New guises of AdS_3 and the entropy of two-dimensional CFT

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Outline

- Fefferman-Graham parametrization of AdS_3 with various boundary metrics
- Stress-energy tensor
- Entropy
- Comments

P. Apostolopoulos, G. Siopsis, N. T. : arxiv:0809.3505[hep-th], Phys. Rev. Lett. 102 (2009) 151301
N. T. : arxiv:0905.2763[hep-th], JHEP 1003 (2010) 040
N. Lamprou, S. Nonis, N. T. : arXiv:1106.1533 [gr-qc]
N. T. : arXiv:1106.2492 [hep-th]

The BTZ black hole

- Metric in Schwarzschild coordinates (ϕ has a period equal to 2π):

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \qquad f(r) = r^2 - \mu. \tag{1}$$

- Temperature, energy and entropy of the black hole ($V = 2\pi$):

$$T = \frac{1}{2\pi}\sqrt{\mu}, \qquad E = \frac{V}{16\pi G_3}\mu, \qquad S = \frac{V}{4G_3}\sqrt{\mu}. \tag{2}$$

- Metric in Fefferman-Graham coordinates:

$$ds^2 = \frac{1}{z^2}\left[dz^2 - \left(1 - \frac{\mu}{4}z^2\right)^2 dt^2 + \left(1 + \frac{\mu}{4}z^2\right)^2 d\phi^2\right]. \tag{3}$$

with

$$z = \frac{2}{\mu}\left(r \mp \sqrt{r^2 - \mu}\right), \qquad r = \frac{1}{z} + \frac{\mu}{4}z. \tag{4}$$

- z takes values $0 < z \leq z_e = 2/\sqrt{\mu}$ and $z_e \leq z < \infty$, covering twice the region outside the event horizon.
  r takes values $r_e = \sqrt{\mu} \leq r < \infty$. Throat at $z = 2/\sqrt{\mu}$.

- General metric in Fefferman-Graham coordinates:

$$ds^2 = \frac{1}{z^2}\left[dz^2 + g_{\mu\nu}dx^\mu dx^\nu\right], \tag{5}$$

where

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu}. \tag{6}$$

- Holographic stress-energy tensor of the dual CFT (Skenderis 2000)

$$\langle T^{(CFT)}_{\mu\nu}\rangle = \frac{1}{8\pi G_3}\left[g^{(2)} - \mathrm{tr}\left(g^{(2)}\right) g^{(0)}\right]. \tag{7}$$

- Apply this to the BTZ black hole with a flat boundary

$$ds_0^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\phi^2. \tag{8}$$

Energy density and pressure:

$$\rho = \frac{E}{V} = -\langle T^t{}_t\rangle = \frac{\mu}{16\pi G_3}, \tag{9}$$

$$p = \langle T^\phi{}_\phi\rangle = \frac{\mu}{16\pi G_3}. \tag{10}$$

AdS_3 with a Rindler boundary

- For the Rindler wedge ($x > 0$), the metric (1) with $\mu = 0$ (AdS_3 in Poincare coordinates) can be put in the form

$$ds^2 = \frac{1}{z^2}\left[dz^2 - a^2x^2\left(1+\frac{z^2}{4x^2}\right)^2 dt^2 + \left(1-\frac{z^2}{4x^2}\right)^2 dx^2\right], \tag{11}$$

with a Rindler boundary

$$ds_0^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -a^2x^2dt^2 + dx^2. \tag{12}$$

- The coordinate transformation does not affect the time coordinate:

$$r(z,x) = a\left(\frac{x}{z}+\frac{z}{4x}\right) \tag{13}$$

$$\phi(z,x) = \frac{1}{a}\log[ax] - \frac{8}{a(4+z^2/x^2)}. \tag{14}$$

- The transformation maps the whole region near negative infinity for ϕ to the neighborhood of zero with $x > 0$. ϕ is not periodic.

