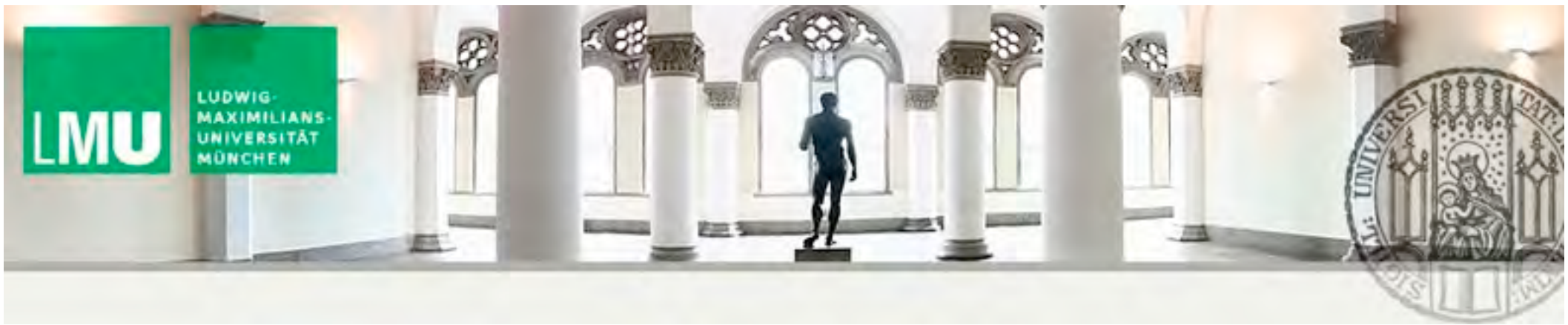
Non-commutative & non-associative closed string geometry from flux compactifications

Dieter Lüst, LMU (Arnold Sommerfeld Center)
and MPI München



6th Regional Meeting in String theory, Milos, 22. June 2011

I) Introduction

Closed string flux compactifications:

- Moduli stabilization ➡ string landscape
- AdS/CFT correspondence
- Generalized geometries
- Here: how does a closed string see space ?
  What is the proper geometrical description of closed string flux background ?

  Non-commutative & non-associative geometry !

D.L., arXiv:1010.1361; R. Blumenhagen, E. Plauschinn, arXiv:1010.1263;

R. Blumenhagen, A. Deser, D.L., E. Plauschinn, F. Rennecke, arXiv:1106.0316;

(See also: L. Cornalba, R. Schiappa (2001); P. Bouwknegt, K. Hannabuss, V. Mathai (2006))

Point particle in a (constant) magnetic field:

Configuration space: $\mathcal{M} = T^*\mathcal{Q}\,, \qquad \vec{B} = \mathrm{rot}\ \vec{A}$

Langrange function: $L = \frac{1}{2}(p_i)^2 = \frac{1}{2}(\dot{x}^i - A^i)^2$

Canonical momenta: $p_i = \frac{\partial L}{\partial \dot{x}^i} = \dot{x}^i - A^i$

$$\pi^{ij} = \{x^i, x^j\} = 0\,, \quad \pi_{ij} = \{p_i, p_j\} = 0\,, \quad \{x^i, p_j\} = \delta^j_i$$

Mechanical momenta: $\overline{p}^i = \dot{x}^i = p^i + A^i$

$$\pi^{ij} = \{x^i, x^j\} = 0\,, \quad \bar{\pi}_{ij} = \{\overline{p}_i, \overline{p}_j\} = \epsilon_{ijk}B^k\,, \quad \{x^i, p_j\} = \delta^j_i$$

Non-commutative (Poisson) algebra

Point particle in the field of a magnetic monopole:

(thanks to Thomas Strobl)

$$\vec{B} \in H^2(\mathcal{Q}),\ H = dB = \star\rho_{magn}$$ (B is non-closed)

ρ_{magn} ... charge density of a magnetic monopole.

$$\pi^{ij} = \{x^i, x^j\} = 0\,, \quad \bar{\pi}_{ij} = \{\overline{p}_i, \overline{p}_j\} = H_{ijk}x^k\,, \quad \{x^i, p_j\} = \delta^j_i$$

This leads to:

$$\bar{\pi}_{ijk} = \{\{\overline{p}_i, \overline{p}_j\}, \overline{p}_k\} + \text{perm.} = H_{ijk}$$

Twisted Poisson structure.

(C. Klimcik, T. Strobl, (2002); A. Alekseev, T. Strobl, (2005); C. Saemann, R. Szabo, arXiv:1106.1890)

As we will see, we will get a twisted Poisson structure for closed strings, however for the position operators instead of the momentum operators.

Non-commutative geometry and string theory (a):

Open strings:

2-dimensional D-branes with 2-form F-flux $\Rightarrow$ coordinates of open string end points become non-commutative:

$$[X_i(\tau), X_j(\tau)] = \epsilon_{ij}\Theta\,, \qquad \Theta = -\frac{2\pi i\alpha' F}{1+F^2}$$

(A. Abouelsaood, C. Callan, C. Nappi, S. Yost (1987);
J. Fröhlich, K. Gawedzki (1993); F. Lizzi, ER. Szabo (1997);
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➢ Non-commutative gauge theories.

(N. Seiberg, E. Witten (1999); J. Madore, S. Schraml, P. Schupp, J. Wess (2000); ....

$$f_1(x) \star f_2(x) \star \ldots \star f_N(x) :=$$

$$\exp\left[i\sum_{m<n}\Theta^{ab}\,\partial_a^{x_m}\,\partial_b^{x_n}\right]\, f_1(x_1)\, f_2(x_2)\ldots f_N(x_N)\Big|_{x_1=\ldots=x_N=x}$$

$$S \simeq \int d^n x\, \mathrm{Tr}\hat{F}_{ab} \star \hat{F}^{ab}$$

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Remark: In the T-dual picture (D1-brane at angle) the coordinates are commutative!

