S and T dualities of S-matrix

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Abstract

We give some examples that show the S-matrix elements in string theory satisfy S and T dualities. Assuming this duality for all S-matrix elements, we then find the tree-level S-matrix elements on the world volume of F_1-string and NS_5-brane, which are related by S-duality to the disk-level S-matrix elements of D_1-string and D_5-brane, respectively. The S-matrix elements indicate that both F_1-string and NS_5-brane have D-string excitations. Inspired by this observation, we then propose a Born-Infeld and Chern-Simons type effective action for both F_1-string and NS_5-brane.

1 Introduction

It is known that the type II superstring theory is invariant under T and S dualities.

Compatibility of a given solution of equations of motion with these dualities can be used to generate new solutions.

In this talk, I would like to apply this compatibility to the other on-shell quantities, *i.e.,* the S-matrix elements.

2 T-duality of S-matrix

The T-duality holds order by order in string loop expansion.

We show that T-duality generates new S-matrix elements from a given S-matrix element. All such S-matrix elements are at the same order in the loop.

Consider the disk level S-matrix element of two gravitons

$$A(D_p; h_1, h_2) \ \sim \ T_p \alpha'^2 K(D_p; h_1, h_2) \frac{\Gamma(-t/4)\Gamma(-s)}{\Gamma(1-t/4-s)} \delta^{p+1}(p_1^a + p_2^a)$$

where $s = -\alpha'(p_1)_a (p_1)_b \eta^{ab}$, and $t = -\alpha'(p_1 + p_2)^2$.

The background metric $\eta_{\mu\nu}$ is the string frame metric.

The kinematic factor can be written as

$$K(D_p; h_1, h_2) \ \sim \ e^{-\phi}\sqrt{-\eta}\Big[R_{1abcd}R_2^{abcd} - 2\hat{R}_{1ab}\hat{R}_2^{ab} - R_{1abij}R_2^{abij} + 2\hat{R}_{1ij}\hat{R}_2^{ij}\Big]$$

where $\hat{R}_{ab} = \eta^{cd} R_{cadb}$ and $\hat{R}_{ij} = \eta^{cd} R_{cidj}$.

The linear curvature tensor is

$$\mathcal{R}_{\mu\nu\rho\lambda} \ = \ \frac{1}{2}(h_{\mu\lambda,\nu\rho} + h_{\nu\rho,\mu\lambda} - h_{\mu\rho,\nu\lambda} - h_{\nu\lambda,\mu\rho})$$

where metric is $\eta_{\mu\nu} + h_{\mu\nu}$ and $h_{\mu\nu}$ is the graviton polarization tensor.

What is the transformation of the above amplitude under T-duaity?

The full set of nonlinear T-duality transformations are

$$e^{2\tilde{\phi}} = \frac{e^{2\phi}}{G_{yy}} \ ; \ \widetilde{G}_{yy} = \frac{1}{G_{yy}} \ ; \ \widetilde{G}_{\mu y} = \frac{B_{\mu y}}{G_{yy}} \ ; \ \widetilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu y}G_{\nu y} - B_{\mu y}B_{\nu y}}{G_{yy}}$$
$$\widetilde{B}_{\mu y} = \frac{G_{\mu y}}{G_{yy}} \ ; \ \widetilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu y}G_{\nu y} - G_{\mu y}B_{\nu y}}{G_{yy}} \ ; \ \widetilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu y} = \mathcal{C}^{(n-1)}_{\mu\cdots\nu} \ ; \ \widetilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu} = \mathcal{C}^{(n+1)}_{\mu\cdots\nu y}$$

If y is identified on a circle of radius R, *i.e.*, $y \sim y + 2\pi R$, then after T-duality the radius becomes $\tilde{R} = \alpha'/R$. The string coupling is also shifted as $\tilde{g} = g\sqrt{\alpha'}/R$.

Suppose we are implementing T-duality along a world volume direction of D_p-brane. Then for the background fields, we have

$$\begin{aligned} e^{-\phi}\sqrt{-\eta} &\longrightarrow e^{-\phi}\sqrt{-\eta} \\ T_p\delta^{p+1}(p_1^a + p_2^a) &\longrightarrow T_{p-1}\delta^p(p_1^a + p_2^a) \\ s, t &\longrightarrow s, t \end{aligned}$$

To generate new n-point functions from a given n-point function, one has to use the linear T-duality transformations which are

$$\tilde{\phi} = \phi - \frac{1}{2}h_{yy},\ \tilde{h}_{yy} = -h_{yy},\ \tilde{h}_{\mu y} = B_{\mu y},\ \tilde{B}_{\mu y} = h_{\mu y},\ \tilde{h}_{\mu\nu} = h_{\mu\nu},\ \tilde{B}_{\mu\nu} = B_{\mu\nu}$$
$$\widetilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu y} = \mathcal{C}^{(n-1)}_{\mu\cdots\nu},\ \ \widetilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu} = \mathcal{C}^{(n+1)}_{\mu\cdots\nu y}$$

