# Special Geometry of Black Hole Horizons

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## Classifying supersymmetric backgrounds

Some of the results so far

- The KSEs of all supergravity theories can be solved for backgrounds preserving one supersymmetry. This includes type II in 10 dim and 11-dim supergravities
- All maximally and nearly maximally supersymmetric backgrounds can be classified in all supergravity theories
- Supergravities for which the KSEs have been solved for backgrounds preserving any number of supersymmetries include:
  - Heterotic
  - $\mathcal{N} = 1, D = 4$ with the most general couplings
  - $(1, 0), D = 6$ with the most general couplings
  - Several others with special couplings

Some properties include

- Supersymmetric backgrounds exhibit special geometric structures. The existence of such structures is related to the question for existence of solutions to non-linear differential equations on manifolds reminiscent to those of the Calabi conjecture, Yamabe problem and Hermitian-Einstein connections.

- Not all BPS configurations are allowed. For example solutions that preserve 31 supersymmetries, allowed by the D=11 algebra of supercharges, do not exist.

An outstanding problem is

- The solution of KSEs of type II and 11-dimensional supergravities in all cases.

## Applications

Some of the applications in physics considered include

- Classification of vacua of compactification with fluxes scenarios
- Classification of the gravitational duals of gauge theories in the context of gauge/gravity correspondences
- Understanding the topology and geometry of black hole horizons and explore uniqueness theorems for black holes
- Find black holes in higher dimensional supergravities which have horizons with exotic topology.

## Uniqueness theorems for black holes

- It is known that the most general asymptotically flat vacuum black hole solution in 4-d is the Kerr black hole [Israel, Carter, Hawking, Robinson]
- In 5-d there is not such a uniqueness theorem. There are black holes with spherical and $S^1 \times S^2$ horizon sections, the BMPV black hole and black ring [Elvang, Emparan, Mateos, Reall], respectively. The possibility of a black hole with horizon section $T^3$ has not been ruled out.
- In higher dimensions, the question remains open. It has been mostly pursued either for static black holes [Gibbons, Ida, Shiromizu; Rogatko; Emparan, Harmark, Niarchos, Obers] or for black holes which preserve some supersymmetry.

## Gaussian null coordinates

Assume that the event horizon is a Killing horizon, ie that the black hole admits a timelike Killing vector field which becomes null at the horizon. Then a coordinate system can be adapted such that a black hole metric is [Friedrich, Racz, Wald]

$$ds^2 = 2du[dr + r\, h_I(r,y)dy^I + r f(r,y)du] + \gamma_{IJ}(y,r)dy^I dy^J$$

Using analyticity in $r$, the near horizon limit can be taken for extreme black holes

$$f(0,y) = 0$$

leading to a near horizon metric

$$ds^2 = 2du[dr + r\, h_I dy^I + r^2\, \Delta du] + \gamma_{IJ} dy^I dy^J$$

where

$$h_I = h_I(0,y)\ , \quad \Delta = \partial_r f|_{r=0}\ , \quad \gamma_{IJ} = \gamma_{IJ}(0,y)$$

- The near horizon metric has two isometries generated by translations in $u$ and the scale transformation

$$u \to \ell^{-1}u\ , \quad r \to \ell r$$

- The two Killing vectors

$$\partial_u\ , \quad -u\partial_u + r\partial_r$$

  do not commute
- The Gaussian null coordinate system can be adapted in the presence of other fields like Maxwell and k-form gauge potentials. Under some assumptions in the extreme case a near horizon geometry can again be defined
- The co-dimension 2 space given by $u = \text{const}$, $r = 0$ is the horizon section, $\mathcal{S}$, and it is required to be compact without boundary.

