Non-Fermi liquids from D-brane constructions

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Based on 1112.3036 and 1207.3352 with O. DeWolfe and C. Rosen

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Contents

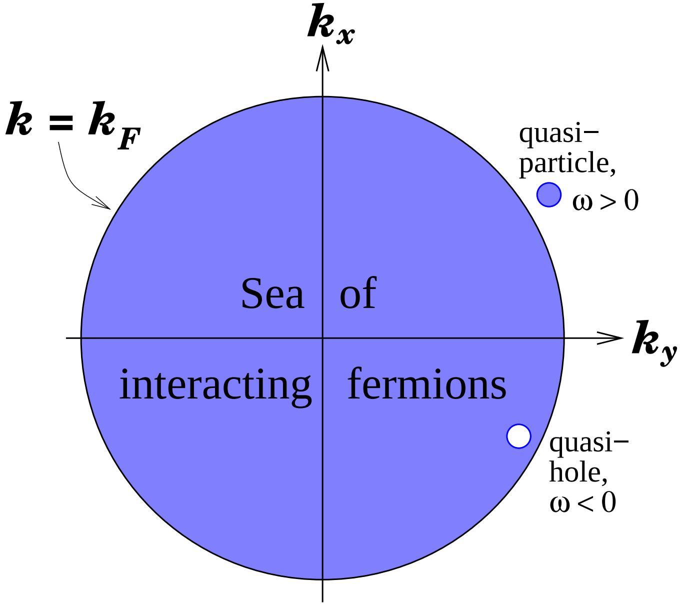
1. Simplest Top-down Fermi Surface constructions

The simplest charged black holes in AdS are purely bosonic backgrounds. To "see" the fermions in the dual description, bounce one more fermion off the normal state black hole and look for Green's function singularities:

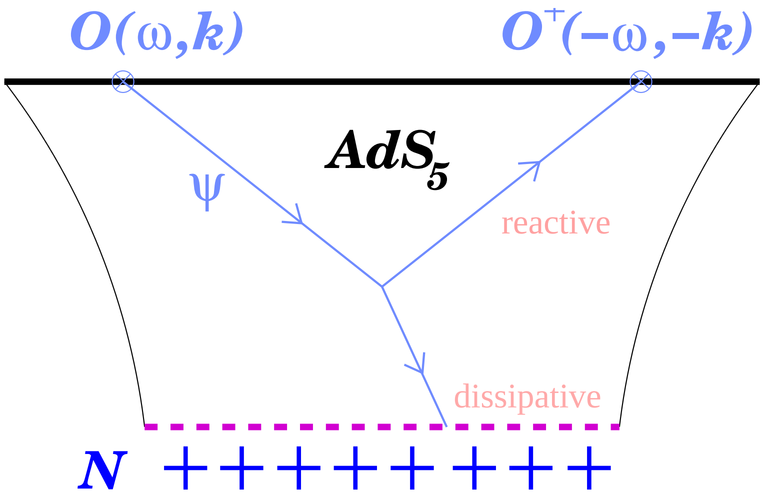
$$G(\omega,k) = \left\langle \mathcal{O}_\chi(\omega,\vec{k})\mathcal{O}_\chi^\dagger(-\omega,-\vec{k})\right\rangle \approx \frac{h_1}{(k-k_F)-\frac{1}{v_F}\omega - h_2 e^{i\gamma}\omega^{2\nu_F}}$$

when $k \approx k_F$ and $\omega \approx 0$.

- A singularity in $G(\omega,k)$ at $\omega = 0$ and finite $k = k_F$ *defines* the presence of a Fermi surface.
- v_F is Fermi velocity.
- Assuming $\nu_F > 1/2$, low-energy dispersion relation is $\omega \approx v_F(k-k_F)$.
- If $\nu_F > 1/2$ or if $e^{i\gamma}$ is nearly real, quasi-particles' width is much smaller than their energy.



The AdS/CFT calculation follows [Lee 0809.3402; Liu, McGreevy, and Vegh 0903.2477; Cubrovic, Zaanen, and Schalm, 0904.1993]:



- As $\omega \to 0$ and $k \to k_F$, we want to see dissipative effects *disappear*.
- Equation solved is a variant of Dirac equation.
- Results from AdS_5 give Green's function in a $3+1$-dimensional field theory.

Fermi surfaces in boundary theory correspond to fermion normal modes in the bulk.

Significant technical difficulties surround the derivation of the appropriate fermion equation of motion [DeWolfe, Rosen, SSG, 1112.3036]: in AdS_5-Reissner-Nordstrom,

$$\left(i\gamma^\mu\nabla_\mu + \frac{5}{L}\gamma^\mu a_\mu - \frac{1}{2L} + \frac{i}{4}f_{\mu\nu}\gamma^{\mu\nu}\right)\chi = 0\,.$$

But final results (for AdS_5-RN) are simple: find two Fermi surfaces, with

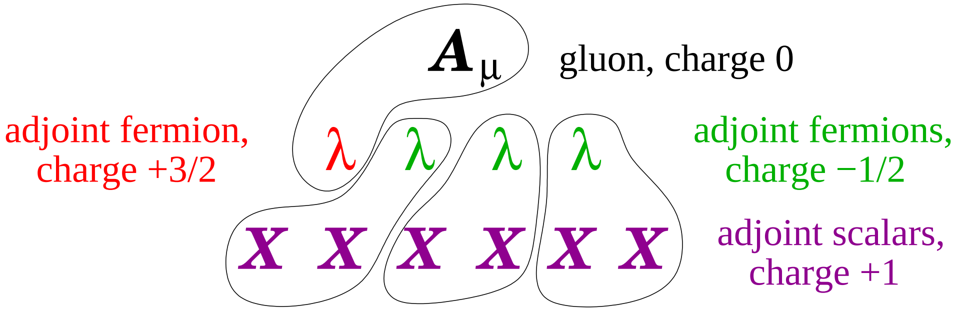
$$\frac{k_F}{\mu} = 2\sqrt{2} \pm 1$$

where μ is chemical potential for fermion charge.

But $\omega \approx (k - k_F)^6$: a *very* non-Fermi-liquid.

Two conundrums:

- Super-Yang-Mills theory has charged bosons as well as charged fermions. Why don't the bosons suck up all the charge in a condensate?



- When $T \to 0$, the charged black holes retain non-zero entropy, $S \propto V N^2 \mu^3$. Huge violation of Nernst's Law. What is the ground state?

The plan of the rest of Part I of the talk:

- Minimal gauged supergravity. $D = 4$, $\mathcal{N} = 2$ is simple and provides a useful warmup, but no Fermi surfaces.
- More gauged supergravity. $D = 4$, $\mathcal{N} = 8$ leads to simple equations and reveals a Fermi surface.
- Yet more gauged supergravity. $D = 5$, $\mathcal{N} = 8$ is extremely complicated but gives the simplest final results.
- Dual field theory. I will *speculate* on the interpretation of the meaning of AdS_5 results in light of scaling arguments and Luttinger-style counting.

Why so much gauged supergravity?

- Gauged SUGRA embeds into various string theory backgrounds, including $AdS_5 \times S^5$.
- SUGRA fixes the correct fermion equations on symmetry principles.
- Solving these equations means we are computing actual correlators in known (SUSY) field theories.

1.1. $D = 4, \mathcal{N} = 2$ gauged supergravity

We couple two Majorana gravitini ψ_μ^i to an $SO(2)$ gauge field $A_\mu^{ij} = \epsilon^{ij} A_\mu$:

$$D_\mu \psi_\nu^i = \nabla_\mu \psi_\nu^i - g A_\mu^{ij} \psi_\nu^j$$

The lagrangian and SUSY transformations are deformations of ungauged SUGRA [Freedman and Das, 1977]:

$$\begin{aligned}\mathcal{L}_{\text{SUGRA}} = & -\frac{1}{2} R - \frac{1}{4} (F_{\mu\nu}^{ij})^2 + 3g^2 \\ & - \epsilon^{\lambda\rho\mu\nu} \bar{\psi}_\lambda^i \gamma_5 \gamma_\mu (\delta^{ij} \nabla_\nu - g A_\nu^{ij}) \psi_\rho^j - \bar{\psi}_\mu^i \left(F^{ij\mu\nu} - \tfrac{i}{2} \gamma_5 \tilde{F}^{ij\mu\nu} \right) \psi_\nu^j \\ & - 2g \bar{\psi}_\mu^i \sigma^{\mu\nu} \psi_\nu^i + \psi^4\end{aligned}$$

$$\delta e_{a\mu} = -i \bar{\epsilon}^i \gamma_a \psi_\mu^i \qquad \delta A_\mu^{ij} = -2 \bar{\epsilon}^{[i} \psi_\mu^{j]}$$

$$\delta \psi_\rho^i = (\delta^{ij} \nabla_\rho - g A_\rho^{ij}) \epsilon^j - \tfrac{i}{2} \sigma^{\mu\nu} F_{\mu\nu}^{ij} \gamma_\rho \epsilon^j + \tfrac{ig}{2} \gamma_\rho \epsilon^i + \psi^2 \epsilon^i \, .$$

The ψ^4 and $\psi^2\epsilon$ have some subtle γ-matrix structure.