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3-dimensional backgrounds with 3-form flux $\Rightarrow$

we will show that coordinates of closed strings become non-commutative:

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and even non-associative:

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Outline:

II) T-duality & twisted tori

III) NC-NA Closed string geometry

IV) Non-associative closed string CFT

II) T-duality & twisted tori

How does a closed string see geometry?

Consider compactification on a circle with radius R:

$$X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma)$$

$$\begin{aligned} X_L(\tau+\sigma) &= \frac{x}{2} + p_L(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau+\sigma)}\,, \\ X_R(\tau-\sigma) &= \frac{x}{2} + p_R(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n e^{-in(\tau-\sigma)} \end{aligned}$$

(KK momenta)

$$\begin{aligned} p_L &= \frac{1}{2}\left(\frac{M}{R} + (\alpha')^{-1}NR\right), \qquad & p &= p_L + p_R = \frac{M}{R} \\ p_R &= \frac{1}{2}\left(\frac{M}{R} - (\alpha')^{-1}NR\right) & \tilde{p} &= p_L - p_R = (\alpha')^{-1}NR \end{aligned}$$

(dual momenta - winding modes)

T-duality: $T: \quad R \longleftrightarrow \frac{\alpha'}{R}, \quad M \longleftrightarrow N$

$$T: \quad p \longleftrightarrow \tilde{p}\,, \quad p_L \longleftrightarrow p_L\,, \quad p_R \longleftrightarrow -p_R\,.$$

- Dual space coordinates: $\tilde{X}(\tau,\sigma) = X_L - X_R$

$(X, \tilde{X})$: Doubled geometry:

(O. Hohm, C. Hull, B. Zwiebach (2009/10))

T-duality is part of diffeomorphism group.

$$T: \quad X \longleftrightarrow \tilde{X}\,, \quad X_L \longleftrightarrow X_L\,, \quad X_R \longleftrightarrow -X_R$$

Compactification on a 2-dimensional torus:

Background: $R_1,\ R_2,\ e^{i\alpha},\ B$

2 complex background parameters:

$$\tau = \frac{e_2}{e_1} = \frac{R_2}{R_1} e^{i\alpha}\,,$$

$$\rho = B + iR_1 R_2 \sin\alpha\,.$$

T-duality transformations:

- $SL(2,\mathbb{Z})_\tau$: $\quad \tau \to \dfrac{a\tau + b}{c\tau + d}$
- $SL(2,\mathbb{Z})_\rho$: $\quad \rho \to \dfrac{a\rho + b}{c\rho + d}$

They act as shifts/rotations on doubled coordinates.

- T-duality in x_1 $\Leftrightarrow$ Mirror symmetry:

$$\tau \leftrightarrow \rho \Longleftrightarrow B \leftrightarrow \Re\tau$$

Three-dimensional backgrounds ⇒ twisted 3-tori:

(A. Dabholkar, C. Hull (2003) ; S. Hellerman, J. McGreevy, B. Williams (2004); J. Derendinger, C. Kounnas, P. Petropoulos, F. Zwirner (2004); J. Shelton, W. Taylor, B. Wecht (2005); G. Dall'Agata, S. Ferrara (2005)...)

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{x^1,x^2} \hookrightarrow M^3 \hookrightarrow S^1_{x^3}$$

The fibration is specified by its monodromy properties.

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Two (T-dual) cases:

(i) Geometric spaces (manifolds)

$$x^3 \to x^3 + 2\pi \;\Rightarrow\; \tau(x^3 + 2\pi) = \frac{a\tau(x^3) + b}{c\tau(x^3) + d}$$

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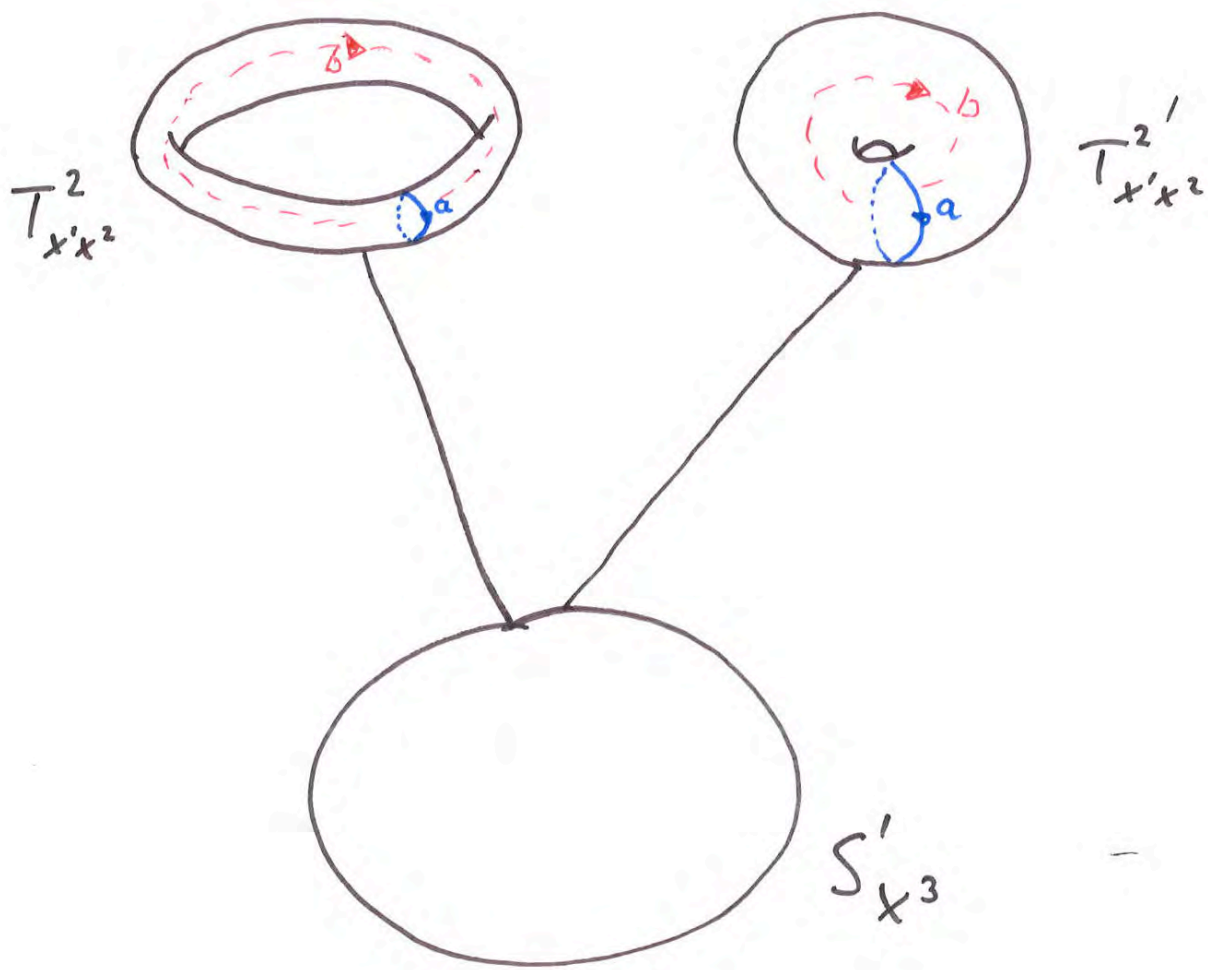