Our strategy for finding the new n-point functions of a D_p-brane is as follows:

$$\mathcal{R}_{a\cdots i\cdots}\mathcal{R}^{a\cdots i\cdots} = \mathcal{R}_{\tilde{a}\cdots i\cdots}\mathcal{R}^{\tilde{a}\cdots i\cdots} + \mathcal{R}_{y\cdots i\cdots}\mathcal{R}^{y\cdots i\cdots}$$

Under T-duality the D_p-brane transforms to D_{p-1}-brane and the above coupling transforms to

$$\rightarrow \tilde{\mathcal{R}}_{a\cdots\tilde{i}\cdots}\tilde{\mathcal{R}}^{a\cdots\tilde{i}\cdots} + \tilde{\mathcal{R}}_{y\cdots\tilde{i}\cdots}\tilde{\mathcal{R}}^{y\cdots\tilde{i}\cdots}$$

The indices are not complete in the T-dual theory. One must add new couplings to the action to have the complete indices in the T-dual theory.

Let us consider each curvature terms in $K(D_p; h_1, h_2)$ separately.

2.1 $\mathcal{R}_{abcd}\mathcal{R}^{abcd}$ term

We first write it as

$$(\mathcal{R}_{abcd})^2 \;=\; (\mathcal{R}_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}})^2 + (h_{\tilde{a}y,\tilde{b}\tilde{c}} - h_{\tilde{b}y,\tilde{a}\tilde{c}})^2 + (h_{yy,\tilde{a}\tilde{b}})^2$$

Our notation is such that *e.g.*, $(\mathcal{R}_{abcd})^2 = \mathcal{R}_{1abcd}\mathcal{R}_2^{abcd}$.

Under T-duality, it transforms to

$$(\mathcal{R}_{abcd})^2 \;\rightarrow\; (\mathcal{R}_{abcd})^2 + (B_{ay,bc} - B_{by,ac})^2 + (h_{yy,ab})^2$$

Because there are incomplete transverse index y, one concludes that the original curvature term is not consistent with T-duality even in the absence of the B-field. One must add some terms to the curvature term to have completed indices in the T-dual theory. The T-dual invariant terms are:

$$(\mathcal{R}_{abcd})^2 + (B_{ai,bc} - B_{bi,ac})^2 - 2(B_{ci,ab})^2 - (h_{cd,ab})^2$$

2.2 $\hat{\mathcal{R}}_{ab}\hat{\mathcal{R}}^{ab}$, $\mathcal{R}_{abij}\mathcal{R}^{abij}$, $\hat{\mathcal{R}}_{ij}\hat{\mathcal{R}}^{ij}$ terms

The T-dual completion in these cases are the following couplings:

$$(\hat{\mathcal{R}}_{ab} - \phi_{,ab})^2 + \frac{1}{2}(B_{ic,ca} - B_{ia,cc})^2 - \frac{1}{2}(B_{ia,cc})^2 - \frac{1}{4}(h_{ab,cc})^2$$
$$(\mathcal{R}_{abij})^2 + \frac{1}{2}(B_{ki,aj} - B_{kj,ai})^2 + \frac{1}{2}(B_{ac,bi} - B_{bc,ai})^2 - \frac{1}{2}(B_{ki,aj})^2 - \frac{1}{2}(B_{ac,bi})^2$$
$$(\hat{\mathcal{R}}_{ij} - \phi_{,ij})^2 + \frac{1}{2}(B_{ab,bj} - B_{aj,bb})^2 + \frac{1}{4}(h_{ab,cc})^2$$

2.3 B-B amplitude

Now adding these terms one finds that the non-tensor graviton terms are canceled.

This means the graviton amplitude is invariant under the T-duality transformations when there is no B-field.

In the presence of B-field, the T-duality predicts the following amplitude:

$$A(D_p; B_1, B_2) \ \sim \ T_{p-1}\alpha'^2 K(D_p; B_1, B_2)\frac{\Gamma(-t/4)\Gamma(-s)}{\Gamma(1-t/4-s)}\delta^{p+1}(p_1^a + p_2^a)$$

where the kinematic factor is

$$\begin{aligned} K(D_p; B_1, B_2) \ = \ e^{-\phi}\sqrt{-\eta}\Big(& B_{ki,aj}B_{kj,ai} + B_{ac,bi}B_{bc,ai} - \frac{1}{2}(B_{ki,aj})^2 - \frac{1}{2}(B_{ac,bi})^2 \\ & -(B_{ic,ca})^2 - 2B_{ab,bi}B_{ai,cc} + (B_{ab,bi})^2 + (B_{ai,bb})^2\Big) \end{aligned}$$

We have checked that this amplitude is exactly reproduced by the disk-level scattering amplitude of two B-fields.