Supergravity dimensional analysis in any D looks like this: $[g_{\mu\nu}] = [A_\mu] = [\phi] = 0$, $[\partial_\mu] = [g] = 1$, $[\psi_\mu] = [\chi] = -[\epsilon] = 1/2$, $[\mathcal{L}] = 2$.

Consider gravitino two-point function in extremal AdS_4-Reissner-Nordstrom:

$$ds^2 = \frac{r^2}{L^2}(f\,dt^2 - d\vec{x}^2) - \frac{L^2}{r^2}\frac{dr^2}{f} \qquad A_0 = \mu\left(1 - \frac{r_0}{r}\right)$$

$$f = 1 - 4\frac{r_0^3}{r^3} + 3\frac{r_0^4}{r^4} \qquad L = \frac{1}{2g} \qquad \mu = \frac{\sqrt{3}r_0}{L}\,.$$

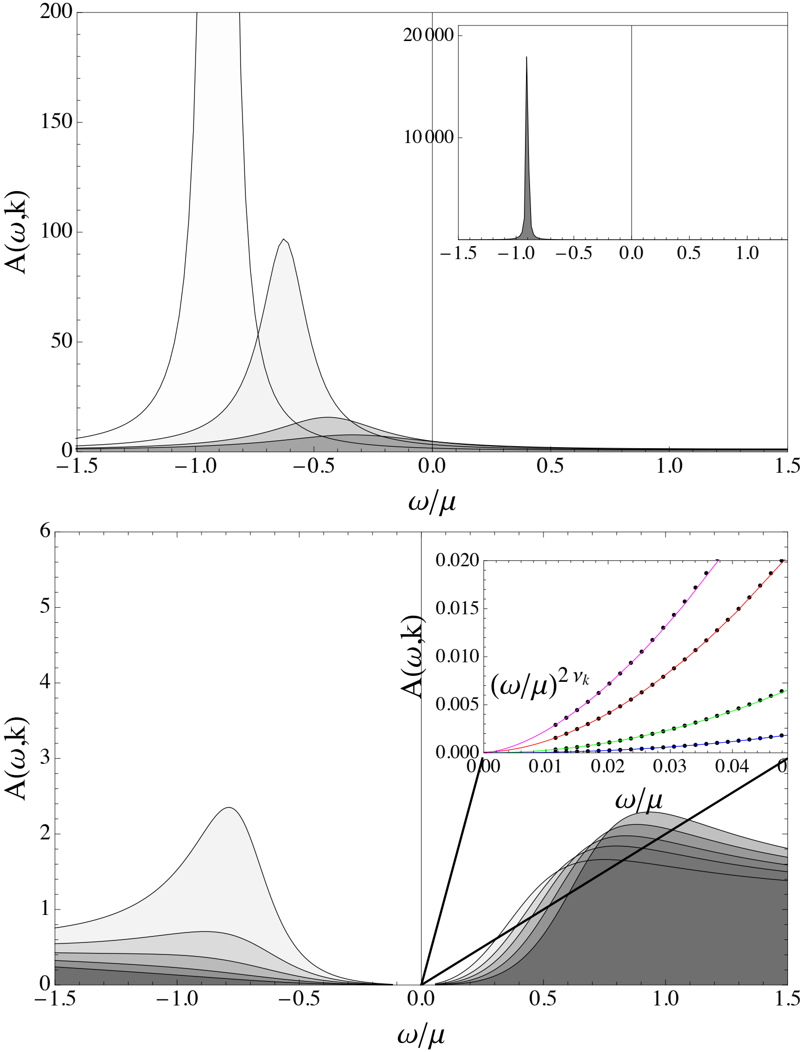
- In $\mathcal{L}_{\text{SUGRA}}$, only the quadratic terms in ψ_μ^i matter since we are after the two-point function G_S of the dual operator $S_m^i \sim \operatorname{tr}\lambda D_m X$, with $\Delta_S = 5/2$.
- There's a scaling form valid for all ω, k:

$$G_S(\omega,k) = \frac{L^2}{\kappa^2}\mu^{2\Delta_S - 3}\hat{G}_S(\hat{\omega},\hat{k}) \qquad \hat{\omega} = \frac{\omega}{\mu}\,, \quad \hat{k} = \frac{k}{\mu}\,,$$

  where $I_{\text{SUGRA}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{g}\,\mathcal{L}_{\text{SUGRA}}$. Usually we quote $G_S(\omega,k)$ with $L = \kappa = r_0 = 1$.
- $\hat{G}_S(\hat{\omega},\hat{k})$ has no free parameters and can be uniquely determined by solving diff EQs in AdS_4-Reissner-Nordstrom.
- Details are challenging because ψ_μ has lots of components and one must be careful about gauge-fixing.

No Fermi surface singularities were found [Gauntlett, Sonner, and Waldram, 1106.4694 & 1108.1205; Belliard, Gubser, and Yarom, 1106.6030]; however, Gauntlett et al. exhibited (see figure)

- Phonino resonances that get sharp as $k \to 0$, with $\omega \approx -\mu$ (relative to expected Fermi energy $\omega = 0$), at $T \sim \mu$ (upper panel).
- Power law depletion of spectral measure $A(\omega, k)$ near $\omega = 0$ for $T \approx 0$ (lower panel).
- Power $\nu_k = \sqrt{\frac{7}{12} + \frac{k^2}{2\mu^2}}$ is always real, in contrast to bottom-up constructions where $\nu_k \sim \sqrt{k^2 - q^2}$ can be imaginary.



1.2. More gauged supergravity: $D = 4, \mathcal{N} = 8$

$D = 4, \mathcal{N} = 8$ gauged supergravity [de Wit and Nicolai, 1982]:

- A semi-pedagogical introduction can be found in [de Wit, hep-th/0212245].
- Field content is: graviton $g_{\mu\nu}$, 8 gravitini ψ^i_μ, 28 gauge fields A^{ij}_μ, 56 Majorana spinors χ^{ijk}, and 70 real scalars ϕ^{ijkl}.
- Eight-valued indices $i, j, \ldots$ characterize either the gauge group $SO(8)$ or the internal symmetry group $SU(8)$.
- Solutions of $D = 4, \mathcal{N} = 8$ can be lifted to solutions of 11-d SUGRA on a (possibly deformed) S^7.
- Scalars parametrize $E_{7(7)}/SU(8)$ and indicate how the S^7 is deformed.
- By specializing to trivial scalars (i.e. round S^7) we are able to ignore difference between $SU(8)$ and $SO(8)$ indices.
- We turn on just one gauge field, $a_\mu = A^{12}_\mu = -A^{21}_\mu$.
- By an $SO(8)$ triality rotation one can describe this equivalently as $A^{12}_\mu = A^{34}_\mu = A^{56}_\mu = A^{78}_\mu$. Before the triality rotation, i is spinorial wrt S^7.