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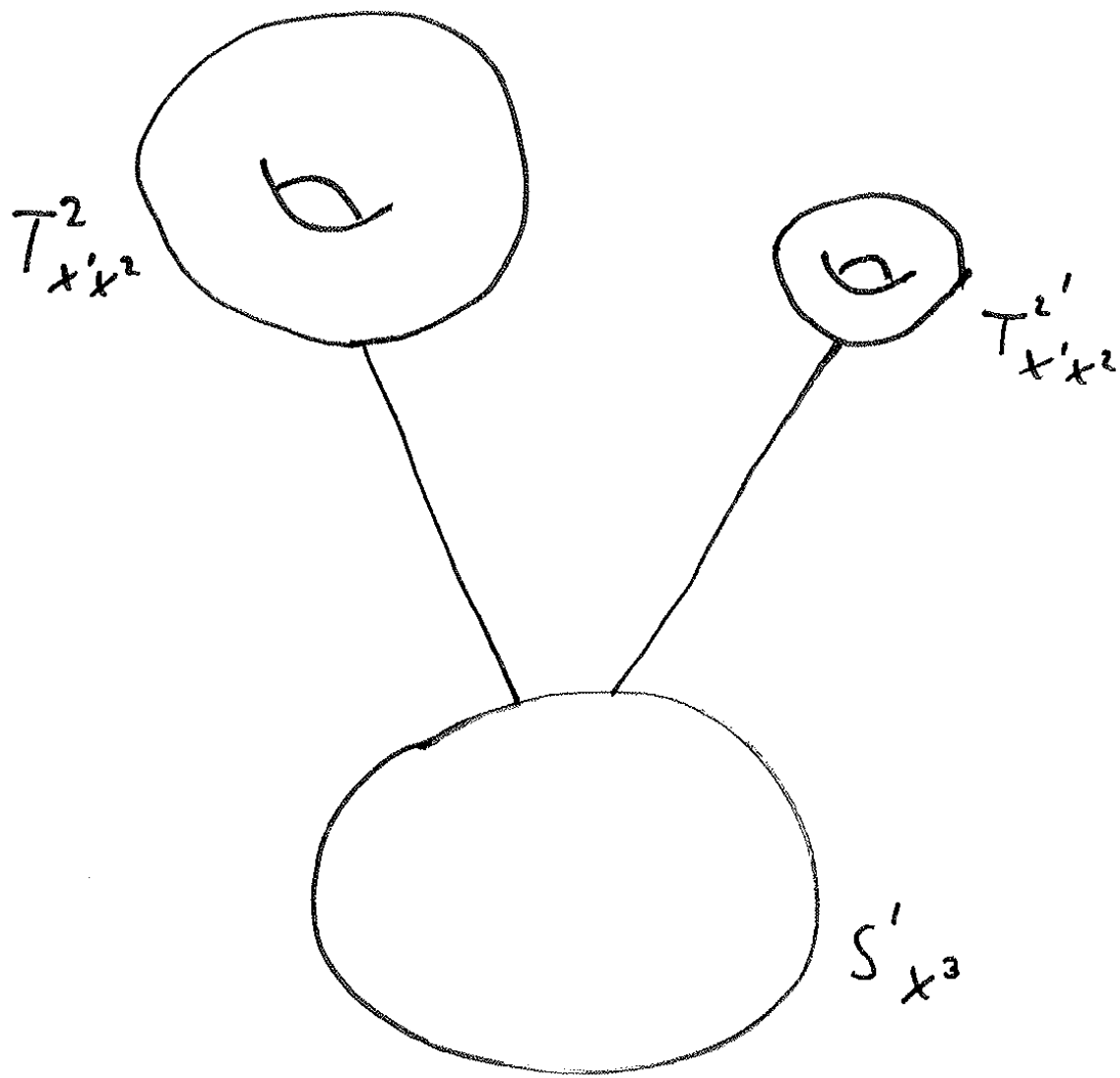

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Fi A ten-dimensional action for non-geometrical fluxes:

(D. Andriot, M. Larfors, D.L. P. Patalong, arXiv:1106.4015)

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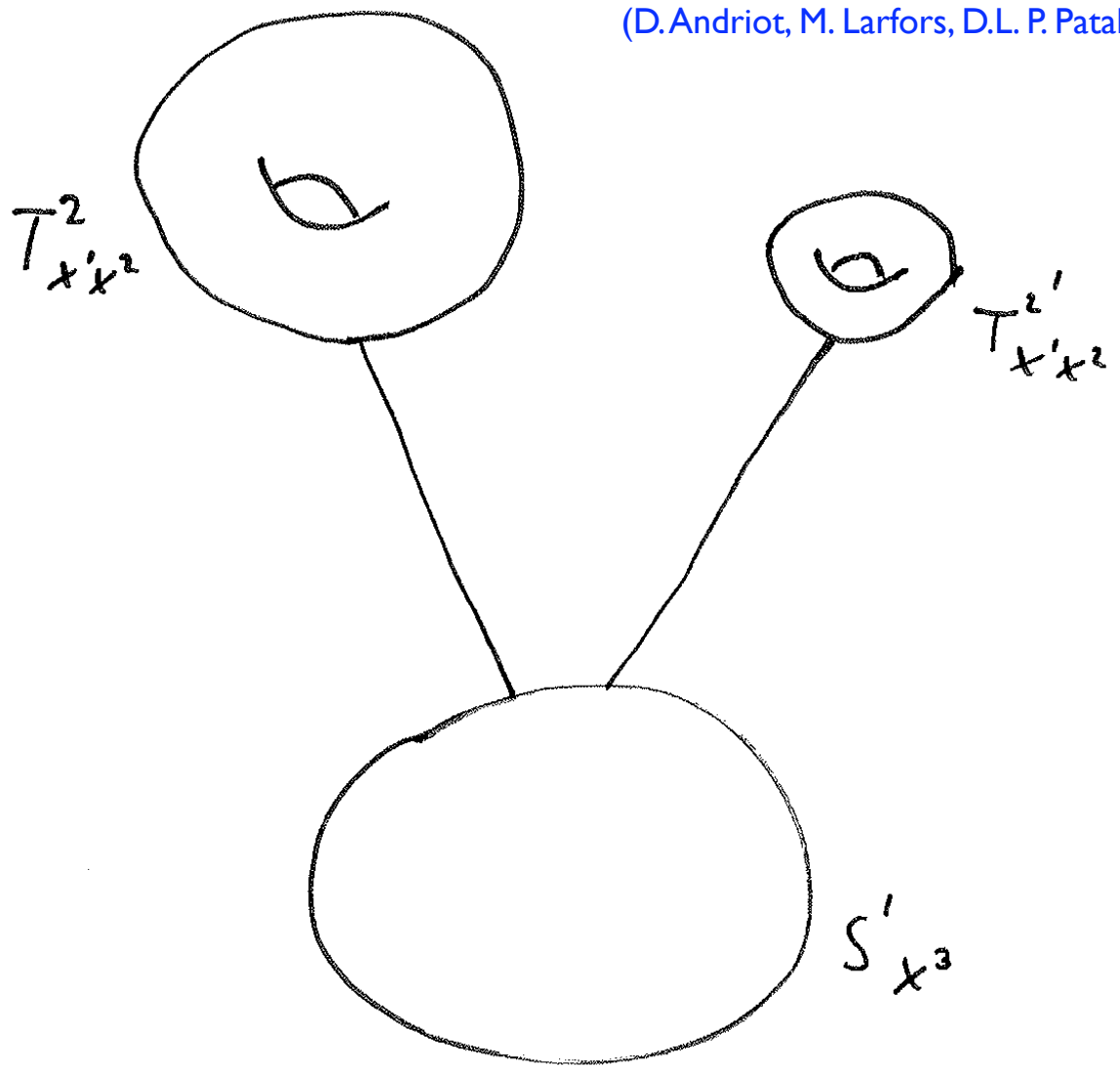

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III) NC-NA Closed string geometry

Can the closed string also see a non-commutative space?

What deformation is needed?

Yes: one needs 3-form flux: $F^{(3)} =$ H/ω/Q/R $(F^{(3)} = \partial F^{(2)})$

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More general:

Doubled geometry: Closed string non-commutativity in $(X, \tilde{X})$-space

Problem:

- Background is non-constant.
- CFT is in general not exactly solvable

Ways to handle:

- Study SU(2) WZW model with H-flux

  (R. Blumenhagen, E. Plauschinn, arXiv:1010.1263)

- Consider monodromy properties and the corresponding closed string boundary conditions
  $\Rightarrow$ Shifted closed string mode expansion

  (D.L., arXiv:1010.1361)

- Consider sigma model perturbation theory for small fluxes

  (R. Blumenhagen, A. Deser, D.L., E. Plauschinn, F. Rennecke, arxiv:1106.0316)

Specific example: elliptic monodromy

C. Hull, R. Reid-Edwards (2009))

(i) Geometric space (ω -flux) $(\omega_{123} \sim \partial_{x^3} g_{x^1 x^2} \sim \partial_{x^3} \Re\tau(x^3))$

$$\tau(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \quad (H \in \frac{1}{4} + \mathbb{Z})$$