For later use, we write the kinematic factor in terms of field strength H

$$K(D_p; B_1, B_2) \ = \ e^{-\phi}\sqrt{-\eta}\big[\frac{1}{6}H_{1ijk,a}H_2^{ijk,a} + \frac{1}{3}H_{1abc,i}H_2^{abc,i} - \frac{1}{2}H_{1bci,a}H_2^{bci,a}\big]$$

2.4 RR-NSNS amplitude

Another example of T-dual S-matrix multiplet is the disk-level S-matrix elements of one RR and one NSNS vertex operators

$$A(D_p;1,2) \ \sim \ T_p\alpha'^2(K_{1T}(D_p;1,2)-K_{2T}(D_p;1,2))\frac{\Gamma(-t/4)\Gamma(-s)}{\Gamma(1-t/4-s)}\delta^{p+1}(p_1^a+p_2^a)$$

The T-dual kinematic factors are

$$K_{1T}(D_p;1,2)=\epsilon^{a_0\cdots a_p}\left(\frac{1}{2!(p-1)!}F_{1\,ia_2\cdots a_p,a}^{(p)}H_{2a_0a_1}{}^{a,i}+\frac{1}{p!}F_{1\,ia_1\cdots a_pj,a}^{(p+2)}R_{2\,a_0}^{a\ \ ij}\right.$$
$$\left.-\frac{1}{3!(p+1)!}F_{1\,ia_0\cdots a_pjk,a}^{(p+4)}H_2^{ijk,a}\right)$$

$$K_{2T}(D_p;1,2)=\epsilon^{a_0\cdots a_p}\left(\frac{1}{2!(p-1)!}F_{1\,aa_2\cdots a_p,i}^{(p)}H_{2a_0a_1}{}^{i,a}+\frac{2}{(p+1)!}F_{1\,a_0\cdots a_pj,i}^{(p+2)}(\hat{R}_2^{ij}-\phi_2^{,ij})]\right)$$

The reason that there are two T-dual kinematic factors is to have consistency with S-duality. Each term is invariant under the RR and the NSNS gauge transformations.

2.5 RR-NSNS-NSNS amplitude

Consider the disk-level S-matrix element of one RR potential $C^{(p-3)}$ and two gravitons

$$A(D_p; C_1^{p-3}, h_2, h_3) \ \sim \ T_p \alpha'^2 K(D_p; C_1, h_2, h_3) \mathcal{J} \delta^{p+1}(p_1^a + p_2^a + p_3^a)$$

where $\mathcal{J}$ is a function of the Mandelstam variables and the kinematic factor is

$$K(D_p; C_1^{p-3}, h_2, h_3) \ = \ \epsilon^{a_0 \cdots a_p} \mathcal{C}^{(p-3)}_{1 a_4 \cdots a_p} \Big[R_{2a_0a_1}{}^{ab} R_{3a_2a_3ba} - R_{2a_0a_1}{}^{ij} R_{3a_2a_3ji} \Big]$$

This amplitude at order α'^2 has only contact terms, which reproduce the curvature corrections to the Chern-Simons action.

This amplitude satisfies the Ward identities corresponding to the gravitons and the RR field.

However, it is not invariant under T-duality. Consequently, the Chern-Simons action is not invariant under the T-duality.

Since the curvature terms have four indices contracted with the volume form, one realizes that the T-dual S-matrix multiplet corresponding to the above amplitude should involve $C^{(p-3)}$, $C^{(p-1)}$, $C^{(p+1)}$, $C^{(p+3)}$, $C^{(p+5)}$.

Since the contracted indices a, b and i, j are not derivative indices, the $C^{(p-3)}$-component is not T-duality invariant when the Killing coordinate is an index of the RR field.

Using the same steps as we have done before, one finds appropriate terms to make the above amplitude invariant under T-duality.

The result satisfy the Ward identity corresponding to the RR-field.

The result, however, does not satisfy the Ward identity corresponding to the B-field.

Hence one must add some other T-duality invariant terms.

The result is

$$A(D_p; C_1^{(p-3)}, B_2, B_3) \ \sim \ T_p \alpha'^2 K(D_p; C_1^{(p-3)}, B_2, B_3) \mathcal{J} \delta^{p+1}(p_1^a + p_2^a + p_3^a) + \cdots$$

where the kinematic factor is

$$K(D_p; C_1^{(p-3)}, B_2, B_3) \ = \ \epsilon^{a_0 \cdots a_p} \mathcal{C}^{(p-3)}_{1 a_4 \cdots a_p} \Big[-\frac{1}{2} H_{2 a_0 a_1 i, a} H_{3 a_2 a_3}{}^{i,a} + \frac{1}{2} H_{2 a_0 a_1 a, i} H_{3 a_2 a_3}{}^{a,i} \Big]$$

The above terms are reproduced by the corresponding disk-level S-matrix element.