Here are the main equations for setting up fermions in RNAdS_4 with only $a_\mu = A_\mu^{12}$ non-zero:

$$D_\mu \chi_{ijk} = \nabla_\mu \chi_{ijk} + 3g A_\mu{}^m{}_{[i}\, \chi_{jk]m} \qquad \text{(even for more general gauge fields)}$$

$$\mathcal{L} = -\frac{1}{2}R - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + 6g^2 + \mathcal{L}_{1/2} \qquad \text{(specialized to round } S^7\text{)}$$

$$\mathcal{L}_{1/2} = -\frac{1}{12}\bar{\chi}^{ijk}(\gamma^\mu D_\mu - \overleftarrow{D}_\mu \gamma^\mu)\chi_{ijk} \qquad \text{(The Dirac kinetic term for } \chi_{ijk}\text{)}$$
$$- \frac{1}{2}\left(F^+_{\mu\nu ij} O^{+\mu\nu ij} + \text{h.c.}\right) \qquad \text{(Eventually can ignore this } F^+O^+ \text{ bit...}$$

$$O^{+\mu\nu ij} \equiv -\frac{\sqrt{2}}{144}\epsilon^{ijklmnpq}\bar{\chi}_{klm}\sigma^{\mu\nu}\chi_{npq} \qquad \text{...which looks like Pauli couplings...}$$
$$- \frac{1}{2}\bar{\psi}_{\rho k}\sigma^{\mu\nu}\gamma^\rho \chi^{ijk} + (\psi_\rho^2 \text{ term}) \qquad \text{...and } \chi\psi \text{ mixing)}\,.$$

To see that you can drop F^+O^+, note that $ij = 12$, so none of klm or npq are 1 or 2: thus χ_{klm}, χ_{npq}, and also $\chi^{ijk} = \chi^{12k}$, are all *uncharged.*

The upshot: Form $\chi = \chi_{1jk} + i\chi_{2jk}$ and find simple massless Dirac equation,

$$\gamma^\mu \left(\nabla_\mu - \frac{i}{\sqrt{2}L} a_\mu\right)\chi = 0\,.$$

$$\gamma^\mu \left(\nabla_\mu - \frac{i}{\sqrt{2}L} a_\mu \right) \chi = 0 \,.$$

This equation is well-analyzed in the bottom-up literature [Liu, McGreevy, and Vegh, 0903.2477; Faulkner et al, 0907.2694; Hartman and Hartnoll, 1003.1918]. There *is* a Fermi surface, near which

$$G_R = \frac{h_1}{k_\perp - h_2 e^{i\gamma_F} \omega^{2\nu_{k_F}}} \qquad \text{where } k_\perp = k - k_F \text{ and } h_2 > 0$$

$$k_F \approx 0.9185 \qquad (\text{with } \kappa = L = r_0 = 1)$$

$$\nu_{k_F} = 0.2393 < 1/2 \qquad \text{rather different from Landau-Fermi liquid}$$

$$\gamma_F \equiv 0.0285 \pmod{2\pi\nu_F} \qquad \text{Really small!}$$

So if the nearest quasi-normal frequency to the origin is $\omega_{\text{QNM}} = \omega_* - i\Gamma$, then

$$\frac{\Gamma}{\omega_*} = \tan \frac{\gamma_F}{2\nu_{k_F}} = \frac{1}{16.8} \qquad \text{for } k_\perp > 0 \quad \text{(quasi-particles)}$$

$$\frac{\Gamma}{\omega_*} = \tan \frac{\gamma_F + \pi}{2\nu_{k_F}} = \frac{1}{2.8} \qquad \text{for } k_\perp < 0 \quad \text{(quasi-holes)} \;.$$

1.3. Yet more gauged supergravity: $D = 5, \mathcal{N} = 8$

$D = 5, \mathcal{N} = 8$ gauged supergravity [Gunaydin, Romans, and Warner, 1986] has a similarly intricate structure, with gauge group $SO(6)$. Realized as type IIB SUGRA on S^5.

With $A_\mu^{12} = A_\mu^{34} = A_\mu^{56} = a_\mu$, find (up to a Chern-Simons term that doesn't matter)

$$\mathcal{L} = -\frac{1}{4}R - \frac{3}{4}f_{\mu\nu}^2 + \frac{3g^2}{4} \qquad ds^2 = \frac{r^2}{L^2}(f\,dt^2 - d\vec{x}^2) - \frac{L^2}{r^2}\frac{dr^2}{f}$$

$$f = 1 - 3\frac{r_0^4}{r^4} + 2\frac{r_0^6}{r^6} \qquad a_0 = \frac{\sqrt{6}r_0}{L}\left(1 - \frac{r_0^2}{r^2}\right) \qquad g = \frac{2}{L}\,.$$

Follows from $SO(6)$ group theory for the most highly charged fermion.

Follows (up to sign) from dimension of dual operator, $\Delta = 5/2$ for $\mathrm{tr}\,\lambda X$.

Really have to know some supergravity to get this term straight.

$$\left(i\gamma^\mu\nabla_\mu + \frac{5}{L}\gamma^\mu a_\mu - \frac{1}{2L} + \frac{i}{4}f_{\mu\nu}\gamma^{\mu\nu}\right)\chi = 0\,.$$

$$\left(i\gamma^\mu \nabla_\mu + \frac{5}{L}\gamma^\mu a_\mu - \frac{1}{2L} + \frac{i}{4} f_{\mu\nu}\gamma^{\mu\nu} \right) \chi = 0$$

(Similar equations were analyzed in [Edalati et al, 1012.3751].) Final results:

$$G_R = \frac{h_1}{k_\perp - h_2 e^{i\gamma_F}\omega^{2\nu_{k_F}}}$$ where $k_F = k_{F,\pm} = 2 \pm \frac{1}{\sqrt{2}}$ $\quad (\kappa = L = r_0 = 1)$

$\nu_{k_F} = 1/12$ So quasi-particle dispersion relation is $\omega_* \propto k_\perp^6$

$\gamma_F = 0.0126$ So $\frac{\Gamma}{\omega_*} \approx \frac{1}{13.2}$ for quasi-particles and quasi-holes.

Also observe a rapidly vanishing residue near Fermi surface, $Z \sim (k_\perp)^{\frac{1}{2\nu_{k_F}} - 1} = k_\perp^5$.

- $k_{F,\pm}$ were determined *numerically*: diff EQ is complicated, and I have no clue how to express the normal mode wave-function in closed form.
- Small ν_{k_F} signals that $k_{F,\pm}$ are close to an oscillatory region with imaginary ν_k.
- Small γ_F owes to AdS_2 effects: Per [Faulkner et al, 0907.2694] (slightly adjusted),

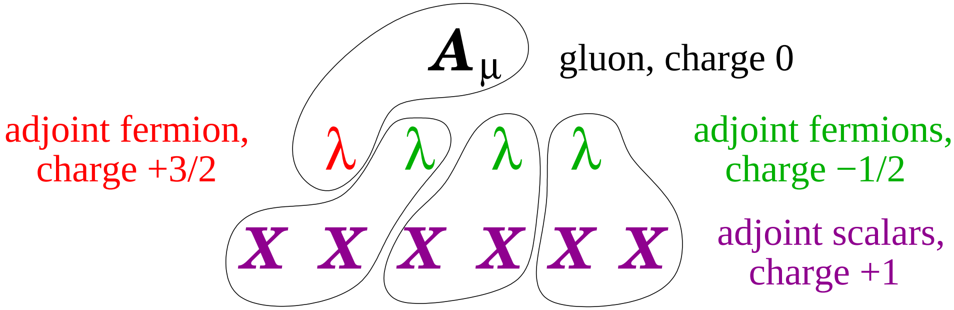
$$\gamma_F = \arg\left(e^{\pi \frac{\sqrt{2}}{3} q} - e^{-2\pi i \nu_{k_F}} \right) \approx \arg\left(41 - e^{-\pi i/6}\right)$$

where $q = 5/2$. So it matters that q is somewhat large.

1.4. Field theory speculation

Dual to $AdS_5 \times S^5$ is $\mathcal{N} = 4$ super-Yang-Mills, whose R-symmetry group is $SO(6)$.

- We specialized from the start to the $U(1) \subset SO(6)$ which is the diagonal combination of the three $U(1)$'s of CSA: $A_\mu^{12} = A_\mu^{34} = A_\mu^{56}$.
- Gauginos in 4 of $SO(6)$ and scalars in the 6 give rise to the following pattern of $U(1)$ charges:



- The way to construct a $q = 5/2$ operator is $\mathcal{O}_\chi = \text{tr}\, \lambda X$. Note $\Delta_{\mathcal{O}_\chi} = 5/2$ as promised. The equality $q = \Delta_{\mathcal{O}_\chi}$ signals that this is a BPS operator.
- You can't turn on a chemical potential for only the fermions: for any choice $U(1) \subset SO(6)$, some scalars will be charged.

In some settings [Huijse and Sachdev, 1104.5022; Iqbal, Liu, and Mezei, 1110.3814; Huijse, Sachdev, and Swingle, 1112.0573], singularities in holographic Green's functions are understood as signaling a Fermi surface of λX mesinos.

