Curved and Fuzzy Geometry of D-branes

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w/ M. Douglas and C. Schweigert `hep-th/0003037`
w/ J. Hoppe and B. Pioline `hep-th/0007067`
w/ M. Petropoulos `to appear`

Some motivation

- Compactifications, brane worlds ...

- Branes in ADS/CFT

- Learn more about fundamental theory in various settings

Some related recent work

Felder, Froehlich, Fuchs, Schweigert
Stanciu, Figueroa-O' Farrill
Alekseev, Mironov, Morozov
Alekseev, Recknagel, Schomerus
Elitzur, Giveon, Kutasov, Rabinovici, Sarkissian
Li
Pawelcyk, S-J Rey
McGreevy, Susskind, Toumbas
Jevicki, Mihailescu, Ramgoolam
Kinar, Loewy, Schreiber, Sonnenschein, Yankielowicz

Does a (quantized) worldvolume magnetic field prevent a brane from shrinking to zero size?

Naively YES ...

$$F \sim \frac{N}{\text{Area}} \Longrightarrow \mathcal{E}_{magn} \sim \int F^2 \sim \frac{N^2}{\text{Area}}$$

so the energy carried by the field diverges when the size shrinks to zero.

But the argument is WRONG for D-branes, because of **strong field ('relativistic') effects**. Indeed,

$$\mathcal{E}_{DBI} \sim T \int \mathrm{det}^{1/2}(g+F)$$

$$\sim T \text{ Area } \sqrt{1+(\frac{N}{\text{Area}})^2}$$

This is finite and minimum when Area $\rightarrow 0$, so the D-brane would still like to shrink.

cf. Chandrasekhar bound for relativistic neutron stars!

So there are no finite-energy, extended and stable branes in flat space-time.

The story is different in the presence of a background (Neveu-Schwarz) B-field.

The simplest example is provided by the **WZW SU(2) model** for which:

$$ds^2 = k\alpha' \left[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta \, d\phi^2)\right] \ ,$$

(*metric of* S^3)

$$H \equiv dB = 2k\alpha' \sin^2\psi \, \sin\theta \, d\psi \, d\theta \, d\phi \ ,$$

($\sim$*volume form of* S^3)

with k the (integer) level of the associated current algebra. A convenient gauge is

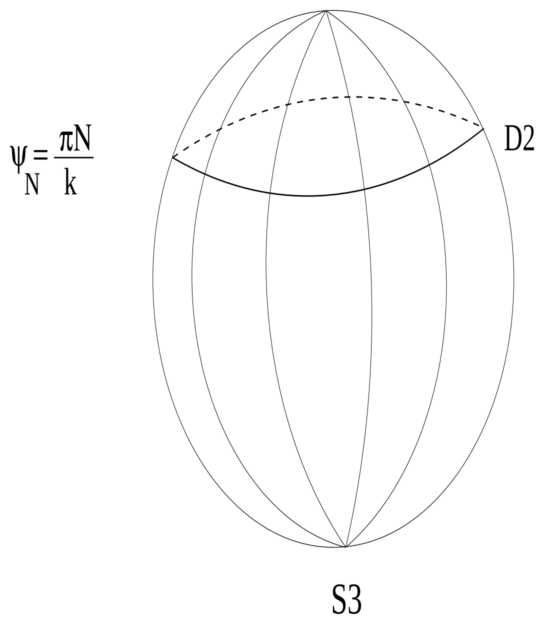
$$B = k\alpha' \left(\psi - \frac{\sin 2\psi}{2}\right) \sin\theta \, d\theta \, d\phi \ .$$

($\sim$*volume form of* S^2)

There are $k-1$ **stable spherical D2 branes** in this background, even though $\pi_2(S^3) = 0$.

Alekseev, Schomerus

These branes carry $N = 1, ..., k-1$ units of magnetic flux.



Stability follows from a simple minimization of the Dirac-Born-Infeld energy functional.