# Gravitational duals from horizons

The near horizon geometries for black holes include the gravitational dual backgrounds, $AdS_n \times_w X$;

$$ds^2 = A^2 \left[e^{2z/\ell}\left(2dudv + \sum_{k>1}(dx^k)^2\right) + dz^2\right] + ds^2(X)$$

After a coordinate transformation

$$v = A^{-2}e^{-2z/\ell}r\,, \quad u = u\,, \quad z = z\,, \quad x^k = x^k\,,$$

it can be rewritten as the near horizon metric

$$ds^2 = 2du(dr - \frac{2r}{\ell}dz - rd\log A^2) + A^2\left(e^{2z/\ell}\sum_{k>1}(dx^k)^2 + dz^2\right) + ds^2(X)$$

Therefore

$$\mathcal{S} = H^{n-2} \times_w X$$

$H^{n-2}$ hyperbolic space.

- Locally, the classification of near horizon geometries include those of the gravitational duals of gauge/gravity correspondences and flux compactifications.

## Heterotic backgrounds

The Killing spinor equations of Heterotic supergravities (10-dimensions) are

$$\hat{\nabla}_\mu \epsilon = \nabla_\mu \epsilon - \frac{1}{8} H_{\mu\nu\rho} \Gamma^{\nu\rho} \epsilon + \mathcal{O}(\alpha') = 0 \,,$$
$$\Gamma^\mu \partial_\mu \Phi \epsilon - \frac{1}{24} H_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \epsilon + \mathcal{O}(\alpha') = 0 \,, \quad \epsilon \in \Delta^+_{\mathbf{16}}$$

These are valid up to 2-loops in the sigma model perturbation theory. Both have been solved in all cases [Gran, Lohrmann, GP; Gran, Roest, Sloane, GP].

Solution of KSE for $dH = 0$. Black hole horizons are indicated in red.

| $L$ | $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$ | $N$ |
|---|---|---|
| 1 | $Spin(7) \ltimes \mathbb{R}^8$ | 1 |
| 2 | $SU(4) \ltimes \mathbb{R}^8$ | , 2 |
| 3 | $Sp(2) \ltimes \mathbb{R}^8$ | , −, 3 |
| 4 | $(\times^2 SU(2)) \ltimes \mathbb{R}^8$ | −, −, −, 4 |
| 5 | $SU(2) \ltimes \mathbb{R}^8$ | −, −, −, −, 5 |
| 6 | $U(1) \ltimes \mathbb{R}^8$ | −, −, −, −, −, 6 |
| 8 | $\mathbb{R}^8$ | −, −, −, −, −, −, −, 8 |
| 2 | $G_2$ | 1, 2 |
| 4 | $SU(3)$ | 1, 2, −, 4 |
| 8 | $SU(2)$ | −, 2, −, 4, −, 6, −, 8 |
| 16 | $\{1\}$ | 8, 10, 12, 14, 16 |

# Heterotic horizons

The near horizon geometry of a supersymmetric black hole, preserving one supersymmetry, in heterotic supergravity can be written [Gutowski, GP] as

$$\begin{aligned} ds^2 &= 2e^-e^+ + d\tilde{s}^2_{(8)} \,, \quad d\tilde{s}^2_{(8)} = \delta_{ij}e^ie^j \,, \\ H &= d(e^- \wedge e^+) + \tilde{H}_{(8)} \,, \quad \tilde{H}_{(8)} = dW \,, \\ \Phi &= \Phi(y) \,, \quad e^- = dr + rh_ie^i \,, \quad e^+ = du \,, \ \ e^i = e^i{}_I dy^I \,. \end{aligned}$$

The dilaton field equation is

$$\tilde{\nabla}^2\Phi - 2\tilde{\nabla}^i\Phi\tilde{\nabla}_i\Phi + \frac{1}{12}(\tilde{H}_{(8)})_{ijk}(\tilde{H}_{(8)})^{ijk} - \frac{1}{2}h_ih^i = 0 \,,$$

- The Killing vector $\partial_u$ is null and $\mathrm{hol}(\hat{\nabla}) \subseteq Spin(7) \ltimes \mathbb{R}^8$.