I am suspicious of a mesino interpretation *in our particular setting* for three reasons:

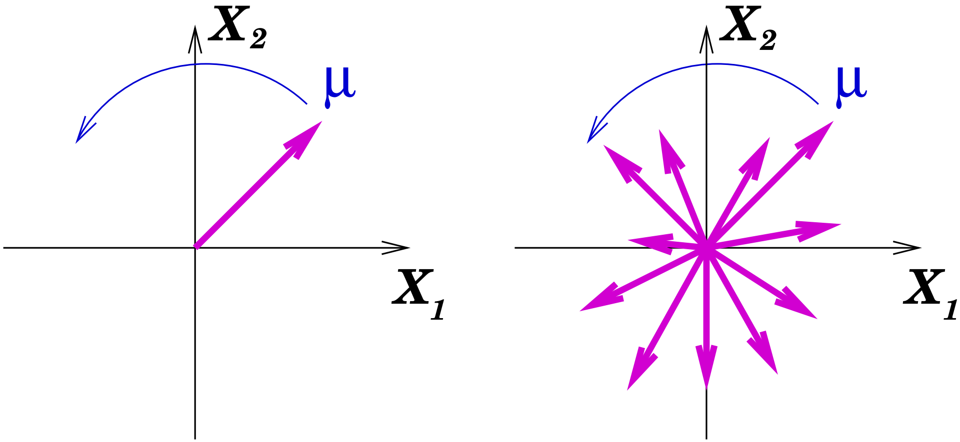
- $\mathcal{N} = 4$ super-Yang-Mills doesn't confine at finite μ as far as I can tell, so it's not clear mesinos should exist.
- If they do exist, I don't see why λX mesinos would be preferred over XX mesons as charge carriers.
- $\langle \mathcal{O}_\chi \mathcal{O}_\chi^\dagger \rangle \sim N^2$, so if you cut the amplitude to find out what states $\mathcal{O}_\chi$ can produce, they're almost certainly colored.

  More precisely: $\langle \mathcal{O}_\chi(x) \mathcal{O}_\chi^\dagger(0) \rangle \sim N^2/x^5$ for $x \ll 1/\mu$ is a non-renormalized, BPS protected result, and N^2 scaling applies equally to residue at $\omega = 0$, $k = k_F$, so $\mathcal{O}(N^2)$ things can be produced by $\mathcal{O}_\chi$ near Fermi surface.

I'd like to consider the following alternative interpretation:

The singularity in $\langle \mathcal{O}_\chi \mathcal{O}_\chi^\dagger \rangle$ is due to a gaugino Fermi surface, co-existing with a scalar condensate which (at large N) leaves the $U(1)$ symmetry unbroken.

Large N allows $U(1)$ to remain unbroken even with non-zero scalar condensate:

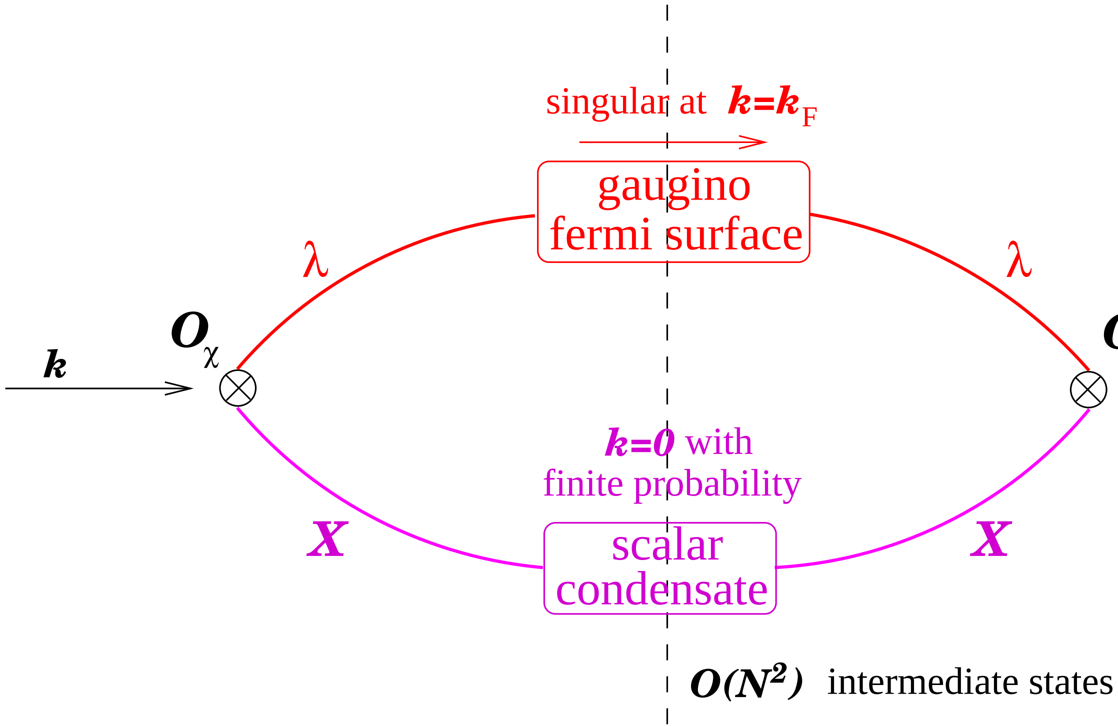


Symmetry broken Symmetry unbroken at large N

A common worry is that scalar condensate can run away along flat directions. But perhaps this is not relevant at large N. Here's why:

- Only a subleading fraction of directions satisfy $[X^I, X^J] = 0$.
- Since RNAdS_5 is finitely far from SUSY limit, it's probably more representative to think of non-commuting directions.
- In non-commuting directions, condensate is limited by $V \sim g^2\,\mathrm{tr}[X^I, X^J]^2$.

So—plausibly—the singularity at $k = k_F$, with residue $\sim N^2$, owes to diagrams roughly like this:



If the above diagram summarizes the right idea, then we should be able to compare total charge density to a Luttinger count of gauginos derived from $k_{F,\pm}$.

Charge density from thermodynamics [Cvetic and Gubser, hep-th/9903132]:

$$j_{\text{total}} = \frac{9\sqrt{6}}{4\pi^2} N^2 \qquad \text{with } \kappa = L = r_0 = 1\,.$$

Charge density from a Luttinger-style count:

charge of the gaugino, $q_\lambda = +3/2$

Degeneracy at each Fermi surface

$k_{F,\pm} = 2 \pm 1/\sqrt{2}$

$g_+ = g_- = N^2$?

$$j_{\text{fermions}} = \sum_{\text{Fermi surfaces}} q_\lambda g_{\text{s}} \int_{|k|<|k_F|} \frac{d^3k}{(2\pi)^3} = \frac{1}{4\pi^2} \left(g_+ |k_{F,+}|^3 + g_- |k_{F,-}|^3 \right)$$

$$= \frac{11}{2\pi^2} N^2 = \sqrt{\frac{242}{243}}\, j_{\text{total}} \qquad \text{who ordered that?}$$

At least this supports the picture of a finite fraction of charge in the scalar condensate; but why should the fraction be so small? Did we get $g_\pm$ right? Maybe $g_\pm < N^2$? Maybe $g_- = -N^2$ if Fermi sea is a thick shell? Other gauginos?

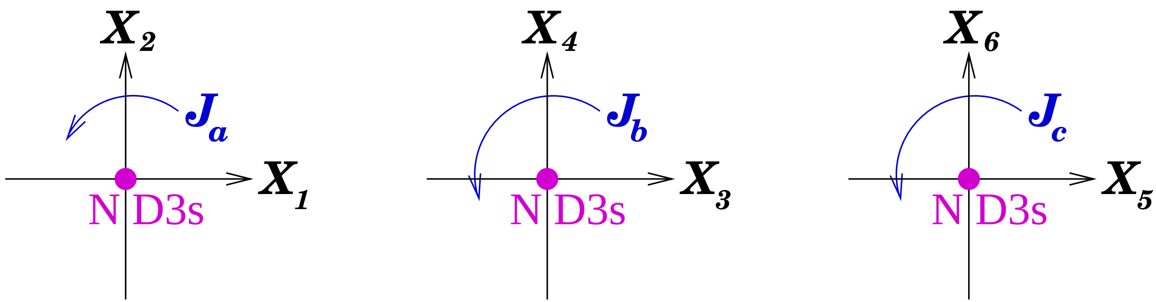
Don't consider the matter settled before someone can come and calculate $k_{F,\pm}$ and g_s in field theory!

2. Fermi surfaces for unequal charge black holes

Given that holographic Fermi surfaces exist in string theory, why continue with more ornate cases?

- Zero-point entropy is very bothersome.
- $\omega_* \propto k_\perp^6$ is far from real CM phenomena.
- The scalar condensate and the gaugino interpretation are guesses; can we gather more field theory data to help our intuition?