The above terms, however, do not satisfy the RR gauge transformation.

So we have to add more T-duality invariant terms to make it symmetric.

In principle, one may find such terms by imposing this symmetry.

It is very difficult to find such terms.

String theory is clever enough to find them.

We have found them by direct calculation in string theory.

The result is

$$
\begin{aligned}
A \;\sim\; & T_p\alpha'^2\epsilon_{a_0\cdots a_p}\mathcal{C}_1^{(p-3)a_4\cdots a_p}\Big[\frac{1}{4}(p_{2i}p_3^i)H_2^{aa_0a_1}H_3^{aa_2a_3}\mathcal{J}-\frac{1}{4}(p_{2a}p_3^a)H_2^{ia_0a_1}H_3^{ia_2a_3}\mathcal{J}\\
&+(p_2)_aH_2^{aa_0a_1}(p_3)_bH_3^{ba_2a_3}\mathcal{J}_3-\frac{1}{2}(p_2)_a(p_2)_bH_2^{aa_0a_1}H_3^{ba_2a_3}\mathcal{J}_1\\
&+\frac{1}{2}(p_2)_a(p_2)_iH_2^{aa_0a_1}H_3^{ia_2a_3}\mathcal{J}_2-(p_1)_i(p_2)_aH_2^{aa_0a_1}H_3^{ia_2a_3}\mathcal{I}_7\\
&-\frac{1}{2}(p_2)_iH_2^{aa_0a_1}(p_3)_aH_3^{ia_2a_3}\mathcal{J}_5-\frac{1}{4}(p_1)_i(p_2)_jH_2^{ia_0a_1}H_3^{ja_2a_3}\mathcal{I}_2\\
&+\frac{1}{4}(p_1)_i(p_1)_jH_2^{ia_0a_1}H_3^{ja_2a_3}\mathcal{I}_1+\frac{1}{4}(p_1)_i(p_2)_aH_2^{ia_0a_1}H_3^{aa_2a_3}\mathcal{I}_3\\
&-\frac{1}{3}(p_2)_aH_2^{a_0a_1a_2}(p_3)_bH_3^{aba_3}\mathcal{J}_4-\frac{1}{6}(p_1)_i(p_2)_aH_2^{a_0a_1a_2}H_3^{iaa_3}\mathcal{I}_2\\
&+\frac{1}{3}(p_2)_iH_2^{a_0a_1a_2}(p_3)_aH_3^{iaa_3}\mathcal{J}_{12}+\frac{1}{6}(p_1)_i(p_2)_jH_2^{a_0a_1a_2}H_3^{ija_3}\mathcal{I}_3\\
&-\frac{1}{3}(p_2)_i(p_2)_aH_2^{a_0a_1a_2}H_3^{iaa_3}(-\mathcal{J}_5+\mathcal{J})\Big]\delta^{p+1}(p_1^a+p_2^a+p_3^a)+\Big[2\leftrightarrow 3\Big]
\end{aligned}
$$

This amplitude satisfies the Ward identities corresponding to both the RR and the B-fields.

The sum of this amplitude and the graviton amplitude satisfy the linear T-duality when one of the indices of the RR potential carries the Killing index.

When the Killing index is carries by graviton/B-field, the amplitude is not invariant under T-duality.

Imposing this T-duality one may find all other components of the T-dual S-matrix multiplet.

That is, one can find the components $C^{(p-1)}$, $C^{(p+1)}$, $C^{(p+3)}$, $C^{(p+5)}$.

We have found the $C^{(p+5)}$-component.

It is reproduced by direct calculation.

3 S-duality of S-matrix

The S-duality is nonperturbative in the string loop expansion.

We show that S-duality generates new S-matrix elements from a given S-matrix element.

In this case, such S-matrix elements are not at the same order in the loop.

We call the set of S-matrix elements in a S-dual combination, the S-dual S-matrix multiplet.

We will show that in some cases the S-dual multiplet has more than one term at the tree-level.

In those cases, the S-duality generates new tree-level S-matrix elements from a given tree-level S-matrix element.

3.1 S-dual multiplets with single tree-level amplitude

The simplest example of S-duality invariant S-matrix element is the following disk-level 1-point coupling:

$$A(D_3; C_1^{(4)}) \ \sim \ T_{D3}\epsilon^{a_0\cdots a_3}C^{(4)}_{a_0\cdots a_3}\delta^4(p_1^a)$$

The RR four-form is invariant under this duality.

The D_3-brane is also invariant under S-duality.

Hence, the above amplitude is invariant under the S-duality.

In this example, the S-duality does not require any tree-level or loop terms.

Let us extend it to 2-point function.

It is given by the disk-level amplitude we discussed in the previous sections.