- For fixed x, there is a minimal value for $r(z,x)$ as a function of z. It is obtained for

$$z_m(x) = 2x \tag{15}$$

and is equal to

$$r_m(x) = a. \tag{16}$$

The corresponding value of ϕ is

$$\phi_m(x) \equiv \phi_m(z_m(x), x) = (\log[ax] - 1)/a. \tag{17}$$

- Bridge connecting the two asymptotic regions (that are copies of each other) at $z \to 0$ and $z \to \infty$.
- The holographic stress-energy tensor is

$$\rho = -\langle T^t{}_t \rangle = -\frac{1}{16\pi G_3}\frac{1}{x^2}, \tag{18}$$

$$p = \langle T^x{}_x \rangle = -\frac{1}{16\pi G_3}\frac{1}{x^2}. \tag{19}$$

It displays the expected singularity at $x = 0$. The conformal anomaly vanishes.

Global coordinates

- AdS_3 can be written as

$$ds^2 = \frac{1}{\cos^2(\tilde{\chi})}\left[-d\tilde{t}^2 + d\tilde{\chi}^2 + \sin^2(\tilde{\chi})\, d\tilde{\phi}^2\right], \tag{20}$$

with $0 \leq \phi \leq 2\pi$, $0 \leq \tilde{\chi} < \pi/2$. The boundary is approached for $\tilde{\chi} \to \pi/2$.

- The Schwarzschild (Poincare) and the global coordinates are related through

$$\tilde{t}(t,r,\phi) = \arctan\left[\sqrt{\frac{r^2-\mu}{r^2}}\frac{\sinh(\sqrt{\mu}\,t)}{\cosh(\sqrt{\mu}\,\phi)}\right] \tag{21}$$

$$\tilde{\chi}(t,r,\phi) = \arctan\sqrt{\frac{r^2}{\mu}\sinh^2(\sqrt{\mu}\phi) + \frac{r^2-\mu}{\mu}\cosh^2(\sqrt{\mu}t)} \tag{22}$$

$$\tilde{\phi}(t,r,\phi) = \arctan\left[\sqrt{\frac{r^2-\mu}{r^2}}\frac{\cosh(\sqrt{\mu}\,t)}{\sinh(\sqrt{\mu}\,\phi)}\right]. \tag{23}$$