The rest of the talk will be devoted to D3-branes with angular momentum in directions perpendicular to their world-volume:



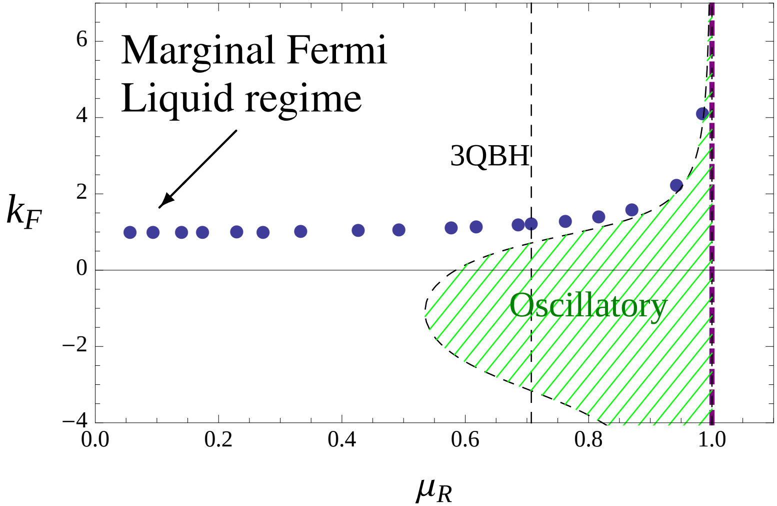
Setting $J_1 = J_a$ and $J_2 = J_b = J_c$ will be general enough. Five-dimensional description is the "2+1-charge" black hole in AdS_5, based on

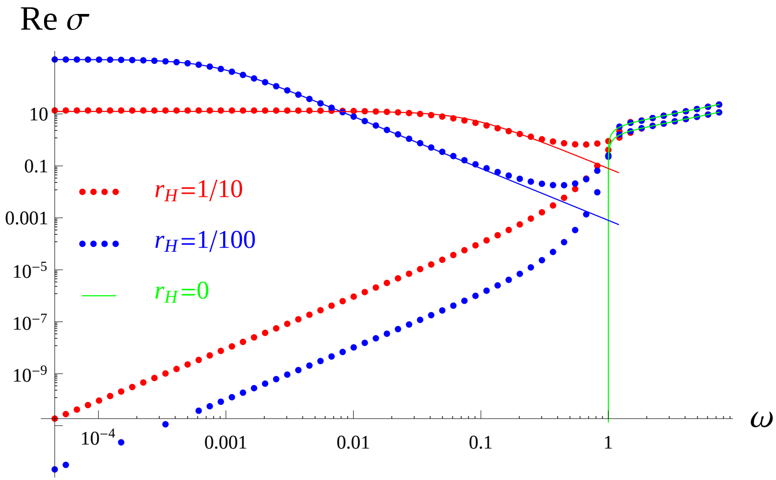
$$\mathcal{L}_{\rm bos} = -R + \frac{(\partial X)^2}{12X^2} + \frac{8X^2}{L^2} + \frac{4}{X^4L^2} \\ - X^8 f_{\mu\nu}f^{\mu\nu} - \frac{2}{X^4}F_{\mu\nu}F^{\mu\nu} - 2\epsilon^{\mu\nu\rho\sigma\tau}f_{\mu\nu}F_{\rho\sigma}A_\tau \,.$$

- X is a scalar in the $\mathbf{20}'$ with $m^2L^2 = -4$ which describes how oblate the S^5 is in the X_1-X_2 v.s. X_3-X_4-X_5-X_6 directions.
- $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ describes J_1 charge.
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ describes J_2 charge.
- We'll avoid situations where the Chern-Simons term is needed.
- $\mathcal{L}_{\rm bos}$ is a truncation of $D = 5, \mathcal{N} = 8$ supergravity, and we'll consider quadratic fermion equations derived from the same theory.
- We'll restrict attention to extremal black holes. Up to rescalings, they are parametrized by the ratio of chemical potentials, $\mu_R = \mu_1/\mu_2$.

Partial summary of results:

- There are several different Fermi surfaces.
- The most interesting one has $k_F \to 1$ as $\mu_R \to 0$ with $r_H = 1$.
- In this limit, $q \gg Q$ and entropy $\to 0$.
- Green's function shows Marginal Fermi Liquid behavior, $\nu_k \to 1/2$ from below.





- $\mu_R \to 0$ limit is related to a Coulomb branch state with non-zero X_1 and X_2.
- This state is a superconductor wrt a_μ and an insulator wrt A_μ.
- Conductivities in the MFL regime show a Drude peak for a_μ and near-insulator behavior for A_μ.

The plan of the rest of Part II of the talk:

- Bosonic backgrounds. Thermodynamics is a bit intricate and exhibits some instabilities.
- Fermions from $D = 5, \mathcal{N} = 8$ supergravity. Hard work with assorted group structures.
- Finding the Fermi surfaces. Numerics supplemented with near-horizon analysis.
- The MFL regime. Analytic results for Coulomb branch solution, superconducting v.s. insulative behavior, some puzzles.

Overall impression: In addition to providing a "field guide" to various Fermi surface phenomena exhibited by D3-branes, we focus in on an MFL regime which is close (in some sense) to a SUSY vacuum state of $\mathcal{N} = 4$ super-Yang-Mills.

We haven't gotten very far in understanding the field theory side.

Scalar condensate is definitely important; not so sure about mesinos versus gauginos.

Zero-point entropy is almost gone.

2.1. Bosonic backgrounds

We want to study charged black branes in the Poincaré patch:

$$\begin{aligned}
ds^2 &= e^{2A}\left[-h(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right] + \frac{e^{2B}}{h}dr^2 \\
A(r) &= \log\frac{r}{L} + \frac{1}{6}\log\left(1+\frac{q^2}{r^2}\right) + \frac{1}{3}\log\left(1+\frac{Q^2}{r^2}\right) \\
B(r) &= -\log\frac{r}{L} - \frac{1}{3}\log\left(1+\frac{q^2}{r^2}\right) - \frac{2}{3}\log\left(1+\frac{Q^2}{r^2}\right) \\
h(r) &= 1 - \frac{r^2(r_H^2+q^2)(r_H^2+Q^2)^2}{r_H^2(r^2+q^2)(r^2+Q^2)^2} \qquad\qquad X(r) = \left(\frac{r^2+q^2}{r^2+Q^2}\right)^{1/6} \\
\phi(r) &= \frac{q(r_H^2+Q^2)}{2Lr_H\sqrt{r_H^2+q^2}}\left(\frac{r_H^2+q^2}{r^2+q^2}-1\right) \\
\Phi(r) &= \frac{Q\sqrt{r_H^2+q^2}}{2Lr_H}\left(\frac{r_H^2+Q^2}{r^2+Q^2}-1\right),
\end{aligned}$$

where to obtain extremal black holes we would set

$$r_H^2 = \frac{1}{4}\sqrt{q^4+8q^2Q^2} - \frac{1}{4}q^2 \,.$$

Some thermodynamic quantities of interest:

$$s = \frac{1}{4G_5 L^3} \sqrt{(r_H^2 + q^2)(r_H^2 + Q^2)^2}$$

$$\mu_1 \equiv 2\phi\Big|_{\text{bdy}} = -\frac{q(r_H^2 + Q^2)}{L r_H \sqrt{r_H^2 + q^2}} \qquad \mu_2 \equiv 2\sqrt{2}\Phi\Big|_{\text{bdy}} = -\frac{\sqrt{2} Q \sqrt{r_H^2 + q^2}}{L r_H}$$

$$\mu_R \equiv \frac{\mu_1}{\mu_2} = \frac{1}{\sqrt{1 + \frac{1}{2}\left(\frac{q}{r_H}\right)^2}} \qquad \text{at extremality.}$$

q and Q are length parameters, *related* to the conserved charge densities ρ_a and ρ_A:

$$\rho_a = \frac{s}{r_H} q \qquad \rho_A = \frac{s}{r_H} Q\,.$$

Expressed in terms of ρ_a, ρ_A, and the energy density, s is *not* uniformly concave, which means there are Gregory-Laflamme instabilities if charges are too large [Gubser, hep-th/9810225; Gubser and Cvetic, hep-th/9903132], e.g. if $\rho_a > \frac{s}{\sqrt{2\pi}}$ when $\rho_A = 0$.

All black holes of interest to us are on the unstable side of Gregory-Laflamme stability line. Are the extremal ones dynamically unstable? Not sure.