In the Einstein frame, $G_{\mu\nu} = e^{\phi/2} g_{\mu\nu}$, the amplitude becomes

$$A(D_3; C_1^{(4)}, h_2) \ \sim \ T_{D3}\alpha'^2 K(D_3; C_1^{(4)}, h_2) \frac{\Gamma(-te^{-\phi/2}/4)\Gamma(-se^{-\phi/2})}{\Gamma(1 - te^{-\phi/2}/4 - se^{-\phi/2})} \delta^4(p_1^a + p_2^a)$$

where the kinematic factor is

$$K(D_3; C_1^{(4)}, h_2) \ = \ \epsilon^{a_0 \cdots a_3} e^{-\phi} \Big[\frac{1}{2!3!} F^{(5)}_{i a_1 \cdots a_3 j, a} \mathcal{R}^{a}{}_{a_0}{}^{ij} - \frac{1}{4!} F^{(5)}_{a_0 \cdots a_3 j, i} \hat{\mathcal{R}}^{ij} \Big]$$

While the Einstein frame metric and the RR four-form are invariant, the dilaton factor is not invariant under the S-duality.

Hence, one has to α'-expand the amplitude to discuss the S-duality at each order of α'.

The expansion is

$$A(D_3; C_1^{(4)}, h_2) \ \sim \ T_{D3}\alpha'^2 K(D_3; C_1^{(4)}, h_2) \left(-\frac{e^{\phi}}{st} + \frac{\pi^2}{24} + O(\alpha'^2 e^{-\phi}) \right)$$

The leading term is invariant under S-duality.

The dilaton factor $e^{-\phi}$ in the α'^2 order terms is not invariant.

Consider the non-holomorphic Eisenstein series defined by

$$2\zeta(2s)E_s(\tau,\bar{\tau}) \;=\; \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

where $\tau = \tau_1 + i\tau_2$.

It is invariant under the $SL(2,Z)$ transformation.

For $s = 1$, this series diverges logarithmically.

The regularized function has the following weak-expansion:

$$2\zeta(2)E_1(\tau,\bar{\tau}) = \zeta(2)\tau_2 - \frac{\pi}{2}\ln(\tau_2) + \pi\sqrt{\tau_2}\sum_{m\neq 0, n\neq 0}\left|\frac{m}{n}\right|^{1/2} K_{1/2}(2\pi|mn|\tau_2)e^{2\pi i mn\tau_1}$$

The first term is the dilaton factor in the disk-level amplitude at order α'^2.

3.1.1 Another example

Consider the disk-level 2-point function of one RR two-form and one B-field.

In the Einstein frame, this amplitude is

$$A(D_3; C_1^{(2)}, B_2^{(2)}) \ \sim \ T_{D3}\alpha'^2 K(D_3; C_1^{(2)}, B_2^{(2)}) \frac{\Gamma(-te^{-\phi/2}/4)\Gamma(-se^{-\phi/2})}{\Gamma(1-te^{-\phi/2}/4-se^{-\phi/2})}\delta^4(p_1^a + p_2^a)$$

where the kinematic factor is

$$K(D_3; C_1^{(2)}, B_2^{(2)}) \ = \ \epsilon^{a_0\cdots a_3} e^{-\phi}\Big[F^{(3)}{}_{ia_2a_3,a} H_{a_0a_1}{}^{a,i} - F^{(3)}{}_{aa_2a_3,i} H_{a_0a_1}{}^{i,a}\Big]$$

Using the S-duality transformation $C^{(2)} \to B^{(2)}$ and $B^{(2)} \to -C^{(2)}$, one finds that it is invariant under the S-duality at order α'^0.

At order α'^2, the dilaton factor is not invariant so it should be again replaced by E_1 to make it invariant.

3.2 S-dual multiplets of D_3-brane with more then one tree-level amplitude

Let us consider the disk-level 2-point function of B-fields on the world volume of D_3-brane.

In the Einstein frame, it is given by

$$A(D_3; B_1^{(2)}, B_2^{(2)}) \ \sim \ T_{D3}\alpha'^2 K(D_3; B_1^{(2)}, B_2^{(2)}) \frac{\Gamma(-te^{-\phi/2}/4)\Gamma(-se^{-\phi/2})}{\Gamma(1-te^{-\phi/2}/4-se^{-\phi/2})}\delta^4(p_1^a+p_2^a)$$

where the kinematic factor is

$$K(D_p; B_1, B_2) \ = \ e^{-\phi}\left(e^{-\phi}\sqrt{-\eta}\Big[\frac{1}{6}H_{1ijk,a}H_2^{ijk,a}+\frac{1}{3}H_{1abc,i}H_2^{abc,i}-\frac{1}{2}H_{1bci,a}H_2^{bci,a}\Big]\right)$$

Obviously the above amplitude can not be extended to the S-duality invariant form by adding only the appropriate loops or the D-instanton effects.

