0)ij} = \eta_{ij}$$

- Then the trace Ward identity leads to

$$\langle T^i_i \rangle = -\epsilon^2 \langle T^t_t \rangle + O(\epsilon^4) \quad \Rightarrow \quad \langle z T^t_t \rangle + \langle T^a_a \rangle = O(\epsilon^4)$$

- This is precisely the condition for Lifshitz invariance!

Outline

The QFT side: remarks

The Lifshitz theory we obtained is markedly different than any other Lifshitz invariant theory that appeared before.

- For example, the scalar theory with action

$$S = \int dt d^3x (\dot{\phi}^2 + \phi(-\partial^2)^z \phi) \quad (1)$$

  which is often used in the literature as an illustrative example (especially when $z = 2$) does not become of the form we find when $z \sim 1 + \epsilon^2$.
- This suggests that this field theory model is unlikely to share key features of the holographic model.
- None of the condensed matter Lifshitz models are of this form, which leads to the question:

➠ Is this type of Lifshitz model special to strongly coupled models or not?

The QFT story

- The answer to this question is:

These models are generic. If one deforms any CFT by any dimension d vector primary, the theory would flow to a Lifshitz invariant fixed point with $z = 1 + \epsilon^2 + O(\epsilon^4)$.

- Whether this fixed point is stable or not depend on the particular CFT/vector primary.

Sketch of proof

Our theory is defined by

$$S = S_{CFT} + \int d^d x \epsilon J^t .$$

where J^i is a conformal vector primary of dimension d.

- The deformed theory is translational invariant
  ➠ the theory still has a conserved stress-energy tensor $\mathcal{T}_{ij}$.
- The theory is not Lorentz invariant ➠ $\mathcal{T}_{ij}$ is not symmetric.

Lifshitz invariance

- A theory with a conserved stress energy tensor $\mathcal{T}_{ij}$ is Lifshitz invariant with dynamical exponent z, if the stress tensor satisfies the trace identity,
$$z\mathcal{T}_t^t + \mathcal{T}_a^a = 0.$$

Proof

The conserved current is given by $l^i = \mathcal{T}_{ij}\xi^i$, where $\delta x^i = \xi^i$ is a Lifshitz transformation,

$$\xi^0 = zx^0, \quad \xi^a = x^a, \quad (a = 1, \ldots, d-1).$$

Taking the divergence one finds.

$$\partial^i l_i = (\partial^i \mathcal{T}_{ij})\xi^j + \mathcal{T}_{ij}\partial^i\xi^j = z\mathcal{T}_t^t + \mathcal{T}_a^a,$$

where we used $\partial_0\xi^0 = z, \partial_a\xi^b = \delta_b^a$.

Non-relativistic stress energy tensor

- In a Lorentz invariant theory one may compute the stress energy tensor by coupling to a metric g_{ij} and differentiating the action w.t.r. g_{ij}.
- In a non-Lorentz invariant theory one should instead couple to a vielbein $e_i^{\hat{k}}$ and obtain the stress energy tensor by

$$T_i^{\hat{k}} = \frac{1}{\det e} \frac{\delta S[e]}{\delta e^i_{\hat{k}}}$$

Back to our model

- In our case,

$$S[e] = S_{CFT}[e] + \epsilon \int d^d x \det e e_i^{\hat{t}} J^i.$$

and therefore

$$T_{\hat{k}\hat{l}} = T^{CFT}_{\hat{k}\hat{l}} - \epsilon \left(\eta_{\hat{k}\hat{l}} J^{\hat{t}} + \delta^{\hat{t}}_{\hat{k}} J_{\hat{l}} - \eta_{\hat{m}\hat{l}} e^j_{\hat{k}} \left(\frac{\delta J^i}{\delta e^j_{\hat{m}}} \right) e^{\hat{t}}_i \right)$$

➢ Taking the trace and using the fact that J^i is a vector primary of dimension d we find that the stress-energy tensor is traceless.

➠ The classical theory is a $z = 1$ non-relativistic CFT.