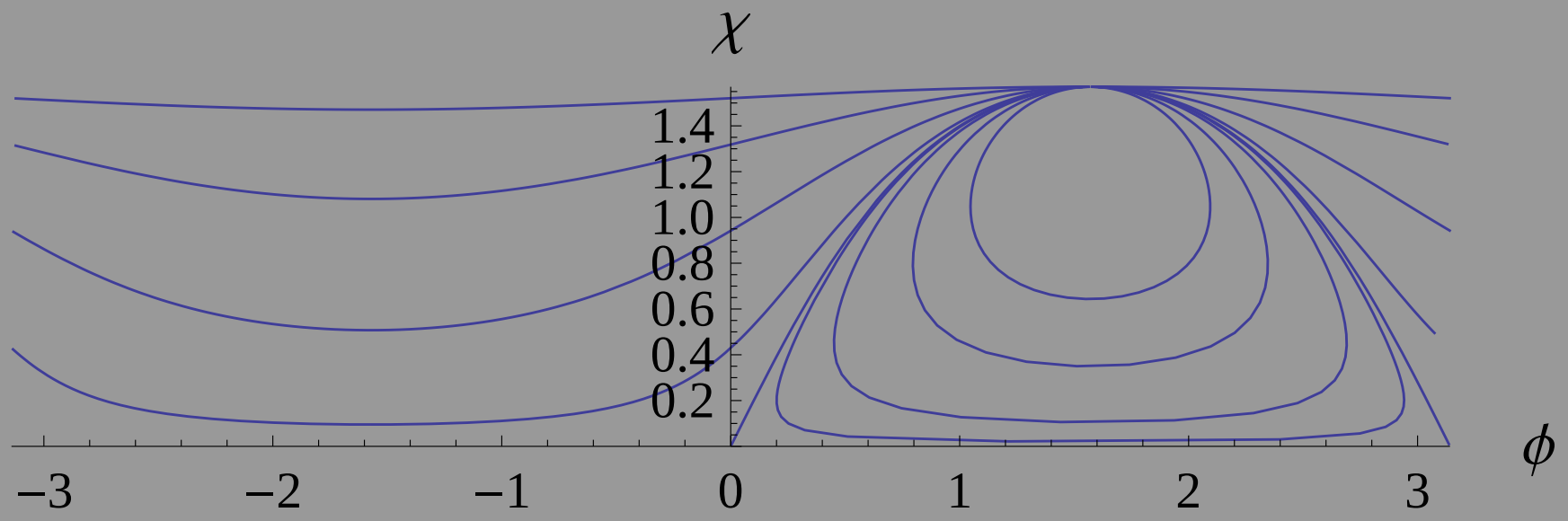

Figure: *Lines of constant r.*

A different state

- The AdS_3 metric can also be put in the form

$$ds^2 = \frac{1}{z^2}\left[dz^2 - a^2x^2d\eta^2 + dx^2\right], \tag{24}$$

with a Rindler boundary. The coordinate transformation that achieves this is given by

$$t(\eta, x) = x\sinh(a\eta) \tag{25}$$

$$r(z) = \frac{1}{z} \tag{26}$$

$$\phi(z, x) = x\cosh(a\eta). \tag{27}$$

- The corresponding stress-energy tensor vanishes.

AdS_3 with de Sitter boundary

- $$ds^2 = \frac{1}{z^2}\left[dz^2 - (1-H^2\rho^2)\left(1+\frac{1}{4}\left[\frac{H^2}{1-H^2\rho^2}-H^2\right]z^2\right)^2 dt^2 + \left(1-\frac{1}{4}\left[\frac{H^2}{1-H^2\rho^2}+H^2\right]z^2\right)^2 \frac{d\rho^2}{1-H^2\rho^2}\right] \quad (28)$$

  with a de Sitter boundary at $z=0$.
- The coordinate transformation is $(-1/H < \rho < 1/H)$

$$r(z,\rho) = \frac{\sqrt{1-H^2\rho^2}}{z} + \frac{H^4\rho^2}{4\sqrt{1-H^2\rho^2}}z \quad (29)$$

$$\phi(z,\rho) = \frac{1}{2H}\log\left[\frac{1+H\rho}{1-H\rho}\right] - \frac{H^2\rho z^2}{2(1-H^2\rho^2+H^4\rho^2z^2/4)} \quad (30)$$

- The transformation maps the region near negative infinity for ϕ to the vicinity of $-1/H$ for $\rho > -1/H$, and the region near positive infinity for ϕ to the vicinity of $1/H$ for $\rho < 1/H$.

- For fixed ρ, there is a minimal value for $r(z,\rho)$, obtained for

$$z_m(\rho) = 2\sqrt{\frac{1-H^2\rho^2}{H^4\rho^2}}. \tag{31}$$

It is given by

$$r_m(\rho) = H^2|\rho|. \tag{32}$$

The corresponding value of ϕ is

$$\phi_m(\rho) \equiv \phi(z_m(\rho),\rho) = \frac{1}{2H}\log\left[\frac{1+H\rho}{1-H\rho}\right] - \frac{1}{H^2\rho}. \tag{33}$$

- Bridge connecting the asymptotic regions at $z \to 0$ and $z \to \infty$.
- Stress-energy tensor:

$$\rho = -\langle T^t_{\ t}\rangle = -\frac{1}{16\pi G_3}\left(\frac{H^2}{1-H^2\rho^2} + H^2\right) \tag{34}$$

$$p = \langle T^\rho_{\ \rho}\rangle = -\frac{1}{16\pi G_3}\left(\frac{H^2}{1-H^2\rho^2} - H^2\right). \tag{35}$$

- The conformal anomaly is $\langle T^{\mu\,(CFT)}_{\ \mu}\rangle = H^2/(8\pi G_3)$.