One needs another disk-level amplitudes as well.

Since the RR two-form and the B-field appear as doublet under the S-duality transformation, the following combination is invariant

$$I = (B^{(2)}, C^{(2)}) \mathcal{M} \begin{pmatrix} B^{(2)} \\ C^{(2)} \end{pmatrix}$$

where the matrix $\mathcal{M}$ is

$$\mathcal{M} = e^{\phi} \begin{pmatrix} |\tau|^2 & -C \\ -C & 1 \end{pmatrix}$$

In component it is

$$I = e^{-\phi} B^{(2)} B^{(2)} + e^{\phi} C^{(2)} C^{(2)} - e^{\phi} C \{B^{(2)}, C^{(2)}\} + e^{\phi} C C B^{(2)} B^{(2)}$$

The dilaton factor is the background field corresponding to the disk-level amplitudes.

If one considers linear S-duality, then the combination of first two term are invariant.

The consistency of the above amplitude with the S-duality then predicts the following disk-level amplitude:

$$A(D_3; C_1^{(2)}, C_2^{(2)}) \ \sim \ T_{D3}\alpha'^2 K(D_3; C_1^{(2)}, C_2^{(2)}) \frac{\Gamma(-te^{-\phi/2}/4)\Gamma(-se^{-\phi/2})}{\Gamma(1-te^{-\phi/2}/4-se^{-\phi/2})}\delta^4(p_1^a+p_2^a)$$

where the kinematic factor is

$$K(D_p; C_1^{(2)}, C_2^{(2)}) \ = \ e^{-\phi}\left(e^{\phi}\sqrt{-\eta}\Big[\frac{1}{6}(F_1^{(3)})_{ijk,a}(F_2^{(3)})^{ijk,a} + \frac{1}{3}(F_1^{(3)})_{abc,i}(F_2^{(3)})^{abc,i} - \frac{1}{2}(F_1^{(3)})_{bci,a}(F_2^{(3)})^{bci,a}\Big]\right)$$

This 2-point function is exactly reproduced by the direct calculation.

Adding the above two disk-level couplings and including the appropriate loops and nonperturbative effects, one can make the whole multiplet to be invariant.

One can extend the above discussion to sphere-level amplitude because the vacumme in this case also is invariant under the S-duality.

3.3 S-matrix elements for F_1-string and NS_5-brane

We now speculate that S-duality may generate the S-matrix elements for F_1-string and NS_5-brane which we don't know how to calculate them by the world-sheet conformal field theory.

Since D_1-string and F_1-string couples linearly to the RR two-form and the B-field, respectively, one concludes that D_1-string is mapped to F_1-string under the S-duality.

In other words the combination of the following standard couplings are invariant under the S-duality:

$$T_{D1} \int C^{(2)} + T_{F1} \int B^{(2)} - T_{D1} \int C^{(2)} - T_{F1} \int B^{(2)}$$

While the coupling of the D_1-string can be confirmed by the disk-level 1-point function in which the RR vertex operator is in $(-1/2, -3/2)$-picture, the coupling of F_1-string has no such description.

Now, what happens when one extends the 1-point function of the D_1-string to 2-point function? The 2-point functions in the Einstein frame is

$$A(D_1; C_1^{(2)}, h_2) \ \sim \ T_{D1}\alpha'^2 K(D_1; C_1^{(2)}, h_2)\frac{\Gamma(-te^{-\phi/2}/4)\Gamma(-se^{-\phi/2})}{\Gamma(1-te^{-\phi/2}/4-se^{-\phi/2})}\delta^2(p_1^a+p_2^a)$$

where the kinematic factor is

$$K(D_1; C_1^{(2)}, h_2) \ = \ \epsilon^{a_0a_1}e^{-\phi}\Big[F^{(3)}_{ia_1j,a}R^a{}_{a_0}{}^{ij} - F^{(3)}_{a_0a_1j,i}\hat{R}^{ij}\Big]$$

In this case, one can not make it invariant under the S-duality by adding the loops and the D-instanton effects. It should be transformed in covariant form. It transforms to the following amplitude on the world-volume of the F_1-string:

$$A(F_1; B_1^{(2)}, h_2) \ \sim \ T_{F1}\alpha'^2 K(F_1; B_1^{(2)}, h_2)\frac{\Gamma(-te^{\phi/2}/4)\Gamma(-se^{\phi/2})}{\Gamma(1-te^{\phi/2}/4-se^{\phi/2})}\delta^2(p_1^a+p_2^a)$$

where the kinematic factor is

$$K(F_1; B_1^{(2)}, h_2) \ = \ \epsilon^{a_0a_1}e^{\phi}\Big[H_{ia_1j,a}R^a{}_{a_0}{}^{ij} - H_{a_0a_1j,i}\hat{R}^{ij}\Big]$$

Adding the above amplitudes one finds S-duality invariant combination.