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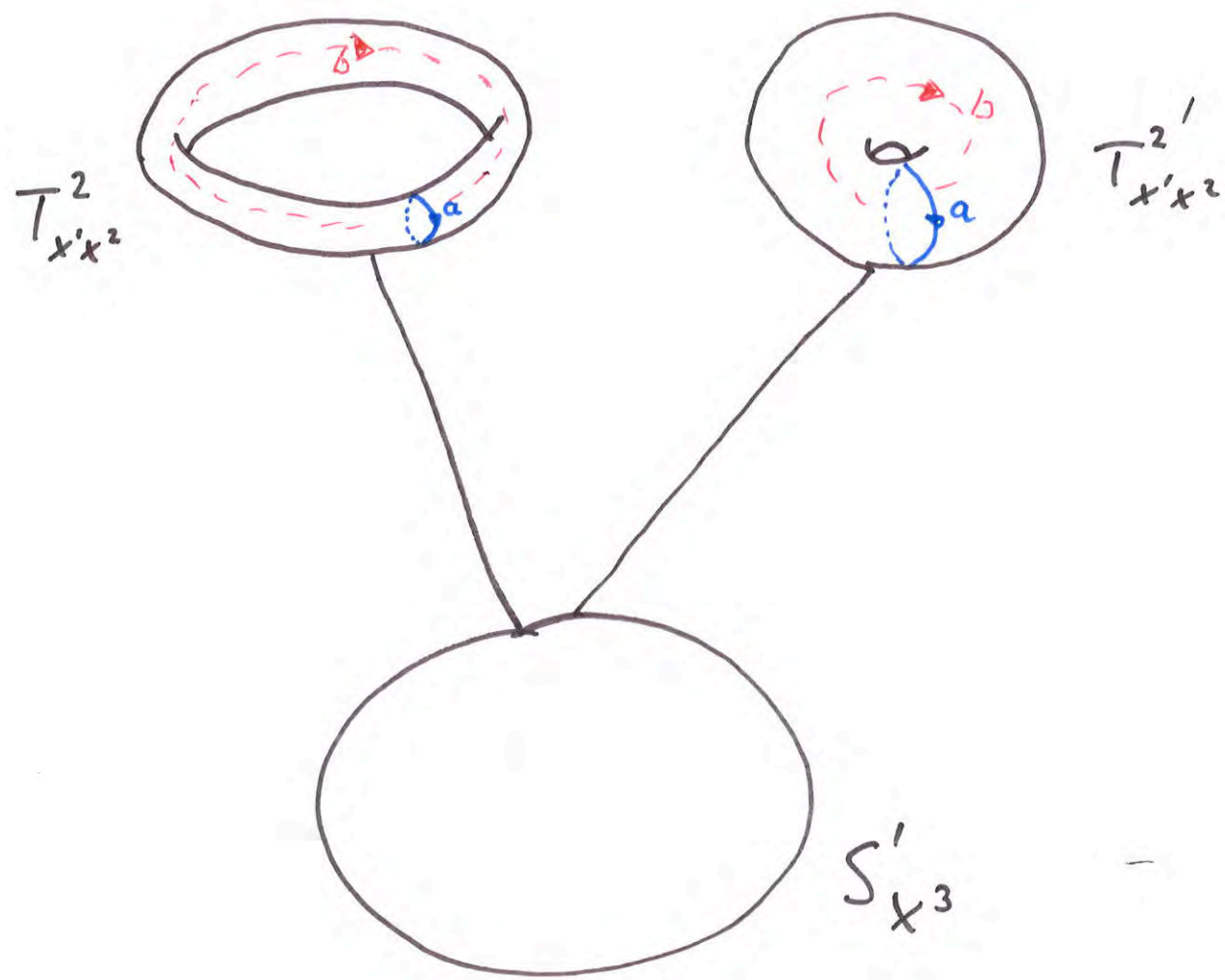

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This induces the following $\mathbb{Z}_4$ symmetric closed string boundary condition:

winding number

$$X^3(\tau, \sigma + 2\pi) = X^3(\tau, \sigma) + 2\pi N_3$$

$$X_L(\tau, \sigma + 2\pi) = e^{i\theta} X_L(\tau, \sigma)\,, \quad \theta = -2\pi N_3 H\,,$$

$$X_R(\tau, \sigma + 2\pi) = e^{i\theta} X_R(\tau, \sigma)\,.$$

L-R symmetric order 4 rotation

(Complex coordinates: $X_{L,R} = X^1_{L,R} + iX^2_{L,R}$)

Corresponding closed string mode expansion $\Rightarrow$

$$X_L(\tau+\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu}\alpha_{n-\nu}e^{-i(n-\nu)(\tau+\sigma)}\,, \qquad \nu = \frac{\theta}{2\pi} = -N_3 H\,,$$

$$X_R(\tau-\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n+\nu}\tilde{\alpha}_{n+\nu}e^{-i(n+\nu)(\tau-\sigma)}$$

(shifted oscillators!)

Then one obtains:

$$[X_L(\tau,\sigma),\bar{X}_L(\tau,\sigma)] = -[X_R(\tau,\sigma),\bar{X}_R(\tau,\sigma)] = \tilde{\Theta}$$

$$\tilde{\Theta} = \alpha'\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu} = -\alpha'\pi\cot(\pi N_3 H)$$

$$[X^1(\tau,\sigma),X^2(\tau,\sigma)] = [X^1_L + X^1_R, X^2_L + X^2_R] = 0$$

T-dual geometry (mirror symmetry): $\tau(x^3) \leftrightarrow \rho(x^3)$

(ii) Non-geometric space (Q-flux)

$$\rho(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \qquad (H \in \frac{1}{4} + \mathbb{Z})$$

$\Rightarrow$ H-field: $H(x^3) = H\,\frac{10 - 12\sin(2Hx^3) - 6\cos(2Hx^3)}{(2\sin(2Hx^3) + \cos(2Hx^3) - 3)^2}$