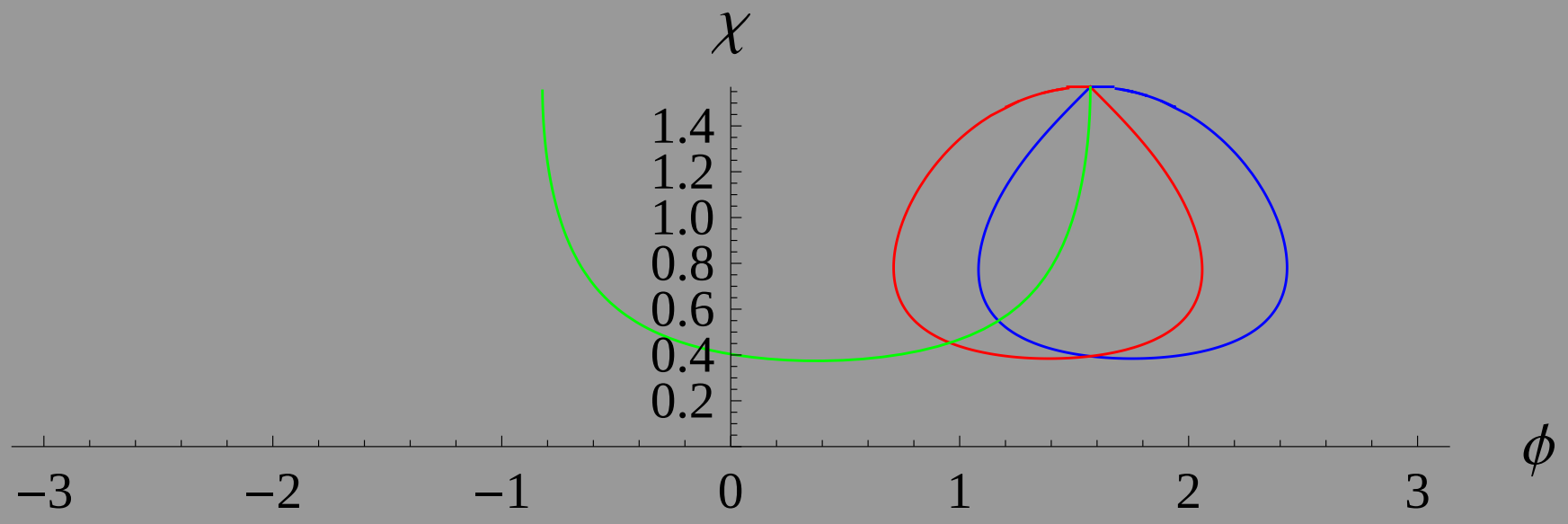

Figure: *Global coordinates.*

A different state

- The metric

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \left(1 - \frac{1}{4}H^2 z^2\right)^2 \left(-(1 - H^2\rho^2)d\eta^2 + \frac{d\rho^2}{1 - H^2\rho^2}\right)\right], \tag{36}$$

  also has a de Sitter boundary.
- The stress-energy tensor is

$$\rho = -\langle T^t_{\ t}\rangle = -\frac{H^2}{16\pi G_3} \tag{37}$$

$$p = \langle T^\rho_{\ \rho}\rangle = \frac{H^2}{16\pi G_3}. \tag{38}$$

- The conformal anomaly is the same as before.

AdS_3 with FRW boundary

- The BTZ metric can be expressed as (ϕ is now periodic)

$$ds^2 = \frac{1}{z^2}\left[dz^2 - \mathcal{N}^2(\tau,z)d\tau^2 + \mathcal{A}^2(\tau,z)d\phi^2\right], \tag{39}$$

with

$$\mathcal{A}(\tau,z) = a(\tau)\left(1 + \frac{\mu - \dot{a}^2(\tau)}{4a(\tau)^2}z^2\right) \tag{40}$$

$$\mathcal{N}(\tau,z) = 1 - \frac{\mu - \dot{a}^2 + 2a\ddot{a}}{4a^2}z^2 = \frac{\dot{\mathcal{A}}(\tau,z)}{\dot{a}}. \tag{41}$$

- The boundary has the form

$$ds_0^2 = g_{\mu\nu}^{(0)}dx^\mu dx^\nu = -d\tau^2 + a^2(\tau)d\phi^2, \tag{42}$$

with $a(\tau)$ an arbitrary function.