We expect in a similar way one can find all tree-level S-matrix elements on the world-volume of F_1-string, *e.g.,*

$$A(F_1;1,2) \;\sim\; T_{F1}\alpha'^2 K(F_1;1,2)\frac{\Gamma(-te^{\phi}/4)\Gamma(-se^{\phi})}{\Gamma(1-te^{\phi}/4-se^{\phi})}\delta^2(p_1^a+p_2^a)$$

where the metric is the string frame metric and the kinematic factor is

$$K(D_1;1,2) \;\stackrel{s}{\longrightarrow}\; K(F_1;1,2)$$

Similarly for all other S-matrix elements.

The amplitude is at strong coupling, however, the S-matrix elements are invariant under the supersymmetry. Hence, we expect them to be valid at weak couplings as well.

The new S-matrix elements can be found by using the linear S-duality transformation on a given disk-level S-matrix element.

The axion and the dilaton, *i.e.*, $\tau = C + ie^{-\phi}$ are the only fields which transformation nonlinearly under the S-duality.

The S-duality transformation that maps $B^{(2)}$ to $C^{(2)}$, maps τ to

$$\tau \xrightarrow{s} -\frac{1}{\tau}$$

At the linear order of axion, it is

$$C \xrightarrow{s} -e^{2\phi}C$$

where $e^{2\phi}$ is the background dilaton factor.

Therefore, the axion state in the disk-level n-point function of D_1-string is mapped to $-e^{2\phi}C$ in the tree-level n-point function of F1-string.

For the magnetic dual couplings, consider

$$T_{D5}\int C^{(6)} + T_{NS5}\int B^{(6)} - T_{D5}\int C^{(6)} - T_{NS5}\int B^{(6)}$$

which are invariant under the S-duality.

Repeating the same steps as we have done in the previous case, one finds the following tree-level 2-point function for the NS_5-brane:

$$A(NS_5;1,2) \;\sim\; T_{NS5}\alpha'^2 K(NS_5;1,2)\frac{\Gamma(-te^{\phi}/4)\Gamma(-se^{\phi})}{\Gamma(1-te^{\phi}/4-se^{\phi})}$$

where the kinematic factor $K(NS_5;1,2)$ is related to the kinematic factor of D_5-brane by the S-duality transformation, *i.e.,*

$$K(D_5;1,2) \;\stackrel{s}{\longrightarrow}\; K(NS_5;1,2)$$

The combination $A(NS_5;1,2) + A(D_5;1,2) + A(\overline{NS}_5;1,2) + A(\overline{D}_5;1,2)$ is then invariant under S-duality.

The gamma functions in the above 2-point functions represent both s- and t-channels.

The poles in the t-channel, which are at $\frac{g_s t}{2} = 0, 2, 4, \cdots$, present the closed D-string couplings to the F_1-string/NS_5-brane.

The poles in the s-channel, which are at $\frac{g_s s}{2} = 0, \frac{1}{2}, 1, \cdots$, present the open D-string excitation of F_1-string/NS_5-brane.

Hence, the S-duality predicts that these objects have D-string excitation at strong and weak couplings.

Let us examine the massless poles in t- and s-channelds.

Consider the following standard coupling in the type IIB supergravity and the linear coupling of the B-field to F_1-string:

$$\int d^{10}x (F^{(3)} + C^{(0)} H)^2 \ \ ; \ \ T_{F1} \int B^{(2)}$$

One can calculate the massless closed D-string pole in the scattering

$$C^{(0)} + \mathrm{F}_1\mathrm{-string} \longrightarrow C^{(2)} + \mathrm{F}_1\mathrm{-string}$$

The Feynman amplitude becomes

$$\mathcal{A}_t(F_1) \sim T_{F1}\frac{F^{(1)\mu}F^{(3)}_{\mu ab}\epsilon^{ab}}{t}$$

On the other hand, the supergravity coupling and the linear coupling of the RR two-form to D_1-string can be used to calculate the massless closed string pole in the following scattering:

$$C^{(0)} + \mathrm{D}_1\mathrm{-string} \longrightarrow B^{(2)} + \mathrm{D}_1\mathrm{-string}$$

The Feynman amplitude in this case becomes

$$\mathcal{A}_t(D_1) \sim T_{D1}\frac{F^{(1)\mu}H_{\mu ab}\epsilon^{ab}}{t}$$

The massless open string pole in the D_1-string scattering can be calculated by using the standard brane couplings $T_{D1}B_{ab}f^{ab}$, $T_{D1}\epsilon^{ab}f_{ab}C^{(0)}$ and $T_{D1}f_{ab}f^{ab}$. The Feynman amplitude becomes

$$\mathcal{A}_s(D_1) \sim T_{D1}\frac{\epsilon^{ab}F_a^{(1)}B_{bc}{}^{,c}}{s}$$

On the other hand, the massless open D-string pole in the F_1-string scattering can be calculated by assuming the brane couplings $T_{F1}C_{ab}f^{ab}$, $T_{F1}\epsilon^{ab}f_{ab}C^{(0)}$ and $T_{F1}f_{ab}f^{ab}$. The Feynman amplitude becomes

$$\mathcal{A}_s(F_1) \sim T_{F1}\frac{\epsilon^{ab}F_a^{(1)}C_{bc}{}^{,c}}{s}$$

Comparing these amplitudes, one finds that they are consistent with our proposal for the string amplitude.