CB, Douglas, Schweigert

Flux stabilization of branes

In a 'minisuperspace' approximation (ψ constant), and for N units of magnetic flux,

$$F = dA = -\frac{N}{2}\,\mathrm{sin}\theta\; d\theta d\phi\ ,$$

the energy reads

$$E_N(\psi) = T\ \int_0^\pi d\theta \int_0^{2\pi} d\phi\ \sqrt{\mathrm{det}\ (\hat{g} + \hat{B} + 2\pi\alpha' F)} + \cdots$$

$$= 4\pi k\alpha'\ T\ \left(\mathrm{sin}^4\psi + (\psi - \tfrac{\mathrm{sin}2\psi}{2} - \tfrac{\pi N}{k})^2\right)^{1/2} + \cdots$$

For $0 < N < k$ this has a unique minimum at

$$\boxed{\psi_N = \frac{\pi N}{k}\ ,} \qquad (1)$$

where it takes the value

$$\boxed{M_N = 4\pi k\alpha'\, T\, \mathrm{sin}\left(\frac{\pi N}{k}\right)\ .} \qquad (2)$$

For values of N outside this range the minimum of the energy is a zero-size brane.

More generally, can consider arbitrary small fluctuations $\delta\psi(\theta,\phi,t)$, $\delta A_\phi(\theta,t)$.

The result for the spectrum is

$$\boxed{\omega^2(r) = j(j+1)/k\alpha' \ ,} \tag{3}$$

with $j = 0, 1, 2, ..$ and r the representation $(j-1) \oplus (j+1)$ of the unbroken SO(3) symmetry.

There is therefore a **triplet of zero modes** $(j = 0)$, corresponding to translations of the S^2 in S^3.

All other modes are positive, which confirms the stability of the branes.

Ramond-Ramond charge

What about the RR one-form charge?

There are two possible definitions:

- $$Q_N = 2\pi\alpha' T \int F \;=\; 4\pi^2\alpha'\, T\, N$$

which is quantized but not gauge invariant, or

- $$Q_N = T \int B + 2\pi\alpha' F \;=\; 2\pi k\alpha'\, T \sin\frac{2\pi N}{k}\ .$$

which is gauge invariant, but not quantized.

This is not, a priori, paradoxical because the corresponding RR field obtains a mass in this background.

The existence of different definitions of 'charge' known in other systems with Chern-Simmons couplings (?)

W. Taylor; Marolf

Comparing with CFT

In the diagonal-invariant bosonic theory there are $(k+1)$ independent boundary states preserving a SU(2) symmetry, i.e. imposing the gluing conditions

$$J^a = g\bar{J}^a g^{-1} \ .$$

These can be decomposed in two different bases:

• **Ishibashi** (character) states $|p \gg_I$ $(p = 0, .., k)$ which couple to closed-string primaries in the representation $p/2 \otimes p/2$,

• **Cardy** states $|q \gg_C$ $(q = 0, .., k)$ with a classical D-brane interpretation (integer-multiplicity of open strings).

Cardy; Pradisi, Sagnotti, Stanev

Comparing with CFT

The two are related by

$$|q \gg_C \equiv \sum_{p=0}^{k} \frac{S_{qp}}{\sqrt{S_{0p}}} |p \gg_I$$

where

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(i+1)(j+1)\pi}{k+2}\right)$$

is the modular-transformation matrix.

We want to identify $|q \gg_C$ with the D2 brane carrying $N = q+1$ units of magnetic flux.

Comparing with CFT

Let us check first the masses of the states, extracted from their static long-range interactions.

Polchinski

In the **supergravity** approximation, the exchange of the (seven-dimensional) graviton and dilaton between two identical D-branes a distance r apart gives

$$\mathcal{E}(r) = 2\kappa_{(7)}^2 \, M_N^2 \, \Delta_{(6)}(r)$$

with $\kappa_{(7)}$ the gravitational coupling in 7d, M_N the mass and $\Delta_{(6)}(r)$ the 6d Green's function.

The string-theory calculation on the other hand gives (here $\alpha' = 1/2$)

$$\mathcal{E}(r) = (2\pi)^4 \frac{|S_{q0}|^2}{S_{00}} \, \Delta_{(6)}(r)$$

where we have projected onto the identity-operator in the closed-string channel.

