What can black holes tell us about microstates?

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Introduction and motivation

There is by now a large body of evidence in string theory that the classical black hole entropy has a statistical interpretation.

Strategy

1. Identify a suitable supersymmetric black hole with certain quantum numbers and calculate its entropy S_{BH} via the Bekenstein-Hawking-Wald formula.

2. Find a microscopic system with the same quantum numbers and count its 'number of states' Ω in the limit when gravity it switched off.

3. Compare S_{BH} with $\ln \Omega$.

This comparison is usually done in the limit when the charges are large.

$\Rightarrow$ the curvature at the horizon is small and hence the Bekenstein-Hawking formula is a reliable approximation.

Also the counting of states simplifies in this limit since we can use asymptotic formula, *e.g.* the Cardy formula.

What happens beyond the large charge limit?

On the microscopic side the low energy dynamics is usually described by a supersymmetric quantum mechanics and there is no difficulty in principle in counting states to arbitrary accuracy.

In a class of N=4 and N=8 supersymmetric string theories one now has exact microscopic results.

Dijkgraaf, Verlinde, Verlinde; Shih, Strominger, Yin; David, Jatkar, A.S.

From this one may be tempted to conclude that the microscopic description is the correct description, and horizon is an ‘emergent phenomenon’ , i.e. an approximate description of the system when there are large number of quantum states.

We shall explore an alternative, more conventional, viewpoint.

Can quantum gravity / closed string theory around the black hole background fully describe the system, providing an alternative dual description?

If so we should be able to compute exact properties of black hole microstates by analyzing quantum gravity in the black hole background.

In particular the description using quantum gravity / closed strings should be able to reproduce the exact result for the number of microstates, instead of just the leading behaviour for large charges.

Some results for the index in heterotic on T^6 from microscopic counting

$(Q^2, P^2)\backslash Q.P$	-2	2	3	4	5	6	7
(2,2)	-209304	648	327	0	0	0	0
(2,4)	-2023536	50064	8376	-648	0	0	0
(2,6)	-15493728	1127472	130329	-15600	972	0	0
(4,4)	-16620544	3859456	561576	12800	3272	0	0
(4,6)	-53249700	110910300	18458000	1127472	85176	-6404	0
(6,6)	2857656828	4173501828	920577636	110910300	8533821	153900	26622
(2,10)	-510032208	185738352	16844421	-2023536	315255	-31104	1620

Q^2, P^2, $Q\cdot P$: T-duality invariant bilinears in the charges.

Question: Can we reproduce these numbers from the analysis of quantum gravity?

For this we need to compute quantum gravity / string theory corrections to the Bekenstein-Hawking-Wald formula exactly.

– remains a part of the wish list.

Nevertheless we shall use the black hole description to make predictions for the microscopic index which can be tested against the explicit results.

Qualitative predictions:

1. Sign of the index

2. Absence of negative discriminant states

Quantitative predictions:

3. Logarithmic corrections to the entropy

The key insight arises from the existence of the AdS_2 factor in the near horizon geometry of extremal black holes.

AdS_2 does not admit any charge or energy carrying excitations since such an excitation will change the asymptotic boundary condition on the gauge fields / metric .

$\Rightarrow$ quantum gravity in the near horizon geometry of extremal black holes describes a microcanonical ensemble of degenerate quantum states.

Sign of the Index:

Typically in the microscopic theory we do not calculate the degeneracy, but an index:

$$\Omega \equiv \mathbf{Tr}'(-1)^{\mathbf{F}} = \mathbf{Tr}'(-1)^{2\mathbf{J_3}}$$

$'$ denotes removal of the trace over fermion zero modes associated with broken supersymmetry.

This is what is protected from corrections when gravity effects are switched on.

For comparison, on the black hole side also we must compute the index, not entropy.

How to compute $\mathrm{Tr}'(-1)^{2J_3}$ for a black hole?

1. SUSY algebra + SL(2,R) isometry of AdS_2
$\Rightarrow$ black holes have spherically symmetric horizon and hence zero average angular momentum.