Note that the string amplitudes have no dilaton factor at order $O(\alpha'^0)$.

One can find the contact terms at order $O(\alpha'^0)$ by subtracting the above massless poles from the α'-expansion of the tree-level 2-point function.

In the case of D_1-string, one finds

$$A(D_1; C^{(0)}, B^{(2)}) - \mathcal{A}_t(D_1) - \mathcal{A}_s(D_1) \ \sim \ T_{D1} C^{(0)} B_{ab} \epsilon^{ab} + O(\alpha'^2)$$

which is a standard term in the Chern-Simons part of the D_1-string action.

Similar calculation for F_1-string gives

$$A(F_1; C^{(0)}, C^{(2)}) - \mathcal{A}_t(F_1) - \mathcal{A}_s(F_1) \ \sim \ T_{F1} C^{(0)} C_{ab} \epsilon^{ab} + O(\alpha'^2)$$

This is a coupling in the world volume of F_1-string.

One can extend this discussion to the n-point functions to find other α'^0-couplings.

4 F_1-string and NS_5-brane action

The α'^0-couplings on the world-volume of D_1-brane and D_5-brane in string frame are given by the following actions:

$$\begin{aligned} S_{D1} &= T_{D1}\int d^2x e^{-\phi}\sqrt{-\det(g_{ab}+B_{ab})} + T_{D1}\int [C^{(2)} + C^{(0)}B^{(2)}] \\ S_{D5} &= T_{D5}\int d^6x e^{-\phi}\sqrt{-\det(g_{ab}+B_{ab})} + T_{D5}\int [C^{(6)} + \\ &\quad + C^{(4)}\wedge B^{(2)} + \frac{1}{2}C^{(2)}\wedge B^{(2)}\wedge B^{(2)} + \frac{1}{3!}C^{(0)}B^{(2)}\wedge B^{(2)}\wedge B^{(2)}] \end{aligned}$$

All the closed string fields in the actions are pull-back of the bulk fields onto the world-volume of branes.

The abelian gauge field can be added to the actions as $B \to B + 2\pi\alpha' f$.

The above actions can be confirmed by the disk-level S-matrix elements on the world-volume of D_1-string and D_5-brane.

We have speculated that the tree-level S-matrix elements of F_1-string and NS_5-brane are related by S-duality to the disk-level S-matrix elements of D_1-string and D_5-brane, respectively.

Using this, we expect the actions of F_1-string and NS_5-brane to be related by the S-duality to the actions of D_1-string and D_5-brane, respectively.

The S-duality transformation are:

$$\begin{aligned} g_{\mu\nu} &\rightarrow e^{-\phi} g_{\mu\nu} \\ \phi &\rightarrow -\phi \\ C^{(2)} &\rightarrow B^{(2)} \\ B^{(2)} &\rightarrow -C^{(2)} \\ C^{(0)} &\rightarrow -e^{2\phi} C^{(0)} \\ C^{(4)} &\rightarrow C^{(4)} \end{aligned}$$

where the metric is the string frame metric.

Using these transformations, one finds that the actions D_1-string and D_5-brane are mapped to

$$\begin{aligned} S_{F1} &= T_{F1}\int d^2x\sqrt{-\det(g_{ab}-C_{ab})}+T_{F1}\int[B^{(2)}+C^{(0)}C^{(2)}] \\ S_{NS5} &= T_{NS5}\int d^6x e^{-2\phi}\sqrt{-\det(g_{ab}-C_{ab})}+T_{NS5}\int e^{-2\phi}[B^{(6)}- \\ &\quad -C^{(4)}\wedge C^{(2)}+\frac{1}{2}B^{(2)}\wedge C^{(2)}\wedge C^{(2)}+\frac{1}{3!}C^{(0)}C^{(2)}\wedge C^{(2)}\wedge C^{(2)}] \end{aligned}$$

The closed string fields are pull-back of the bulk fields onto the world-volume of branes.

The abelian gauge field can be added to the actions as $C^{(2)}\to C^{(2)}+2\pi\alpha' f$.

The gauge field f_{ab} and the transverse scalar fields in the definition of the pull-back operation are the massless open D-string excitation of F_1-string and NS_5-brane.

Since the D-brane action is supersymmetric, we expect the above action to be valid for both strong and weak couplings.

Thank you

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