Comparing with CFT

Comparing the two expressions gives

$$M_N = \frac{2\sqrt{2}\pi^2}{\kappa_{(7)}} \frac{|S_{q0}|}{\sqrt{S_{00}}} = \frac{(2\pi)^{3/2}(2k+4)^{1/4}}{\kappa_{(7)}} \sin\left(\frac{N\pi}{k+2}\right)$$

Using the relations

$$T = \frac{\sqrt{2\pi^3}}{\kappa_{(10)}} \ , \quad \text{and} \quad \kappa^2_{(10)} = \kappa^2_{(7)} 2\pi^2 R^3$$

one finds

$$\boxed{M_N = 4\pi\alpha' T \ \left(\frac{(k+2)R^6}{\alpha'^3}\right)^{1/4} \sin\left(\frac{\pi N}{k+2}\right)}$$

This agrees with the semiclassical result in the $k \to \infty$ limit. The agreement is actually exact, if we shift the radius of S^3 to

$$R^2 = (k+2)\alpha' \ .$$

Comparing with CFT

Consider next the spectrum of quadratic fluctuations. The light open-string states are

$$|\text{open}>= J^a_{-1}|j>$$

where $|j>$ is a integer spin-j primary.

These transform in the $(j-1)\oplus j\oplus(j+1)$ representations, but imposing the Virasoro constraint will project the representation j out. The mass squared is $j(j+1)/(k+2)$. These results agree with the semiclassical analysis.

However: the CFT spectrum is truncated at

$$j < \min(N, k/2-N)$$

This is missed by the semiclassical analysis.

Comparing with CFT

Finally, the RR charge is given by the coefficient of the $|p = 1 \gg_I$ character state. Indeed, the exchanged RR closed string transform in the $(p/2 \otimes 1/2, p/2 \otimes 1/2)$ representation because of the fermionic zero modes, so $p = 1$ contains the singlet.

A calculation similar to the one for the mass gives

$$Q_N = \frac{(2\pi)^{3/2}(2k+4)^{1/4}}{2\kappa_{(7)}} \sin\left(\frac{2N\pi}{k+2}\right) \ ,$$

in perfect agreement again with the semiclassical result for gauge-invariant charge.

In summary: mass, charge, multiplicity and small fluctuation spectrum of semiclassical analysis are exact, up to the $k \to k+2$ shift and the truncation of the latter at N.

'Fuzzy bound states'

The mass of the Nth D-brane is less than the mass of N free D-particles,

$$M_N = 4\pi k\alpha'\, T \sin\left(\frac{\pi N}{k}\right) \; < (4\pi^2\alpha' T)\, N \; .$$

There should thus exist a 'T-dual' description in terms of **D-particle bound states.**

This is analogous to the bound states in a constant RR four-form field, $C_{0ij} \sim \epsilon_{ijk}X^k$. The non-abelian WZ coupling

$$\sim \mathrm{tr} \int C_{0ij}[X^i, X^j] \; ,$$

together with the non-abelian potential

$$\sim \mathrm{tr} \int [X^i, X^j]^2$$

imply that the minima of the energy obey $[X^i, X^j] \propto \epsilon_{ijk}X^k$. Here X^i are the $N\times N$ matrix coordinates of the D-particles.

Myers

'Fuzzy bound states'

A similar calculation can be done in principle in the case at hand, using Myers' DBI couplings

CB, Bain; Das, Trivedi, Vaidya

Deriving exact results is however difficult, because the full non-abelian DBI action is not known. The system could on the other hand be used to put constraints on the NADBI action.

Note also that the **spectrum truncation** is automatic in this dual picture, because the total number of degrees of freedom is N^2.

D-strings in $AdS_5 \times S^5$

Up to now did not worry about stability of the closed-string background. One way to embedd this into a completely consistent string context is to consider

$$SU(2)\, WZW \oplus Feigin/Fuchs \oplus R^6$$

which is the near-horizon geometry of NS five-branes. D-strings stretching radially inwards will then form such (supersymmetric) bound states.