2. SInce extremal black holes describe a microcanonical ensemble, all states in the ensemble have $J_3 = 0$.

Thus black holes have

$$\mathrm{Tr}'(-1)^{2J_3} = \mathrm{Tr}'(1) = e^{S_{BH}}$$

S_{BH}: black hole entropy (after stringy and quantum corrections)

A.S.; Dabholkar, Murthy, Gomes, A.S.

$$\mathrm{Tr}'(-1)^{2J_3} = \mathrm{Tr}'(1) = e^{S_{BH}}$$

This explains why the index on the microscopic side can be compared with $e^{S_{BH}}$ on the black hole side.

$$\Omega \Leftrightarrow e^{S_{BH}}$$

But this also make a non-trivial prediction:

$$\Omega \equiv \mathrm{Tr}'(-1)^{2J_3} > 0$$

Microscopic index must be positive

(no *a priori* reason for this on the microscopic side).

Absence of negative discriminant states:

In heterotic string theory on T^6 supersymmetric black hole solutions do not exist for

$$\Delta \equiv Q^2 P^2 - (Q.P)^2 < 0$$

Since these black holes describe microcanonical ensemble of states carrying fixed charges, this implies that

the microscopic index must vanish for $\Delta < 0$.

– another non-trivial prediction for the microscopic index.

A caveat:

Macroscopic arguments hold for single centered black holes, but the total index receives contribution from single and two centered black hole solutions.

2-centered solutions can have negative index and also negative discriminant, spoiling the earlier arguments

Dabholkar, Gaiotto, Nampuri

microscopic index = 1-centered index + 2-centered index

1-centered index= microscopic index – 2-centered index

Strategy: Compute separately the contribution to the index from two centered black holes, subtract this from the microscopic index and then test the predictions from AdS_2 geometry.

Given a 2-centered configuration we can compute its contribution to the index

Example: Index of (Q, 0) + (0, P) is

$$(-1)^{Q.P+1}\,|Q.P|\,f(Q^2/2)\,f(P^2/2)$$

f(n) defined through

$$\sum_n f(n) e^{2\pi i n\tau} = \eta(\tau)^{-24}$$

Furthermore supergravity analysis tells us what 2-centered configurations exist at any given point in the moduli space. Denef; Denef, Moore

– can use this to compute total 2-centered contribution to the index.

Results for the total index in heterotic on T^6 from microscopic counting

$(Q^2, P^2)\backslash Q.P$	-2	2	3	4	5	6	7
(2,2)	-209304	648	327	0	0	0	0
(2,4)	-2023536	50064	8376	-648	0	0	0
(2,6)	-15493728	1127472	130329	-15600	972	0	0
(4,4)	-16620544	3859456	561576	12800	3272	0	0
(4,6)	-53249700	110910300	18458000	1127472	85176	-6404	0
(6,6)	2857656828	4173501828	920577636	110910300	8533821	153900	26622
(2,10)	-510032208	185738352	16844421	-2023536	315255	-31104	1620

Red entries: Negative index

Blue entries: Negative discriminant states

Result for the index after subtracting the contribution from two centered black holes

$(Q^2, P^2)\backslash Q.P$	-2	2	3	4	5	6	7
(2,2)	648	648	0	0	0	0	0
(2,4)	50064	50064	0	0	0	0	0
(2,6)	1127472	1127472	25353	0	0	0	0
(4,4)	3859456	3859456	561576	12800	0	0	0
(4,6)	110910300	110910300	18458000	1127472	0	0	0
(6,6)	4173501828	4173501828	920577636	110910300	8533821	153900	0
(2,10)	185738352	185738352	16844421	16491600	0	0	0

1. No more negative index

2. No negative discriminant states.

Such tests have been carried out for many other N=4 supersymmetric string theories where the exact dyon spectrum is known.

General results:

1. Absence of negative discriminant states in (microscopic index - 2-centered index) can be proved in general using properties of Siegel modular forms which are the generating functions of the microscopic index.

2. The positivity of the index has been proved in the limit of large charges and tested in many examples, but a general proof is still missing.