- The horizon section $r = 0, u =$ const is a compact 8-dimensional manifold.
- If $h = 0$, the dilaton field equation and compactness imply that $M = \mathbb{R}^{1,1} \times X_8$, where $X_8$ is a product of Berger manifolds that admit parallel spinors, $H = 0$ and $\Phi$ is constant.
- If $h \neq 0$, $N = 1$ supersymmetry and compactness of $\mathcal{S}$ imply that $\hat{\tilde{\nabla}}_{(8)} h = 0$.
- The holonomy of $\hat{\nabla}$ reduces to $G_2$, $\mathrm{hol}(\hat{\nabla}) \subseteq G_2$, and $M$ admits 2 supersymmetries.

Heterotic horizons, $h \neq 0$, always preserve even number of supersymmetries.

| $N$ | $\mathrm{hol}(\hat{\nabla})$ |
|---|---|
| 2 | $G_2$ |
| 4 | $SU(3)$ |
| 6 | $SU(2)$ |
| 8 | $SU(2)$ |

# $N = 4, SU(3)$

The near horizon geometries are $SL(2,\mathbb{R}) \times U(1)$ fibrations over a Hermitian 6-dimensional manifold $B^6$ with structure group $U(3)$ and equipped with a compatible connection with skew-symmetric torsion.
The metric and torsion are

$$\begin{aligned} ds^2 &= \eta_{ab}\lambda^a\lambda^b + ds^2_{(6)} \\ H &= CS(\lambda) + H_{(6)} , \end{aligned}$$

where the connection 1-forms are

$$\lambda^- = e^- , \quad \lambda^+ = e^+ - \frac{1}{2}k^2u^2e^- - uh ,$$
$$\lambda^1 = k^{-1}\left(h + k^2ue^-\right) , \quad \lambda^6 = k^{-1}\ell ,$$

where $h^2 = k^2$, $k$ constant.
The Lie algebra of the associated vector fields is

$$[\xi_+, \xi_-] = -k\xi_1, \qquad [\xi_+, \xi_1] = k\xi_+, \qquad [\xi_-, \xi_1] = -k\xi_- , \qquad [\xi_a, \xi_6] = 0$$

which is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$.

The Killing spinors are

$$\begin{aligned}\epsilon^1 &= 1 + e_{1234}\ , \quad \epsilon^2 = -k^2 u(1+e_{1234}) + h_i \Gamma^{+i}(1+e_{1234})\ , \\ \epsilon^3 &= i(1-e_{1234})\ , \quad \epsilon^4 = -ik^2 u(1-e_{1234}) + ih_i \Gamma^{+i}(1-e_{1234})\ ,\end{aligned}$$

The curvature of $\lambda$ is

$$\begin{aligned}\mathcal{F}^+ &= -u(1+\frac{1}{2}k^2 ru)dh\ , \quad \mathcal{F}^- = rdh\ , \\ \mathcal{F}^1 &= k^{-1}(1+k^2 ru)dh \quad \mathcal{F}^6 = k^{-1}d\ell\ ,\end{aligned}$$

where $h$ and $\ell$ are Hermitian-Einstein

$$dh \in \mathfrak{su}(3)\ , \quad d\ell \in \mathfrak{u}(3)$$

with $d\ell_{ij}\omega^{ij}_{(6)} = -2k^2$. In particular,
The horizon section $\mathcal{S}$ is a holomorphic $T^2$ fibration over $B^6$

$$ds^2_{(8)} = k^{-2}h^2 + k^{-2}\ell^2 + ds^2_{(6)}\ , \quad H_{(8)} = k^{-2}h\wedge dh + k^{-2}\ell\wedge d\ell + H_{(6)}$$

The conditions that remain be solved to find a solutions are

$$dh\wedge\omega^2_{(6)}=0\,,\quad d\ell\wedge\omega^2_{(6)}=-\frac{k^2}{3}\,\omega^3_{(6)}\,,\quad d\big(e^{-2\Phi}\omega^2_{(6)}\big)=0\,,$$
$$\hat{\tilde{\rho}}_{(6)}-d\ell=0\,,\quad k^{-2}dh\wedge dh+k^{-2}d\ell\wedge d\ell-di_{I_{(6)}}d\omega_{(6)}=0\,.$$

where

$$\hat{\tilde{\rho}}_{(6)}=\frac{1}{4}\hat{\tilde{R}}_{k\ell,}{}^i{}_j(I_{(6)})^j{}_i\,e^k\wedge e^\ell=-i\partial\bar\partial\log\det g_{(6)}+4i\partial\bar\partial\Phi\,,$$