2.2. Fermions from $\mathcal{D}=5, \mathcal{N}=8$ supergravity

Scalar coset in $\mathcal{D}=5, \mathcal{N}=8$ is $E_{6(6)}/USp(8)$, and fermions are conveniently expressed in terms of $USp(8)$ representations: ψ_μ^a in the **8** and χ^{abc} in the **48**:

$$48 = \binom{8}{3} - 8 \qquad \text{because} \qquad \chi^{abc} = \chi^{[abc]} \quad \text{and} \quad \Omega_{ab}\chi^{abc} = 0\,.$$

We focus on the spin-$1/2$ fields. They satisfy a symplectic Majorana condition,

$$\chi^{abc} = C(\bar{\chi}^{abc})^T \qquad \text{where} \qquad \bar{\chi}^{abc} = (\chi_{abc})^\dagger \gamma^0 \quad \text{and} \quad (\gamma^\mu)^T = C\gamma^\mu C^{-1}\,.$$

To compare with $\mathcal{N}=4$ super-Yang-Mills we need to understand the $SO(6)$ group content.

$$USp(8) \supset SO(6) \qquad \text{so that} \qquad \mathbf{8} = \mathbf{4} + \bar{\mathbf{4}}\,.$$

For χ^{abc}, $\mathbf{48} = \mathbf{20} + \overline{\mathbf{20}} + \mathbf{4} + \bar{\mathbf{4}}$, and

$$\chi_{\mathbf{20}} \sim \mathrm{Tr}(\lambda X) \qquad \text{with} \qquad m = \frac{1}{2L}$$

$$\chi_{\mathbf{4}} \sim \mathrm{Tr}(\lambda F_+) \qquad \text{with} \qquad m = \frac{3}{2L}\,.$$

If $SO(6)$ were unbroken, then $\mathbf{8} = \mathbf{4} + \overline{\mathbf{4}}$ and $\mathbf{48} = \mathbf{20} + \overline{\mathbf{20}} + \mathbf{4} + \overline{\mathbf{4}}$ would take us a long way toward figuring out which χ modes don't mix with gravitinos.

$U(1)_a \times U(1)_b \times U(1)_c \subset SO(6)$ is unbroken, and carefully tracking the charges helps identify decoupled χ modes and their explicit duals.

Practice a little charge counting:

$$Z_1 = X_1 + iX_2 = Z_1^{(1,0,0)} \qquad \lambda_1 = \lambda_1^{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} \qquad \lambda_2 = \lambda_2^{(\frac{1}{2},-\frac{1}{2},-\frac{1}{2})}$$

$$\text{“maximal”} \quad \chi^{(\frac{3}{2},\frac{1}{2},\frac{1}{2})} \leftrightarrow \operatorname{Tr} \lambda_1 Z_1 \qquad \chi^{(\frac{3}{2},-\frac{1}{2},-\frac{1}{2})} \leftrightarrow \operatorname{Tr} \lambda_2 Z_1$$

$$\text{“overlapping”} \quad \chi^{(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})} \quad \begin{array}{c}\text{eight charge choices,}\\ \text{three-fold degenerate}\end{array} \quad \leftrightarrow \quad \begin{array}{c}\text{various } \operatorname{Tr} \lambda X \text{ and}\\ \operatorname{Tr} \lambda F_+ \text{ operators}\end{array}$$

“Maximal” fermions (24 in all) are the ones where one $U(1)$ charge is $\pm 3/2$. They can't mix with each other because they all have distinct (q_a, q_b, q_c). And they can't mix with gravitini because $\mathbf{4} + \overline{\mathbf{4}}$ has all $U(1)$ charges $\pm 1/2$.

Of the “overlapping” fermions, with (q_a, q_b, q_c) quantum numbers identical to gravitinos, 8 indeed mix with gravitinos (super-Higgs), while the other 16 decouple completely.

The upshot: There are 8 massive gravitinos, which we didn't study (!), plus 40 decoupled spin-$1/2$ fields, each of which satisfies an equation of motion of the form

$$\Big[i\gamma^\mu\nabla_\mu - g\left(m_1X^2 + \frac{m_2}{X^4}\right) + gq_1\gamma^\mu a_\mu + gq_2\gamma^\mu A_\mu \\ + ip_1X^4 f_{\mu\nu}\gamma^{\mu\nu} + i\frac{p_2}{X^2}F_{\mu\nu}\gamma^{\mu\nu}\Big]\chi = 0\,,$$

where $g = \frac{2}{L}$ and $(m_1, m_2, q_1, q_2, p_1, p_2)$ are parameters which differ among the 40 cases. Two interesting cases:

$\chi^{q_aq_bq_c}$	m_1	m_2	q_1	q_2	p_1	p_2	type	Dual operator
$\chi^{(\frac{3}{2},\frac{1}{2},\frac{1}{2})}$	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{2}$	1	$-\frac{1}{4}$	$\frac{1}{2}$	maximal	$\mathrm{Tr}\,\lambda_1 Z_1$
$\bar\chi^{(\frac{3}{2},-\frac{1}{2},\frac{1}{2})}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	1	$-\frac{1}{4}$	$-\frac{1}{2}$	maximal	$\mathrm{Tr}\,\bar\lambda_3 Z_1$

To get hold of $m_1, \ldots, p_2$, one needs several terms in $D = 5, \mathcal{N} = 8$ lagrangian:

$$\mathcal{L}_\chi = \underbrace{\frac{i}{12}\bar\chi^{abc}\gamma^\mu D_\mu\chi_{abc}}_{\boxed{q_1,\, q_2}} + \frac{ig}{2}\bar\chi^{abc}\underbrace{\left(\frac{1}{2}A_{bcde} - \frac{1}{45}\Omega_{bd}T_{ce}\right)}_{\boxed{m_1,\, m_2}}\chi_a{}^{de} + \underbrace{\frac{i}{8}F_{\mu\nu}{}^{ab}\bar\chi_{acd}\gamma^{\mu\nu}\chi_b{}^{cd}}_{\boxed{p_1,\, p_2}}$$

2.3. Finding the Fermi surfaces

We want to find singularities in two-point functions of operators dual to χ, e.g. $\mathrm{Tr}\,\lambda X$.

Let's briefly rehearse some well-established methods, e.g. from [Faulkner et al, 0907.2694].

$$\chi = e^{-2A} h^{-1/4} e^{-i\omega t + ikx}\Psi \qquad\qquad \Psi = \begin{pmatrix} \Psi_{1+} \\ \Psi_{1-} \\ \Psi_{2+} \\ \Psi_{2-} \end{pmatrix}$$

$$\left.\begin{aligned} \left(\partial_r + \frac{me^B}{\sqrt{h}}\right)\Psi_{\alpha-} &= \frac{e^{B-A}}{\sqrt{h}}\left[u(r) + (-1)^\alpha k - v(r)\right]\Psi_{\alpha+} \\ \left(\partial_r - \frac{me^B}{\sqrt{h}}\right)\Psi_{\alpha+} &= \frac{e^{B-A}}{\sqrt{h}}\left[-u(r) + (-1)^\alpha k - v(r)\right]\Psi_{\alpha-} \end{aligned}\right\} \qquad \alpha = 1, 2$$

where

$$u(r) = \frac{1}{\sqrt{h}}(\omega + gq_1\phi + gq_2\Phi) \qquad v(r) = 2e^{-B}\left(p_1 X^4 \partial_r\phi + p_2 \frac{1}{X^2}\partial_r\Phi\right) .$$

Near the boundary, for $|mL| \neq 1/2$, we have

$$\Psi_{\alpha+} \sim A_\alpha(k) r^{mL} + B_\alpha(k) r^{-mL-1} , \qquad \Psi_{\alpha-} \sim C_\alpha(k) r^{mL-1} + D_\alpha(k) r^{-mL} ,$$

while for $mL = 1/2$ we have

$$\Psi_{\alpha -} \sim C_\alpha(k) r^{-1/2} \log r + D_\alpha(k) r^{-1/2} .$$

A spinorial Green's function $G_\alpha(k) \sim \langle \mathcal{O}_\alpha(k) \overline{\mathcal{O}}_\alpha(-k) \rangle$ can be defined through

$$D_\alpha(k) = G_\alpha(k) A_\alpha(k)$$

A singularity occurs when $A \to 0$ without $D \to 0$. ($C \sim A$ and $B \sim D$ always). Corresponds to a normal mode of χ.

Near-horizon analysis is equally important and equally familiar: $AdS_2 \times \mathbf{R}^2$ region with new scaling exponents:

$$ds^2 = \frac{L_2^2}{\zeta^2}(-d\tau^2 + d\zeta^2) + K_x^2 d\vec{x}^2 \qquad a_\tau = \frac{e_1}{\zeta} \quad A_\tau = \frac{e_2}{\zeta}$$

where

$$r - r_H = \frac{K_r}{\zeta} \qquad t = K_\tau \tau$$

and $(K_x, K_r, K_\tau, e_1, e_2)$ are somewhat complicated combinations of q, Q, r_H, and L.