A similar situation arises if we replace the NS5-branes by D3-branes. The AdS part of the near-horizon geometry is

$$ds^2 = L^2\ \frac{dy^2 + dx_\mu dx^\mu}{y^2}\ , \qquad C_{y0ab} = L^2\ \frac{\epsilon_{abc} x^c}{y^5}\ .$$

Here (y, x^μ) are the usual Poincaré coordinates, with $y = 0$ the AdS boundary.

D-strings in $AdS_5 \times S^5$

The worldvolume theory of N D-strings (parametrized by x^0 and y) contains non Abelian DBI and WZ terms, which must combine into an energy functional

$$\boxed{\mathcal{E} = TL^2 \int \frac{dy}{2y^6}\ \mathrm{Tr}\ \left(y^2 \frac{\partial X^a}{\partial y} - \frac{1}{2}\epsilon^{abc}[X^b, X^c] \right)^2 + \cdots}$$

CB, Hoppe, Pioline

Can guess this from the **superconformal symmetry** of the problem: susy $\Longrightarrow \mathcal{E}$ is perfect square, while scale invariance ($y \to \lambda y,\ X^\mu \to \lambda X^\mu$) fixes the powers of y.

Alternatively, recognize the square terms as coming from SYM reduction. The cross term would be a total derivative in flat space-time, but here contributes a bulk term. After integration by parts, this is the non Abelian WZ coupling.

NB: Could have used susy argument to discover the non Abelian WZ coupling.

D-strings in $AdS_5 \times S^5$

Susy configurations given by solutions to **Nahm's equations**

$$y^2 \frac{dX^a}{dy} = \frac{1}{2}\epsilon^{abc}[X^b, X^c] \ ,$$

but **with unusual boundary conditions:**

$$X^a \sim -y\rho_+^a + \text{subleading} \quad \text{at} \quad y \to 0$$

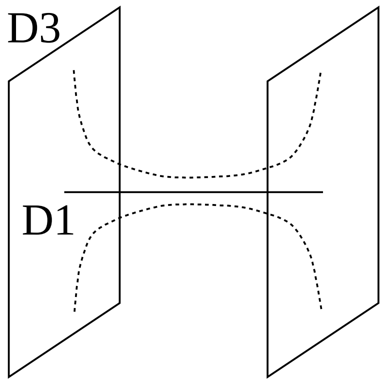
$$X^a \sim -y\rho_-^a + \text{subleading} \quad \text{at} \quad y \to \infty$$

Here $\rho_\pm^a$ are, a priori, any N-dimensional representations of SU(2).

D-strings in $AdS_5 \times S^5$

Recall that for a D1-D3 system, N stretched D-strings are charge-N **BPS monopoles** in the $SU(2)$ theory on the D3's. The $1/r$ divergence of the Higgs field can be interpreted as a D1 spike on the D3 worldvolume.

Callan, Maldacena; Gibbons



Conversely, the coordinates of the stretched D strings satisfy Nahm's equations on the interval $s = 1/y \in [0, 1]$,

Diaconescu

with standard boundary conditions,

$$X^a(s \to 0) \sim \frac{\rho^a}{s} \;, \qquad X^a(s \to 1) \sim \frac{\rho^a}{1-s} \;.$$

These can be motivated by the fact that the D-strings blow up into a **fuzzy D3-brane** at the endpoints.

Constable; Myers; Tafjord

D-strings in $AdS_5 \times S^5$

The modified b.cs in our case are imposed by scale invariance.

Same mathematical problem arises in study of instantons in $S^3 \times R$

Kronheimer

and for domain walls of $N = 1^*$ theory

CB, Hoppe, Pioline

Consider the trivial solutions

$$X^a = -y\rho^a \quad \text{for} \quad \text{all} \quad y \ .$$

Can specify a N-dimensional rep. of SU(2) by a partition

$$n_1 + n_2 + ... + n_k = N$$

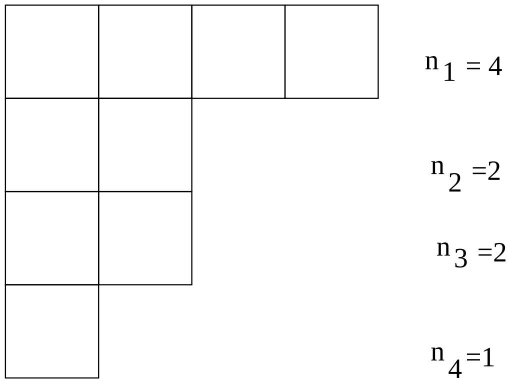
(we take $n_1 > n_2 ... > n_k$)

The above solution corresponds to k distinct bound states of n_1 ,..., n_k D-strings each.