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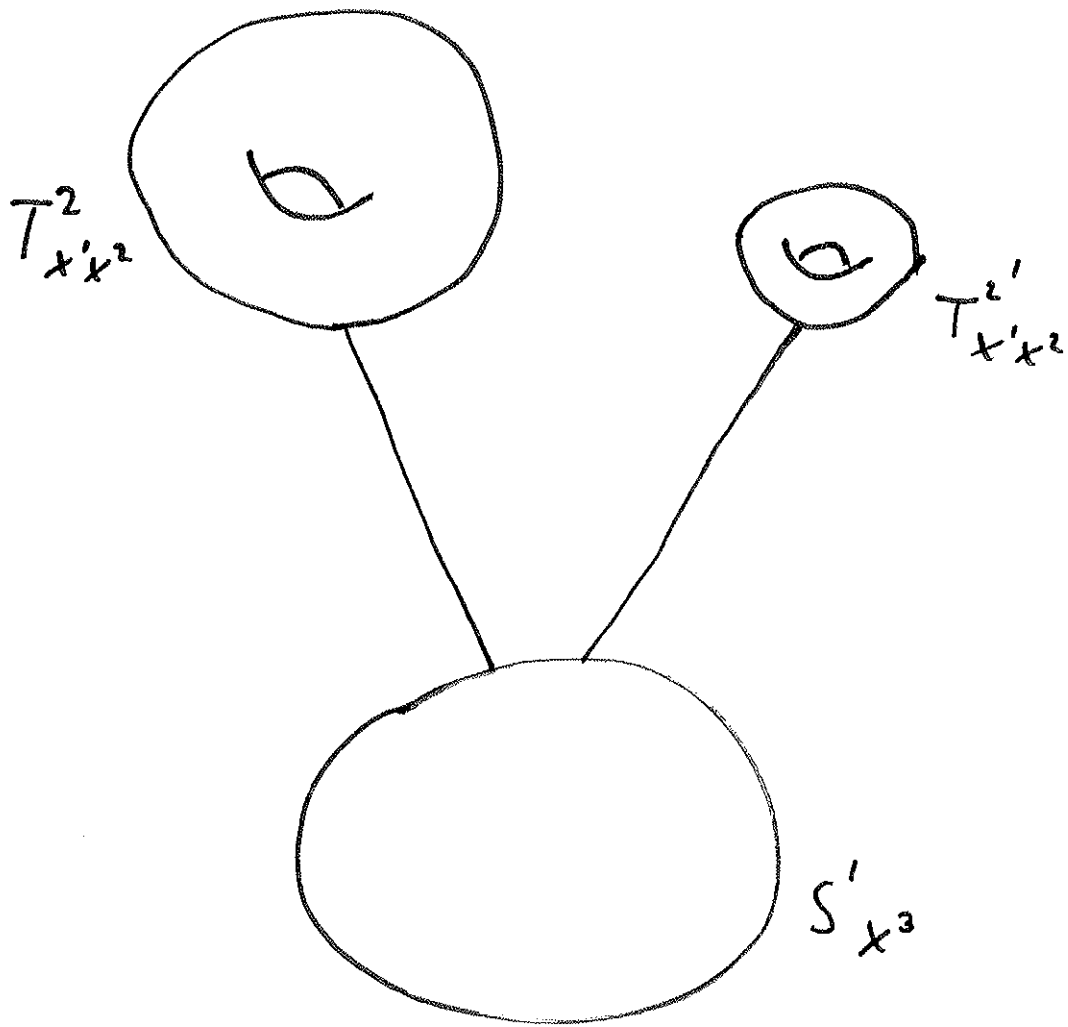

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$$\rho(x^3) = \frac{(1+i)\cos(Hx^3) + \sin(Hx^3)}{\cos(Hx^3) - (1+i)\sin(Hx^3)} \qquad (H \in \frac{1}{4} + \mathbb{Z})$$

$\Rightarrow$ H-field: $H(x^3) = H\,\frac{10 - 12\sin(2Hx^3) - 6\cos(2Hx^3)}{(2\sin(2Hx^3) + \cos(2Hx^3) - 3)^2}$

Monodromy: $\rho(x^3 + 2\pi) = -1/\rho(x^3)$

This induces the following $\mathbb{Z}_4$ asymmetric closed string boundary condition:

$$X^3(\tau, \sigma + 2\pi) = X^3(\tau, \sigma) + 2\pi N_3$$

$$\begin{aligned} X_L(\tau, \sigma + 2\pi) &= e^{i\theta} X_L(\tau, \sigma)\,, \qquad \theta = -2\pi N_3 H\,, \\ X_R(\tau, \sigma + 2\pi) &= e^{-i\theta} X_R(\tau, \sigma)\,. \end{aligned}$$

L-R a-symmetric order 4 rotation

Corresponding closed string mode expansion $\Rightarrow$

$$X_L(\tau+\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n-\nu}\alpha_{n-\nu}e^{-i(n-\nu)(\tau+\sigma)}\,, \qquad \nu = \frac{\theta}{2\pi} = -N_3 H\,,$$

$$X_R(\tau-\sigma) = i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\frac{1}{n+\nu}\tilde{\alpha}_{n-\nu}e^{-i(n-\nu)(\tau-\sigma)}$$

Then one finally obtains:

$$[X_L(\tau,\sigma),\bar{X}_L(\tau,\sigma)] = [X_R(\tau,\sigma),\bar{X}_R(\tau,\sigma)] = \tilde{\Theta}$$

$$[X^1(\tau,\sigma),X^2(\tau,\sigma)] = [X^1_L + X^1_R, X^2_L + X^2_R] = i\tilde{\Theta}$$

T-duality in x^3 - direction $\Rightarrow$ R-flux

Winding no. $N_3 \Longleftrightarrow$ Momentum no. M_3

$$[X^1(\tau,\sigma), X^2(\tau,\sigma)] = i\Theta$$

$$\Theta = \alpha' \sum_{n\in\mathbb{Z}} \frac{1}{n-\nu} = -\alpha'\pi\cot(\pi M_3 H)$$

Act on wave functions $\Rightarrow$ replace momentum number

by momentum operator:

$$M_3 \equiv \sqrt{\alpha'}\, p^3\,, \qquad N_3 \equiv \sqrt{\alpha'}\, \tilde{p}^3$$

Then one obtains the following non-commutative algebra:

$$[X^1, X^2] \simeq i l_s^3 F^{(3)} p^3 \qquad ([X^i, X^j] \simeq i\epsilon^{ijk} F^{(3)} p^k)$$

Use $\quad [p^3, X^3] = -i$

$$\Longrightarrow \quad [[X^1, X^2], X^3] + \text{perm.} \simeq F^{(3)} l_s^3$$

Non-associative algebra (twisted Poisson structure)!