Singularities occur at $\omega = 0$, and allowed solution in $AdS_2 \times \mathbf{R}^2$ is

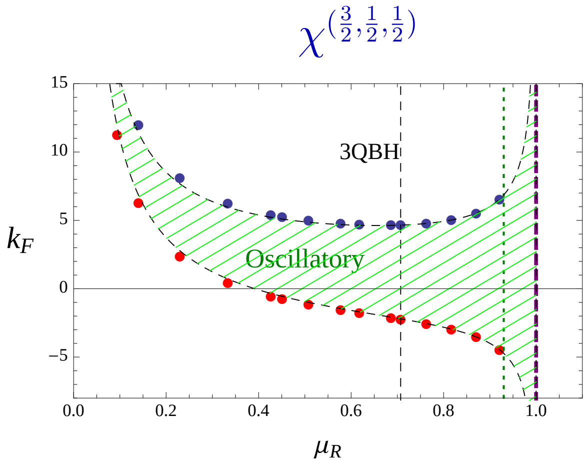
$$\Psi_\alpha \propto \zeta^{-\nu_k} \qquad \text{where} \quad \nu_k = \sqrt{m_2^2 L_2^2 - g^2(q_1 e_1 + q_2 e_2)^2}$$

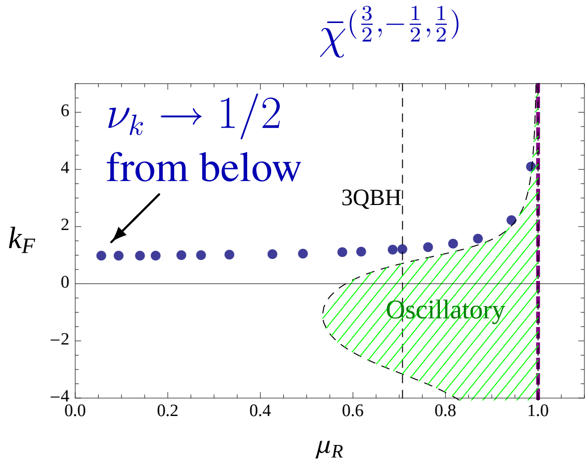
$$m_2^2 = m^2(X_H) + \frac{\tilde{k}^2}{K_x^2} \qquad \tilde{k} = k - (-1)^\alpha \frac{2K_x}{L_2^2}\left(p_1 X_H^4 + p_2 \frac{1}{X_H^2}\right) .$$

Two contrasting cases:

- ν_k imaginary: purely infalling mode, non-zero flux into the horizon at $\omega = 0$, can't be a normal mode. "Oscillatory."
- ν_k real and positive: $\Psi_\alpha \to 0$ as $\zeta \to \infty$, which is $r \to r_H$. "Normalizable."

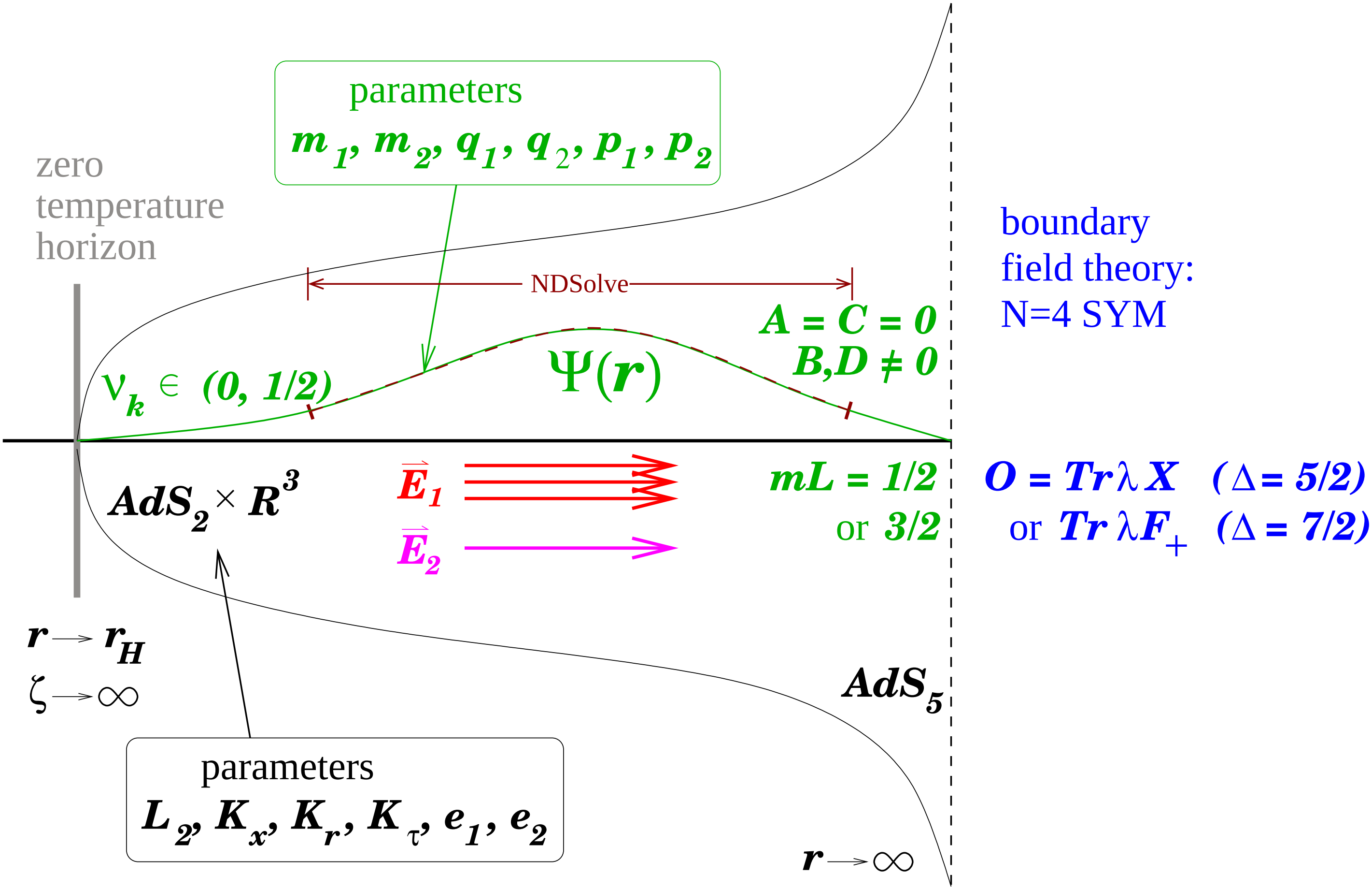
Results with $r_H = 1$:





(For three-charge black hole, 3QBH, $r_H = \frac{1}{\sqrt{3}} r_0$ due to change in definition of r.)

Each point is a normalizable bulk fermion mode, indicating a Fermi surface.



2.4. The MFL regime

We found $0 \leq \nu_k < 1/2$ for all Fermi surfaces we found: *all* are non-Fermi liquids, where Fermi velocity is not formally defined.

$$G_R = \frac{h_1}{k_\perp - \frac{1}{v_F}\omega - h_2 e^{i\gamma_F}\omega^{2\nu_F}} \qquad \text{close to the Fermi surface, with} \quad k_\perp = k - k_F\,.$$

MFL limit is where $\nu_k \to 1/2$ from below, so that Fermi velocity is almost defined:

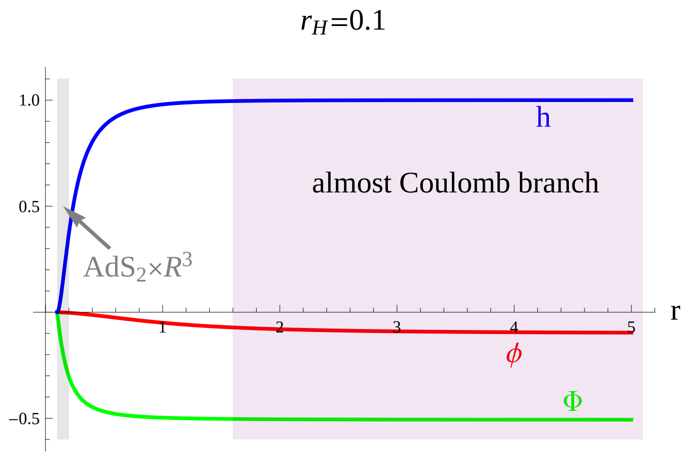
$$G_R \approx \frac{h_1}{k_\perp + \tilde{c}_1 \omega \log \omega + c_1 \omega}\,.$$

MFL theory has had notable successes in describing normal state of cuprates near optimal doping [Varma, Littlewoord, Schmitt-Rink, Abrahams, Ruckenstein 1989], but its microscopic underpinnings are not well understood.