Necessary conditions for the existence of solutions are

$$c_1(P)\wedge[e^{-2\Phi}\omega^2_{(6)}]=0\,,\quad c_1(Q)\wedge[e^{-2\Phi}\omega^2_{(6)}]=-\frac{k^2}{6\pi}\,[e^{-2\Phi}\omega^3_{(6)}]\,,$$
$$c_1(B^6)-c_1(Q)=0\,,\quad c_1(P)\wedge c_1(P)+c_1(Q)\wedge c_1(Q)=0\,,$$

- $\mathcal{S}$ is a holomorphic $T^2$-fibration over $B^6$ and the fibre connection is $(h, \ell)$.
- $\mathcal{S}$ is a complex manifold with Hermitian form $\omega_{(8)} = k^{-2} h \wedge \ell + \omega_{(6)}$

A comparative table of the properties of $\mathcal{S}$ and $B^6$ are

| Geometry | $B^6$ | $\mathcal{S}$ |
|---|---|---|
| Hermitian | yes | yes |
| $\theta = 2d\Phi$ | yes | no |
| $\mathrm{hol}(\hat{\nabla}) \subseteq SU(3)$ | no | yes |
| $\mathrm{hol}(\hat{\nabla}) \subseteq U(3)$ | yes | no |
| $dH_{(n)} = 0$ | no | yes |

# Examples

Explicit examples include

- $S^3 \times S^3 \times T^2$
- $SU(3)$
- $S^1 \times S^3 \times K_3$

## $\mathcal{S}$ as $T^4$-fibrations

Many examples can be constructed by taking $B^6$ to be a holomorphic $T^2$ fibration over a 4-d Kähler manifold $X^4$ with Kähler from $\kappa$.

Thus $\mathcal{S}$ is a holomorphic $T^4$-fibration over $X^4$, the supersymmetry conditions can be written as

$$dh^1\wedge\kappa=0\,,\quad dh^2\wedge\kappa=0\,,\quad dh^3\wedge\kappa=0\,,\quad d\ell\wedge\kappa=-\frac{k^2}{2}\,e^{2\Phi}\kappa^2\,,$$
$$-i\partial\bar\partial\log\det(i\kappa)-d\ell=0\,,\quad d\kappa=0\,,$$
$$dh^1\wedge dh^1+dh^2\wedge dh^2+dh^3\wedge dh^3+d\ell\wedge d\ell+2k^2 i\partial\bar\partial e^{2\Phi}\wedge\kappa=0\,,$$

where $h^1=h$, and $h^2$ and $h^3$ are along the two new fibre directions.

- These are 6 equations for 6 unknown functions on $X^4$.
- One of the equations is the Monge-Amperé equation but the system cannot be separated.

The associated co-homological system is

$$c_1(P_1)\wedge[\kappa]=0\,,\quad c_1(P_2)\wedge[\kappa]=0\,,\quad c_1(P_3)\wedge[\kappa]=0\,,$$
$$c_1(Q)\wedge[\kappa]=-\frac{k^2}{4\pi}\,[e^{2\Phi}\kappa^2]\,,\quad c_1(X)-c_1(Q)=0\,,$$
$$\sum_{r=1}^{3}c_1(P_r)\wedge c_1(P_r)+c_1(Q)\wedge c_1(Q)=0\,.$$

Suppose that there are manifolds $X^4$ such that the cohomological system has solutions.

Question: Are these cohomological conditions necessary and sufficient for the existence of heterotic horizons?