The quantum theory

- Since $\epsilon \ll 1$, we can analyze the quantum theory using conformal perturbation theory.

$$
\begin{aligned}
Z[\epsilon] &= Z_{CFT} - \epsilon \int d^d x A_{(0)i} \langle \mathcal{J}^i(x) \rangle_{CFT} \\
&+ \frac{1}{2}\epsilon^2 \int d^d x \int_{|x-y|>\Lambda} d^d y A_{(0)i}(x) A_{(0)j}(y) \langle \mathcal{J}^i(x) \mathcal{J}^j(y) \rangle_{CFT}
\end{aligned}
$$

  where $1/\Lambda$ is a UV cut-off.
- The first non-trivial effects are at order ϵ^2. To compute this we may use the OPE of $\mathcal{J}^i \mathcal{J}^j$.

Renormalization

- Inserting the OPE

$$\mathcal{J}_i(x)\mathcal{J}_j(0) \sim \sum C_{ij}^k \frac{\mathcal{O}_k}{x^{2d-\Delta_k}},$$

  where Δ_k is the dimension of the operator $\mathcal{O}_k$, one finds infinities when

$$\Delta_k \leq d$$

- To remove these infinities we need to renormalize the sources of $\mathcal{O}_k$. [If we do not have couplings to these operators we have to add them at this point.]

OPE of $\mathcal{J}_i\mathcal{J}_j$

- This OPE contains the following universal terms

$$\mathcal{J}_i(x)\mathcal{J}_j(0) \sim \frac{I_{ij}}{x^{2d}} + \cdots + \mathcal{A}_{ij}{}^{kl}\frac{\mathcal{T}_{kl}}{x^d} + \cdots ,$$

where $I_{ij} = \delta_{ij} - 2x_i x_j/x^2$.

- ➢ In $d = 2$, the OPE coefficient $\mathcal{A}_{ij}{}^{kl}$ is completely fixed by conformal invariance.
- ➢ In $d > 2$, the OPE coefficient $\mathcal{A}_{ij}{}^{kl}$ is fixed up to two constants by conformal invariance.

Renormalization: leading order term

- The leading order term gives a power-law divergence

$$\frac{1}{2}\epsilon^2 \int d^d x \int_{|x-y|>\Lambda} d^d y \, A_{(0)i}(x) A_{(0)j}(y) \frac{I_{ij}}{x^{2d}} \sim \epsilon^2 \int d^d x \Lambda^d A_{(0)i} A^i_{(0)} \quad (2)$$

- This is removed by adding the counterterm:

$$S_{ct} \sim \int \sqrt{-g}\gamma^{ij} A_i A_j$$

where $\Lambda = e^{r_0}$.

- This is one of counterterms we found earlier.

Renormalization: subleading order

- The subleading order term gives a logarithmic divergence

$$\frac{1}{2}\epsilon^2 \int d^d x \int_{|x-y|>\Lambda} d^d y \, A_{(0)i}(x) A_{(0)j}(y) \mathcal{A}_{ij}{}^{kl} \frac{\mathcal{T}_{kl}}{x^d}$$
$$\sim \epsilon^2 \int d^d x \log \Lambda A_{(0)i} A_{(0)} j \mathcal{T}^{ij}$$

- We need to renormalize the source of $\mathcal{T}_{ij}$, i.e. the spacetime metric:

$$g_{ij}(x, \Lambda) = \eta_{ij} + \log \Lambda \epsilon^2 A_i A_j = \eta_{ij} + r_0 \epsilon^2 A_i A_j$$

- This is precisely the form of the metric we found earlier.

Beta function and Lifshitz invariance

- It follows that there is beta function,

$$\beta_g^{ij} \equiv \frac{dg_{ij}}{d\log\Lambda} = \epsilon^2 A^i A^j$$

- The dilatation Ward identity is now equal to

$$\langle T_i^i \rangle = \beta_g^{ij} T_{ij}$$

  - This is precisely the Ward identity we found at strong coupling using holographic renormalization!
  - The same argument we gave earlier establishes Lifshitz invariance with $z = 1 + \epsilon^2$.