Quantitative tests: Logarithmic corrections

The exact microscopic results for the index in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric string theories allows us to compute systematic correction to the index Ω beyond the large charge limit.

Can we reproduce these corrections from the macroscopic side?

Microscopic results in the limit when all components of the charge are taken to be large:

$$\begin{aligned} \ln\Omega &= \pi\sqrt{\Delta} + \mathcal{O}(1) \quad \text{for N=4} \\ &= \pi\sqrt{\Delta} - 2\ln\Delta + \mathcal{O}(1) \quad \text{for N=8} \end{aligned}$$

$$\Delta \equiv Q^2P^2 - (Q.P)^2$$

Note: This is different from the Cardy limit results when only one component becomes large keeping the other components fixed:

$$\begin{aligned} \ln\Omega &= \pi\sqrt{\Delta} - \frac{m+2}{4}\ln\Delta + \mathcal{O}(1) \quad \text{for N=4} \\ &= \pi\sqrt{\Delta} - 2\ln\Delta + \mathcal{O}(1) \quad \text{for N=8} \end{aligned}$$

m: number of matter multiplets

Strategy for computing S_{BH}:

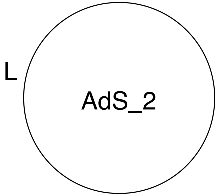
Euclidean near horizon geometry has the form

$AdS_2 \times S^2 \times K$ with flux through various cycles

K: 6-dimensional compact space of string scale size

$$ds^2 = a^2(d\eta^2 + \sinh^2\eta\, d\theta^2) + a^2(d\psi^2 + \sin^2\psi\, d\phi^2) + ds_K^2$$

a: a constant that scales with the charges



L: regulated length of the boundary of AdS_2

Let Z_{AdS_2} be the partition function of string theory in this background

Then

$$Z_{AdS_2} = e^{S_{BH} - E_0 L}$$

E_0: energy, S_{BH}: entropy

Once we compute Z_{AdS_2}, we can extract S_{BH} from it.

$$Z_{AdS_2} = e^{S_{BH} - E_0 L}$$

Classical contribution to S_{BH} gives us back the Wald entropy $\pi\sqrt{\Delta}$

One can show that logarithmic corrections to S_{BH}, if present, must come from one loop contribution of massless fields to Z_{AdS_2}

This involves two types of contributions:

1. Determinant of the kinetic operator of massless fields after removing the zero modes

2. Contribution from integration over the zero modes.

1. Determinant of the kinetic operator of massless fields

- Find the quadratic action of massless fields expanded around the near horizon geometry with fluxes.

- Find the eigenvalues of the kinetic operator.

- Take the product of non-zero eigenvalues.

2. Zero mode contribution

- Identify the asymptotic symmetries responsible for the zero modes.

- Change integration over the zero modes to integration over parameters labelling the (super-)group of asymptotic symmetries.

- The Jacobian for this change of variables gives the zero mode contribution to Z_{AdS_2}.

Given Z_{AdS_2} we can isolate the 'infinite part' $e^{-E_0 L}$ and finite part $e^{S_{BH}}$ easily and compute S_{BH}.

Results for logarithmic term in S_{BH}:

The theory	non-zero mode contribution	zero mode contribution	total contribution
N=4	$\frac{1}{4}(6+m)\ln\Delta$	$-\frac{1}{4}(6+m)\ln\Delta$	0
N=8	$5\ln\Delta$	$-7\ln\Delta$	$-2\ln\Delta$

m: number of matter multiplets

The final result is in perfect agreement with the microscopic results.

Banerjee, Gupta, A.S.; Banerjee, Gupta, Mandal, A.S.