- The coordinate transformation is

$$r(\tau, z) = \frac{\mathcal{A}(\tau, z)}{z} = \frac{a}{z} + \frac{\mu - \dot{a}^2}{4}\frac{z}{a}. \tag{43}$$

- The coordinates (τ, z) do not span the full BTZ geometry. They cover the two regions outside the event horizons, located at

$$z_{e1} = \frac{2a}{\sqrt{\mu} + \dot{a}}, \qquad z_{e2} = \frac{2a}{\sqrt{\mu} - \dot{a}}. \tag{44}$$

  The quantities z_{e1}, z_{e2} are the two roots of the equation $r(\tau, z) = r_e = \sqrt{\mu}$.
- The coordinates also cover part of the regions behind the horizons. For constant τ, the minimal value of $r(\tau, z)$ is obtained for

$$z_m(\tau) = \frac{2a}{\sqrt{\mu - \dot{a}^2}}, \tag{45}$$

  corresponding to

$$r_m(\tau) = \sqrt{\mu - \dot{a}^2}. \tag{46}$$

  Clearly, $r_m \leq r_e$. Time-dependent throat.

- The transformation of the time coordinate for $z < z_{e1}$ or for $z > z_{e2}$ is

$$t(\tau, z) = \frac{\epsilon}{2\sqrt{\mu}} \log \left[\frac{4a^2 - \left(\sqrt{\mu} + \dot{a}\right)^2 z^2}{4a^2 - \left(\sqrt{\mu} - \dot{a}\right)^2 z^2} \right] + \epsilon\, c(\tau), \tag{47}$$

  where the function $c(\tau)$ satisfies $\dot{c} = 1/a(\tau)$ and $\epsilon = \pm 1$.
- For $z_{e1} < z < z_{e2}$ the transformation is

$$t(\tau, z) = \frac{\epsilon}{2\sqrt{\mu}} \log \left[\frac{-4a^2 + \left(\sqrt{\mu} + \dot{a}\right)^2 z^2}{4a^2 - \left(\sqrt{\mu} - \dot{a}\right)^2 z^2} \right] + \epsilon\, c(\tau). \tag{48}$$

- The transformation is singular on the event horizons.

- The coordinate ϕ remains unaffected by the transformation. It is periodic, with periodicity 2π.
- Dual picture: thermalized CFT on an expanding background, with a scale factor $a(\tau)$.
- Stress-energy tensor:

$$\rho = \frac{E}{V} = -\langle T^{\tau}_{\ \tau}\rangle = \frac{1}{16\pi G_3}\frac{\mu - \dot{a}^2}{a^2} \tag{49}$$

$$P = \langle T^{\phi}_{\ \phi}\rangle = \frac{1}{16\pi G_3}\frac{\mu - \dot{a}^2 + 2a\ddot{a}}{a^2}, \tag{50}$$

- Casimir energy $\sim \dot{a}^2/a^2$.
- Conformal anomaly:

$$\langle T^{\mu\,(CFT)}_{\ \mu}\rangle = \frac{1}{8\pi G_3}\frac{\ddot{a}}{a}. \tag{51}$$

- Observation: The entropy is proportional to the narrowest part of the throat or bridge.
- This line defines the boundary of the part of the bulk geometry that is not covered by the Fefferman-Graham parametrization. In a sense, it determines the part of the bulk that is not included in the construction of the dual theory.
- In quantitative terms:

$$S = \frac{1}{4G_3} A, \tag{52}$$

with A the length of the narrowest part of the throat or bridge at a given time.