D-strings in $AdS_5 \times S^5$

What is the dual holographic interpretation of these bound states?

Conjecture that they correspond to heavy extrenal sources (Wilson-'t Hooft lines) in the



representation of $SU(\infty)$.

Evidence? : correct U(1) charge, and minimum number of blown up D3-branes. No other (gauge-invariant) interpretation could think of. How to check?

$N = 1^*$ *Domain walls*

The $\mathcal{N} = 4$ Yang-Mills theory in 4d has a deformation lifting all of its moduli space while preserving N-1 susy:

$$W = \mathrm{Tr} \sum \left(\frac{1}{6} \epsilon^{abc} \Phi^a [\Phi^b, \Phi^c] - \frac{1}{2} m \Phi^a \Phi^a \right) .$$

The supersymmetric classical vacua of the theory obey

$$\frac{\partial W}{\partial \Phi^a} = \frac{1}{2} \epsilon^{abc} [\Phi^b, \Phi^c] - m \Phi^a = 0 \ , \quad D = 0$$

with $D = \sum [\Phi^i, \Phi^i] = 0$. and are hence in one-to-one correspondence with N-dimensional **unitary representations of** $SU(2)$,

Vafa,Witten

$$\Phi^a = m \rho^a \ , \quad [\rho^a, \rho^b] = \epsilon^{abc} \rho^c \ , \quad \rho^a = -(\rho^a)^\dagger$$

The number of classical vacua grows as $\sim e^{\sqrt{N}}$. The unbroken gauge symmetry is

$$G_{(\rho)} \equiv \prod U(n_j)/U(1)$$

Quantum mechanically, the vacua with $U(n)$ non-Abelian factors split into n distinct vacua.

Are there SUSY domain walls interpolating between classical vacua ?

$N = 1^*$ *Domain walls*

Such walls must obey

$$\left(\frac{d\Phi^a}{dx}\right)^\dagger = -\frac{\partial W}{\partial \Phi^a} , \quad D = 0 .$$

Choosing antihermitian matrices to satisfy the D-term constraint leads to

$$\frac{d\Phi^a}{dx} = \frac{1}{2}\epsilon^{abc}[\Phi^b, \Phi^c] - m\Phi^a ,$$

with boundary conditions

$$\begin{array}{l} \Phi^a(x \to -\infty) \to m\rho_-^a \\ \Phi^a(x \to +\infty) \to m\rho_+^a \end{array} , \quad W(\rho_-) > W(\rho_+)$$

Changing variables

$$\Phi^a = -e^{-mx}X^a \qquad \text{and} \qquad s = e^{-mx}/m ,$$

gives precisely our previous Nahm problem.

$N = 1^*$ *domain walls*

This mathematical problem can be solved with the following results:

Kronheimer; CB, Hoppe, Pioline; Frey

• There exist BPS walls for any initial state $\{n_i\}$ and final $\{m_j\}$ such that

$$n_1 \geq m_1; \quad n_1 + n_2 \geq m_1 + m_2 \quad \cdots$$

(this is a partial ordering).

ex: no wall for $\{10, 5, 1, 1\} \to \{8, 8, 1\}$

• The dimension of the moduli space is additive,

$$\text{dim}(\rho, \rho'') = \text{dim}(\rho, \rho') + \text{dim}(\rho', \rho'')$$

NB: See Frey for the dual supergravity description of these domain walls

Concluding remarks

Fuzzy geometries are generic in presence of **'electric or magnetic' backgrounds as in AdS**. Brane tension can be balanced by flux leading to stable states. Alternatively, branes may bind into higher-dimensional ones, gaining energy by a 'dielectric' effect.

This 'fuzziness' has been argued to be at the heart of the string exclusion principle. It is an example of 'IR/UV mixing'. It is very likely an important feature of the underlying theory.