This nicely agrees with the non-associative closed string structure found by Blumenhagen, Plauschinn in the SU(2) WZW model: arXiv:1010.1263

Summary: Chain of three T-dualities:

Flux	Commutators	Three-brackets
H-flux	$[\tilde{X}^1, \tilde{X}^2] \simeq w_3$	$[\tilde{X}^1, \tilde{X}^2, \tilde{X}^3]$
ω-flux	$[X^1, \tilde{X}^2] \simeq w_3$	$[X^1, \tilde{X}^2, \tilde{X}^3]$
Q-flux	$[X^1, X^2] \simeq w_3$	$[X^1, X^2, \tilde{X}^3]$
R-flux	$[X^1, X^2] \simeq p_3$	$[X^1, X^2, X^3]$

IV) Non-associative closed string CFT

R. Blumenhagen, A. Deser, D.L., E. Plauschinn, F. Rennecke, arXiv:1106.0316

General idea: Consider a flat background plus linear B-field:

$$B_{ab} = \frac{1}{3} H_{abc}\, X^c$$

CFT-perturbation theory linear in H:

(Theory is still conformal at linear order in H.)

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1, \qquad \mathcal{S}_1 = \frac{1}{2\pi\alpha'} \frac{H_{abc}}{3} \int_\Sigma d^2 z\, X^a \partial X^b\, \partial X^c$$

Correlation function:

$$\begin{aligned} \langle \mathcal{O}_1 \ldots \mathcal{O}_N \rangle &= \frac{1}{\mathcal{Z}} \int [dX]\, \mathcal{O}_1 \ldots \mathcal{O}_N\, e^{-\mathcal{S}[X]} \\ &= \langle \mathcal{O}_1 \ldots \mathcal{O}_N \rangle_0 - \langle \mathcal{O}_1 \ldots \mathcal{O}_N\, \mathcal{S}_1 \rangle_0 + \ldots \end{aligned}$$

Solution of classical equations of motion:

$$\partial\overline{\partial}X^a = \tfrac{1}{2}H^a{}_{bc}\,\partial X^b\,\overline{\partial}X^c$$

$$X_0^a(z,\overline{z}) = \mathsf{X}_L^a(z) + \mathsf{X}_R^a(\overline{z})$$

$$\mathsf{X}^a(z,\overline{z}) = \mathsf{X}_0^a(z,\overline{z}) + \tfrac{1}{2}H^a{}_{bc}\,\mathsf{X}_L^b(z)\,\mathsf{X}_R^c(\overline{z})$$

Tachyon vertex operator:

$$\mathcal{V}(z,z) = :\exp\big(ik_L\cdot\mathcal{X}_L + ik_R\cdot\mathcal{X}_R\big)$$

$$k_L^a = p^a + \frac{w^a}{\alpha'}\,, \qquad k_R^a = p^a - \frac{w^a}{\alpha'}$$

T-duality: $\mathcal{X}^a_L(z)$ $\xrightarrow{\text{T-duality}}$ $+\mathcal{X}^a_L(z)$,

$\mathcal{X}^a_R(\overline{z})$ $\qquad\qquad$ $-\mathcal{X}^a_R(\overline{z})$.

Allowed momenta and winding:

H-flux		ω-flux		Q-flux		R-flux	
$\langle p_1, p_2, p_3\rangle^-$	✓	$\langle p_1, p_2, w_3\rangle^-$	✓	$\langle p_1, w_2, w_3\rangle^-$	✓	$\langle w_1, w_2, w_3\rangle^-$	✓
$\langle w_1, w_2, w_3\rangle^+$	✓	$\langle w_1, w_2, p_3\rangle^+$	✓	$\langle w_1, p_2, p_3\rangle^+$	✓	$\langle p_1, p_2, p_3\rangle^+$	✓

T-duality: $\mathcal{X}^a_L(z)$ $\xrightarrow{\text{T-duality}}$ $+\mathcal{X}^a_L(z)$,

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Allowed momenta and winding:

H-flux		ω-flux		Q-flux		R-flux	
$\langle p_1, p_2, p_3\rangle^-$	✓	$\langle p_1, p_2, w_3\rangle^-$	✓	$\langle p_1, w_2, w_3\rangle^-$	✓	$\langle w_1, w_2, w_3\rangle^-$	✓
$\langle w_1, w_2, w_3\rangle^+$	✓	$\langle w_1, w_2, p_3\rangle^+$	✓	$\langle w_1, p_2, p_3\rangle^+$	✓	$\langle p_1, p_2, p_3\rangle^+$	✓

Remark: At higher order in H, the theory will flow to some new CFT:

$$\mu\,\frac{\partial g^{ab}}{\partial\mu} = -\frac{\alpha'}{4} H^a{}_{pq}\, H^{bpq}$$

H-flux: Flow to the SU(2) WZW model.

R-flux: Flow to an L-R asymmetric „SU(2) WZW“ model.