Some details of the computation

$\mathrm{AdS_2} \times \mathrm{S^2}$ metric

$$ds^2 = a^2(d\eta^2 + \sinh^2\eta\, d\theta^2) + a^2(d\psi^2 + \sin^2\psi\, d\phi^2)$$

Suppose $\Delta\mathcal{L}_{\rm eff}$ is the one loop effective Lagrangian density on $\mathrm{AdS_2} \times \mathrm{S^2}$

Then one loop correction to the effective action is

$$\Delta\mathcal{S}_{\rm eff} = \int \sqrt{\det g}\,\Delta\mathcal{L}_{\rm eff} = 8\pi^2 a^4(\cosh\eta_0 - 1)\Delta\mathcal{L}_{\rm eff}$$

η_0: cut-off on η to make $\mathrm{AdS_2}$ volume finite

Length of the boundary: $L = 2\pi a \sinh\eta_0$

$$\Delta\mathcal{S}_{\rm eff} = 8\pi^2 a^4\left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1})\right)\Delta\mathcal{L}_{\rm eff}$$

$$\Delta\mathcal{S}_{eff} = 8\pi^2 a^4 \left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1})\right)\Delta\mathcal{L}_{eff}$$

$\Rightarrow$ **one loop multiplicative contribution to Z_{AdS_2}:**

$$\exp\left[8\pi^2 a^4 \left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1})\right)\Delta\mathcal{L}_{eff}\right]$$

Comparing with $Z_{AdS_2} = \exp[S_{BH} - E_0 L]$ we get

$$\Delta S_{BH} = -8\pi^2 a^4 \Delta\mathcal{L}_{eff}$$

We calculate $\Delta\mathcal{L}_{eff}$ using heat kernel method.

Let $\{\psi_r\}$ denote the set of fluctuating massless fields around the near horizon background.

Let the eigenfunctions of the kinetic operator be:

$$\psi_r = f_r^{(n)}(x)$$

with eigenvalue κ_n.

Heat kernel:

$$K(x, x', s) = \sum_n e^{-\kappa_n s} f_r^{(n)}(x) f_r^{(n)}(x')$$

$$\Delta\mathcal{S}_{eff} = -\frac{1}{2}\sum_n \ln \kappa_n = -\frac{1}{2}\int_\epsilon^\infty \frac{ds}{s} e^{-\kappa_n s}$$

ϵ: a string scale UV cut-off.

$$\mathbf{K}(\mathbf{x},\mathbf{x}',\mathbf{s}) = \sum_{\mathbf{n},\mathbf{r}} \mathbf{e}^{-\kappa_\mathbf{n}\mathbf{s}} \mathbf{f}_\mathbf{r}^{(\mathbf{n})}(\mathbf{x}) \mathbf{f}_\mathbf{r}^{(\mathbf{n})}(\mathbf{x}')$$

$$\Delta\mathcal{S}_{\mathbf{eff}} = -\frac{1}{2}\int_\epsilon^\infty \frac{\mathbf{ds}}{\mathbf{s}} \mathbf{e}^{-\kappa_\mathbf{n}\mathbf{s}} = -\frac{1}{2}\int \mathbf{d}^4\mathbf{x}\,\sqrt{\det \mathbf{g}} \int_\epsilon^\infty \frac{\mathbf{ds}}{\mathbf{s}} \mathbf{K}(\mathbf{x},\mathbf{x};\mathbf{s})$$

$$\Rightarrow \Delta\mathcal{L}_{\mathbf{eff}} = -\frac{1}{2}\int_\epsilon^\infty \frac{\mathbf{ds}}{\mathbf{s}} \mathbf{K}(\mathbf{x},\mathbf{x};\mathbf{s})$$

The terms proportional to $\ln \mathbf{a}$ come from integration over the range $\epsilon << \mathbf{s} << \mathbf{a}^2$

We explicitly find $(\mathbf{f}_\mathbf{r}^{(\mathbf{n})}, \kappa_\mathbf{n})$, calculate $\mathbf{K}(\mathbf{x},\mathbf{x};\mathbf{s})$ and its behaviour in the range $\epsilon << \mathbf{s} << \mathbf{a}^2$.

Note: $\kappa_\mathbf{n} = 0$ modes must be removed.

Zero mode contribution:

The path integral over the fields is defined with the standard general coordinate invariant measure, *e.g.* **for gauge fields:**

$$\int [DA_\mu] \exp\left[-\int d^4x \sqrt{\det g}\, g^{\mu\nu} A_\mu A_\nu \right] = 1$$

SInce $\sqrt{\det g}\, g^{\mu\nu} \sim a^2$ this shows that $[aA_\mu]$ has a independent measure.