What's going on as we approach the MFL limit in our construction?

- $T = 0$ is held fixed—by construction.
- Convenient to hold $q = 1$ fixed too. Thus $r_H \to 0$. Also set $L = 4G_5 = 1$.
- $s \approx 2r_H^2 \to 0$ and $Q \approx r_H \to 0$.

- $\rho_a \approx 2r_H \to 0$ and $\rho_A \approx 2r_H^2 \to 0$.
- ρ_a/ρ_A is big, but $\mu_R = \mu_1/\mu_2$ is small: Weird.
- $AdS_2 \times \mathbf{R}^2$ region gets squeezed, since it goes out only to $\sim 2r_H$.
- Outer geometry converges essentially to Coulomb branch solution.



- Basically, take the SUSY Coulomb branch solution at finite q and "dope" with a little bit of Q—but remember, ρ_a and ρ_A are the conserved charge densities.
- Can demonstrate analytically that $k_F \approx r_H$ at small r_H for $\bar{\chi}^{(\frac{3}{2},-\frac{1}{2},\frac{1}{2})}$.
- $\chi^{(\frac{3}{2},\frac{1}{2},\frac{1}{2})}$ has finite k_F/μ_2 and $\nu_k < 1/2$ (non-MFL, but co-existing with MFL).

The Coulomb branch solution has a long history [Kraus, Larsen, Trivedi hep-th/9811120; Freedman, Gubser, Pilch, Warner hep-th/9906194; Bianchi, DeWolfe, Freedman, Pilch, hep-th/0009156]. The main things to know are:

- The 5-d geometry lifts to the background of a uniform disk of D3-branes spread uniformly in the X_1-X_2 directions out to a radius q.
- Two-point functions characteristically exhibit a continuum above a gap $\Delta_g \equiv q/L^2$. Surprising because BPS spectrum extends down continuously to 0.

In particular, fermion two-point functions exhibit the gap Δ_g.

Is this an insulator band-gap? (Remember we haven't "doped" yet.) Or is it a superconducting gap?

The claim: The Coulomb branch state is an insulator wrt A_μ and a superconductor wrt a_μ.

Superconductivity is subtle to see in 5-d because the only scalar involved (our friend X) is *neutral*—like a dilaton.

Let's finish with an examination of conductivities to demonstrate the **claim**.

Conductivities are simple for the Coulomb branch because the linear perturbations

$$a_x = e^{-i\omega t} b_x(r) \qquad\qquad A_x = e^{-i\omega t} B_x(r)$$

decouple from all other perturbations. (Usually h_{tx} couples, but here the background *has no charge.*)

$$b_x'' + \frac{3r^2 - q^2}{r^3 + q^2 r} b_x' + \frac{\omega^2 L^4}{r^4 + q^2 r^2} b_x = 0 \qquad\qquad B_x'' + \frac{3}{r} B_x' + \frac{\omega^2 L^4}{r^4 + q^2 r^2} B_x = 0\,.$$

Solutions are easy:

$$b_x = \frac{1}{X^6} B_x = \frac{\Gamma\left(\frac{1+\sqrt{1-\omega^2}}{2}\right)\Gamma\left(\frac{3+\sqrt{1-\omega^2}}{2}\right)}{\Gamma\left(1+\sqrt{1-\omega^2}\right)} r^{1+\sqrt{1-\omega^2}} \times$$
$$ {}_2F_1\left(\frac{1+\sqrt{1-\omega^2}}{2}, \frac{3+\sqrt{1-\omega^2}}{2}; 1+\sqrt{1-\omega^2}; -r^2\right)$$

where we've taken either the more regular solutions at $r \to 0$, or the purely infalling ones.

How can b_x possibly describe a superconductor while $B_x = X^6 b_x$ describes an insulator?

Expand both at large r and small ω to get low frequency conductivity. $q = L = 1$.

$$b_x = \left(1 - \frac{1}{r^2}\right) + \frac{1 + 2\log r}{4r^2}\omega^2 + \mathcal{O}(\omega^4) + \mathcal{O}(1/r^4)$$
$$B_x = 1 + \frac{1 + 2\log r}{4r^2}\omega^2 + \mathcal{O}(\omega^4) + \mathcal{O}(1/r^4)\,.$$

- The $\frac{\log r}{r^2}\omega^2$ term gets canceled by $S_{\text{c.t.}} \propto \int d^4x\ (f_{mn}^2 + 2F_{mn}^2)$.
- We then read off Green's function from $b_x = 1 + G_a^R(\omega)/2r^2$.
- $\lim_{\omega\to 0} G_a^R(\omega) = -2$ while $\lim_{\omega\to 0} G_A^R(\omega) = 0$.

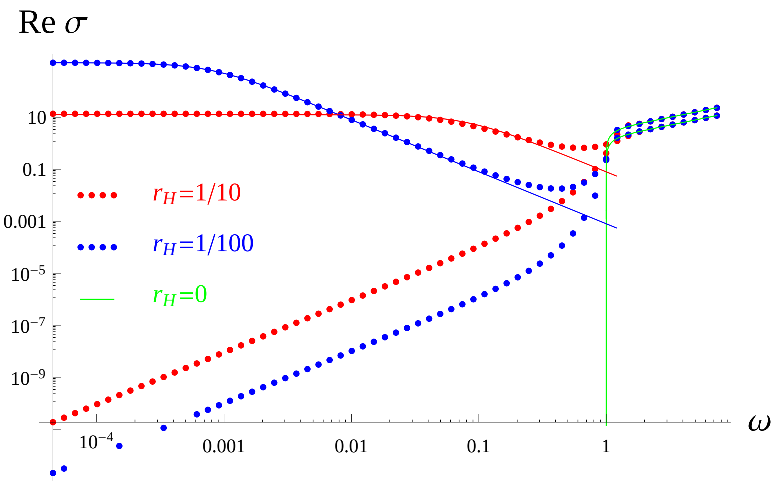
Now we can extract the low-frequency conductivity:

$$\sigma(\omega) \equiv \frac{G^R(\omega)}{i(\omega + i\epsilon)} \approx \frac{G^R(0)}{i(\omega + i\epsilon)} = G^R(0)\left[-i\mathcal{P}\frac{1}{\omega} - \pi\delta(\omega)\right]$$

So the $-\frac{1}{r^2}$, coming precisely from $1/X^6$, is just what we need to go from insulative behavior for A_μ to superconducting behavior for a_μ.

Full spectral measure shows hard-gapped s-wave superconductivity for a_μ:

$$\operatorname{Re}\sigma_a(\omega) = 2\pi\delta(\omega) + \frac{\pi\omega}{2}\theta(\omega^2-1)\tanh\frac{\pi\sqrt{\omega^2-1}}{2}$$
$$\operatorname{Re}\sigma_A(\omega) = \pi\omega\,\theta(\omega^2-1)\tanh\frac{\pi\sqrt{\omega^2-1}}{2}\,.$$



We also worked out conductivities in the MFL regime: small r_H quantifies slight "doping" of Coulomb branch configuration.

- Gauge field perturbations now mix with each other and the metric.

- Small conductivity is mostly A_μ; large one is mostly a_μ.
- $\delta(\omega)$ behavior partially broadens to a Drude peak, but we're not sure we understand the full small ω behavior when $r_H \neq 0$.

3. Summary

- There are various Fermi surfaces in charged black holes dual to $\mathcal{N} = 4$ super-Yang-Mills.
- They are hard to find because supergravity is complicated. But sometimes k_F and ν_k are nice numbers.
- After initial study of gravitinos, we focused entirely on decoupled spin-$1/2$ particles.
- The charged black holes have some weird thermodynamics, including Gregory-Laflamme instabilities.
- We found a Marginal Fermi Liquid regime approaching a SUSY vacuum state (on the Coulomb branch) at zero temperature, along a "doping" axis. Could a field theory construction of MFL be within reach?
- The Coulomb branch state is a hard-gapped s-wave superconductor.
- $\mathcal{O}(N^2)$ scaling of two-point functions suggests that the Fermi surfaces are for adjoint fermions—like gauginos? But we're having trouble accounting for full range of Fermi surface behaviors without some recourse to bound states.