Stability and consistent truncation

- If there are other singular terms in the OPE, renormalization would induce couplings to relevant operators and the Lifshitz critical point would be unstable.
- The same terms are also linked to the absence of consistent truncation involving the metric and massive vector only.

➠ For any such term, there would be a cubic coupling in the bulk that is quadratic in the gauge field and linear in the field dual to the corresponding relevant operator: one cannot consistently truncate this field.

Outline

Lifshitz black holes

- In the condensed matter applications one would be primarily interested in the behavior of the system at finite temperature.
- To study this holographically we need black hole solutions with Lifshitz asymptotics.
- To date most such black holes were either obtained numerically or they contain a running dilaton.
- Here we will present analytic solutions with $z \approx 1$.

Lifshitz black holes

We obtain the solution perturbatively in ϵ.

- At order ϵ^0 we just have pure gravity and the solution is the standard neutral black brane

$$ds^2 = \frac{d\rho^2}{\rho^2(1-\frac{\rho_0^d}{\rho^d})} + \rho^2\left(-(1-\frac{\rho_0^d}{\rho^d})dt^2 + dx\cdot dx\right),$$

- At order ϵ^1 the massive vector is turned on,

$$A_t(y) = \epsilon A_{(0)t} y(1-\frac{1}{y^d})\ {}_2F_1(\frac{1}{d},\frac{d-1}{d};2;1-\frac{1}{y^d}),$$

where $y=\rho/\rho_0$. This is the unique solution that has a non-zero source at the boundary and vanishes at the horizon.

Lifshitz black holes

- At order ϵ^2 the massive vector backreacts onto the metric which becomes

$$ds^2 = \frac{dy^2}{c(y)} - dt^2 c(y) b(y)^2 + y^2 dx \cdot dx,$$

where

$$c(y) = (y^2 - \frac{1}{y^{d-2}}) + \Delta c(y); \qquad b(y) = 1 + \Delta b(y).$$

- $\Delta c(y)$ and $\Delta b(y)$ satisfy decoupled first order equations. For example,

$$\partial_y (y^{d-2} \Delta c) = -\frac{\epsilon^2}{2\rho_0^2} \frac{y^{d-1}}{y^2 - y^{2-d}} \left(A_t^2 + \frac{(y^2 - y^{2-d})(\partial_y A_t)^2}{(d-1)} \right).$$

Holographic data

- $d = 2$

$$\langle \mathcal{J}^t \rangle = \frac{(\epsilon A_{(0)t})}{32\pi G_3}(1 - 4\log 2),$$
$$\langle T_{tt} \rangle = \frac{1}{16\pi G_3}(1 + \frac{1}{2}(\epsilon A_{(0)t})^2(\tilde{c} + 2\log 2)),$$
$$\langle T_{xx} \rangle = \frac{1}{16\pi G_3}(1 + \frac{1}{2}(\epsilon A_{(0)t})^2(\tilde{c} - 2\log 2 + 2))$$

- They satisfy the correct Ward identity

$$z\langle \mathcal{T}_t^t \rangle + \langle \mathcal{T}_x^x \rangle = \epsilon A_{(0)t} \langle \mathcal{J}^t \rangle.$$

- Similar results hold for $d = 3$.

Outline

Conclusions

- We show that the Lifshitz solution with $z \approx 1$ is dual to a deformation of a relativistic CFT by the time component of a vector conformal primary of dimension d.
- Such a deformation of a relativistic CFT always leads to a Lifshitz invariant theory with $z \approx 1$.

Conclusions

- We formulated necessary conditions for existence of top-down models: the spectrum of the AdS critical point should have massive vector modes with mass in a specific range.
- This condition would be sufficient if these modes consistently truncate from the rest.
- Consistent truncation is also linked to stability of the Lifshitz critical point.
- We obtained analytically a Lifshitz invariant black hole.