Basic three-point function:

$$\left\langle \mathcal{X}^a(z_1,z_1)\,\mathcal{X}^b(z_2,z_2)\,\mathcal{X}^c(z_3,z_3)\right\rangle$$
$$= -\frac{\alpha'^2}{12}\,H^{abc}\left[L\left(\frac{z_{12}}{z_{13}}\right)+L\left(\frac{z_{23}}{z_{21}}\right)+L\left(\frac{z_{13}}{z_{23}}\right)-\text{c.c.}\right]$$

$$(z_{ij}=z_i-z_j)$$

(agrees with WZW-model computation: R. Blumenhagen, E. Plauschinn, arXiv:1010.1263)

Rogers dilogarithm: $L(x)=\mathrm{Li}_2(x)+\frac{1}{2}\log(x)\log(1-x)$

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Rogers dilogarithm: $L(x)=\mathrm{Li}_2(x)+\frac{1}{2}\log(x)\log(1-x)$

This function is discontinuous when $z_1\to z_2=1,\ z_1\to z_3=0$

It develops a jump when all three points approach each other, i.e. $z_1\to z_3=0,\ z_2\to z_3=0$

3-tachyon amplitudes:

(i) 3 momentum states in H-background:

$$\langle\, \mathcal{T}_1\, \mathcal{T}_2\, \mathcal{T}_3\, \rangle^- = \int \prod_{i=1}^3 d^2 z_i\, \delta^{(2)}(z_i - z_i^0)\, \delta(p_1 + p_2 + p_3) \times \exp\Big[-i\theta^{abc}\, p_{1,a} p_{2,b} p_{3,c} \big[\mathcal{L}\big(\tfrac{z_{12}}{z_{13}}\big) - \mathcal{L}\big(\tfrac{\bar{z}_{12}}{\bar{z}_{13}}\big)\big]\Big]_\theta .$$

(ii) 3 momentum states in R-background:

(corresponds to 3 winding states in H-background)

$$\langle\, \mathcal{T}_1\, \mathcal{T}_2\, \mathcal{T}_3\, \rangle^+ = \int \prod_{i=1}^3 d^2 z_i\, \delta^{(2)}(z_i - z_i^0)\, \delta(p_1 + p_2 + p_3) \times \exp\Big[-i\theta^{abc}\, p_{1,a} p_{2,b} p_{3,c} \big[\mathcal{L}\big(\tfrac{z_{12}}{z_{13}}\big) + \mathcal{L}\big(\tfrac{\bar{z}_{12}}{\bar{z}_{13}}\big)\big]\Big]_\theta .$$

Behavior under permutations of the vertex operators:

$$\langle\, \mathcal{V}_{\sigma(1)} \mathcal{V}_{\sigma(2)} \mathcal{V}_{\sigma(3)} \rangle^\epsilon = \exp\Big[\, i\, \big(\tfrac{1+\epsilon}{2}\big)\, \eta_\sigma\, \pi^2\, \theta^{abc}\, p_{1,a}\, p_{2,b}\, p_{3,c}\Big] \langle \mathcal{V}_1\, \mathcal{V}_2\, \mathcal{V}_3 \rangle^\epsilon$$

$(\epsilon = \mp 1$ for H or R)

N-tachyon amplitudes:

$$\langle \mathcal{V}_1\, \mathcal{V}_2\, \ldots\, \mathcal{V}_N \rangle^{\mp} = \langle V_1\, V_2\, \ldots\, V_N \rangle_0^{\mp} \;\times$$

$$\exp\Big[-i\theta^{abc}\sum\nolimits_{1\leq i<j<k\leq N} p_{i,a}\, p_{j,b}\, p_{k,c}\Big[\mathcal{L}\big(\tfrac{z_{ij}}{z_{ik}}\big) \mp \mathcal{L}\big(\tfrac{z_{ij}}{z_{ik}}\big)\Big]\Big]_\theta$$

E.g. the fluxed Virasoro-Shapiro amplitude with N=4:

$$\langle \mathcal{V}_1\, \mathcal{V}_2\, \mathcal{V}_3\, \mathcal{V}_4 \rangle^{\mp} = \langle V_1\, V_2\, V_3\, V_4 \rangle_0^{\mp} \times$$

$$\exp\Big[-i\theta^{abc}\, p_{1,a}\, p_{2,b}\, p_{3,c}\, \Big[\mathcal{L}\big(\tfrac{z_{12}}{z_{13}}\big) - \mathcal{L}\big(\tfrac{z_{12}}{z_{14}}\big) + \mathcal{L}\big(\tfrac{z_{13}}{z_{14}}\big) - \mathcal{L}\big(\tfrac{z_{23}}{z_{24}}\big) \mp \text{c.c.}\Big]\Big]_\theta$$

The phase factor appearing when commuting two vertex operators can be decoded in a deformed N-product:

$$V_{p_1}(x) \,{}_N\, \ldots \,{}_N\, V_{p_N}(x) \stackrel{\text{def}}{=} \exp\Big(-i\,\frac{\pi^2}{2}\,\theta^{abc} \sum_{1\leq i<j<k\leq N} p_{i,a}\,p_{j,b}\,p_{k,c}\Big)\, V_{\sum p_i}(x)$$

Non-associative $\triangle$ - product for functions:

$$f_1(y) \triangle f_2(y) \triangle \ldots \triangle f_N(y) :=$$

$$\exp\left[\sum_{m<n<r} F^{abc}\,\partial_a^{y_m}\,\partial_b^{y_n}\,\partial_c^{y_r}\right]\, f_1(y_1)\,f_2(y_2)\ldots f_N(y_N)\Big|_{y_1=\ldots=y_N=y}$$

(see also: K. Savvidy (2002))

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- Is there are non-commutative (non-associative) theory of gravity? Is there a map to commutative gravity (like SW-map for gauge theories)?

(Non-commutative geometry & gravity: P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess (2005))