- BTZ black hole with a flat boundary:

$$A = 2\pi\sqrt{\mu}, \qquad S_{th} = \frac{\pi}{2G_3}\sqrt{\mu}. \tag{53}$$

AdS_3 with Rindler boundary

- The throat is located at:

$$z_m(x) = 2x \tag{54}$$

  in Fefferman-Graham coordinates.
- Equivalently, it is located at

$$r_m(x) = a, \qquad \phi_m(x) \equiv \phi_m(z_m(x), x) = (\log[ax] - 1)/a \tag{55}$$

  in Schwarzschild (Poincare) coordinates.
- The entropy is

$$S = \frac{1}{4G_3}\int_{-\infty}^{\infty} a d\phi = \frac{1}{4G_3}\int_0^{\infty}\frac{dx}{x} = \frac{1}{4G_3}\int_0^{\infty}\frac{dz}{z}. \tag{56}$$

- The infinities come from the endpoints, where the line approaches the boundary.
- Regulate !

$$S = \frac{2}{4G_3}\int^{1/\epsilon} ds, \tag{57}$$

- For a $(d+1)$-dimensional bulk metric

$$ds^2 = \frac{1}{u^2}\left(du^2 - dt^2 + d\vec{x}^2\right), \tag{58}$$

the effective Newton's constant G_d is

$$\frac{1}{G_d} = \frac{1}{G_{d+1}} \int_0^\infty \frac{du}{u^{d-1}}. \tag{59}$$

- For $d = 2$ and $u = 1/r$

$$\frac{1}{G_2} = \frac{1}{G_3} \int_0^\infty \frac{dr}{r}. \tag{60}$$

A line integral in AdS_3, starting and ending at the boundary.

- $G_2 = 0$. But, regulate !

$$\frac{1}{G_2} = \frac{2}{G_3} \int^{1/\epsilon} ds \tag{61}$$

- This gives

$$S_R = \frac{1}{4G_2}. \tag{62}$$

AdS_3 with de Sitter boundary

- The bridge approaches the boundary four times, so that

$$S = \frac{4}{4G_3} \int^{1/\epsilon} ds. \tag{63}$$

- This gives

$$S = \frac{1}{2G_2}. \tag{64}$$

BTZ black hole with time-dependent boundary

- The throat has $r_m(\tau) = \sqrt{\mu - \dot{a}^2}$, so that

$$S = \frac{\pi}{2G_3}\sqrt{\mu - \dot{a}^2}. \tag{65}$$

- The asymptotic symmetries of (2+1)-dimensional Einstein gravity with a negative cosmological constant correspond to a pair of Virasoro algebras, with central charges $c = \tilde{c} = 3/(2G_3)$ (Brown, Henneaux 1986). For the BTZ black hole, the eigenvalues Δ, $\tilde{\Delta}$ of the generators L_0, $\tilde{L}_0$ are

$$\Delta = \tilde{\Delta} = \frac{\mu}{16G_3}. \tag{66}$$

- The Cardy formula gives

$$S = 2\pi\sqrt{\frac{c}{6}\left(\Delta - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{\tilde{c}}{6}\left(\tilde{\Delta} - \frac{\tilde{c}}{24}\right)} = \frac{\pi}{2G_3}\sqrt{\mu - 1}. \tag{67}$$

For $\mu \gg 1$, it reproduces correctly the entropy of the thermalized CFT.

- The shift of the mass term by 1 is a result of the influence of the Casimir energy on the entropy.
- Our result gives a generalization of the Cardy formula for a time-dependent background, with Casimir energy $\sim \dot{a}^2/a^2$.

- AdS_3 with a Milne boundary: zero entropy
- AdS_3 with a steady-state boundary: zero entropy

Comments

- The de Sitter entropy was calculated through holographic means in:
  Hawking, Maldacena, Strominger 2001
  Iwashita, Kobayashi, Shiromizu, Yoshiho 2006
  The Randall-Sundrum construction was used.
- A general framework for the calculation of entanglement entropy for a flat boundary through holography was given in:
  Ryu,Takayanagi 2006 Hubeny,Rangamani,Takayanagi 2007
- Connection with minimal surfaces ?
- Higher dimensions ?