Zero modes of A_μ are of the form $\partial_\mu \Lambda$ with Λ not vanishing at ∞.

Changing variables from aA_μ to $\Lambda \Rightarrow$ 'a' per zero mode

Net contribution to Z_{AdS_2} from gauge field zero modes is a^{N_z} where N_z is the number of zero modes.

Computation of N_z:

Let

$$A_\mu(x) = g_\mu^{(k)}(x)$$

be the zero mode wave functions

$$N_z = \sum_k 1 = \int d^4x \sqrt{\det g}\, g^{\mu\nu} \sum_k g_\mu^{(k)}(x) g_\nu^{(k)}(x)$$

$n_z \equiv g^{\mu\nu} \sum_k g_\mu^{(k)}(x) g_\nu^{(k)}(x)$ is independent of x after summing over k.

$$N_z = 8\pi^2 a^4 n_z (\cosh \eta_0 - 1) = 8\pi^2 a^4 n_z \left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1}) \right)$$

$$N_z = 8\pi^2 a^4 n_z (\cosh \eta_0 - 1) = \left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1}) \right)$$

$\Rightarrow$ **gauge field zero mode contribution to Z_{AdS_2}:**

$$a^{N_z} = \exp \left[8\pi^2 a^4 n_z \ln a \left(\frac{L}{2\pi a} - 1 + \mathcal{O}(L^{-1}) \right) \right]$$

Comparing with $Z_{AdS_2} = e^{S_{BH} - E_0 L}$ we get the logarithmic contribution to S_{BH} from the zero modes:

$$\Delta S_{BH} = -8\pi^2 a^4 n_z \ln a$$

Contributions from other zero modes can be found similarly.

Given this success, we would like to generalize this to N=2 supersymmetric string theories.

The main bottleneck is the absence of reliable results on the microscopic side.

Nevertheless one can try to make progress on the macroscopic side so that comparison with the microscopic data may be made if and when the latter is available.

Progress has been made on several fronts.

1. A general formula relating the total index to the index associated with single centered black holes has been found.

Manschot, Pioline, A.S.

– required for testing positivity of the index etc.

2. Logarithmic corrections have been computed in the STU model and was found to vanish.
– consistent with the earlier proposal for the index

David; David, de Wit, Cardoso, Mahapatra

3. Computation of logarithmic corrections to half BPS black holes in generic N=2 supersymmetric string theory is in progress.

– would constrain the measure in the OSV integral.

Computation of logarithmic corrections can also be extended to non-supersymmetric extremal black hole, *e.g.* extremal Reissner-Nordstrom, extremal Kerr, extremal BTZ etc.

A.S., work in progress

These results would put strong constraint on any microscopic theory that attempts to provide a statistical interpretation of the entropy.

Some preliminary results

Theory	logarithmic correction
extremal Reissner-Nordstrom + n_s massless scalars, n_f massless Dirac fields, n_v additional massless vector	$-\frac{1}{180}(? + n_s + 62n_v + 11n_f)\ln A_H$
extremal Kerr + n_s massless scalars, n_f massless Dirac fields, n_v massless vector	$\frac{1}{180}(64 + 2n_s - 26n_v + 7n_f)\ln A_H$
extremal BTZ +n_v Chern-Simons vector	$-\frac{1}{2}(n_v + 3)\ln A_H$ $\surd$
Half BPS in N=2 sugra + n_h hypermultiplet n_v vector multiplet	?

Summary

Quantum gravity in the near horizon geometry can make non-trivial prediction for the microstates which can be tested by explicit microscopic calculation.

1. **Positivity of the index**
2. **Absence of negative discriminant states**
3. **Coefficient of the logarithmic correction to the Bekenstein-Hawking-Wald formula in the large charge limit.**

This indicates that quantum gravity in the near horizon geometry could provide us with an exact dual description of black hole microstates instead of being merely an emergent phenomenon in the large charge limit.