# $X^4 = S^2 \times S^2$

Take as metric and hermitian form on $\mathcal{S} = S^3 \times S^3 \times T^2$

$$\begin{aligned} ds^2_{(8)} &= (\sigma^3)^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\rho^3)^2 + (\rho^1)^2 + (\rho^2)^2 + (\tau^1)^2 + (\tau^2)^2 \,, \\ \omega_{(8)} &= -\sigma^3 \wedge \rho^3 - \sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2 - \tau^1 \wedge \tau^2 \,, \end{aligned}$$

$S^3 \times S^3 \times T^2$ is not (conformally) balanced.
Set

$$\omega = -\sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2 - \tau^1 \wedge \tau^2 \,, \quad \chi = \frac{1}{2\sqrt{2}}(\sigma^1 + i\sigma^2) \wedge (\rho^1 + i\rho^2) \wedge (\tau^1 + i\tau^2) \,,$$

on $S^3 \times S^3 \times T^2$, and

$$h = -(\sigma^3 - \rho^3) \,, \quad \ell = (\sigma^3 + \rho^3) \,.$$

Moreover

$$i_{\sigma^3}\omega = i_{\rho^3}\omega = 0 \,, \quad i_{\sigma^3}\chi = i_{\rho^3}\chi = 0 \,,$$

and

$$\mathcal{L}_{\sigma^3}\omega = \mathcal{L}_{\rho^3}\omega = 0 \,, \quad \mathcal{L}_{\sigma^3+\rho^3}\chi = 2i\chi \,, \quad \mathcal{L}_{\sigma^3-\rho^3}\chi = 0 \,.$$

Thus s $\omega_{(6)} = \omega$ on $B^6 = S^2 \times S^2 \times T^2$. $d\omega_{(6)} = 0$, so $B = S^2 \times S^2 \times T^2$ is Kähler
The dilaton is constant.

- The canonical bundle of $S^2 \times S^2 \times T^2$ is not trivial but it is trivial on $S^3 \times S^3 \times T^2$.
- Since $B = S^2 \times S^2 \times T^2$, the 4-dimensional Kähler manifold $X$ is $X = S^2 \times S^2$. with Kähler form

$$\kappa = -\sigma^1 \wedge \sigma^2 - \rho^1 \wedge \rho^2 \, .$$

- $S^3 \times S^3 \times T^2$ is a $T^4$ fibration over $S^2 \times S^2$ with principal bundle connection $\ell = \ell$, $h^1 = h$, $h^2 = \tau^1$ and $h^3 = \tau^2$.
- The horizon is isometric to $(AdS_3 \times S^3 \times S^3)/S^1 \times T^2$.

# del Pezzo

$X^4 = dP_{9-k}$ del Pezzo surface, ie $CP^2$ blown up at $k < 9$ points. The second cohomology of $dP_{9-k}$ has a basis

$$-K = 3H - E_1 - \cdots - E_k \ , \quad \alpha_i = E_i - E_{i+1} \ , \quad i < k \ , \quad \alpha_k = H - E_1 - E_2 - E_3 \ ,$$

where $E_i$ are the exceptional divisors and $K$ the canonical divisor.
The intersection matrix is

$$K \cdot \alpha_i = 0 \ , \quad \alpha_i \cdot \alpha_j = -A_{ij} \ , \quad i,j = 1, \dots, k \ ,$$

where $(A_{ij})$ is the Cartan matrix of exceptional Lie algebras $\mathbf{E}_k$.

| $k$ | $\mathbf{E}_k$ |
|---|---|
| 1 | $A_1$ |
| 2 | $A_1 \oplus A_1$ |
| 3 | $A_2 \oplus A_1$ |
| 4 | $A_4$ |
| 5 | $D_5$ |
| 6 | $\mathbf{E}_6$ |
| $k$ | $\mathbf{E}_k, k > 6$ |

- There is a solution of the cohomological conditions for every del Pezzo surface
- There are no solutions for $k > 9$ unless $dH \neq 0$
- The horizon sections have topologies which include $\big((k-1)S^2 \times S^4 \# kS^3 \times S^3\big) \times T^2$

For example for $k = 3$

$$c_1(Q) = K \ , \quad c_1(P_1) = c_1(P_2) = c_1(P_3) = E_1 - E_2 \ .$$

The solution for the del Pezzo surface $X^4 = S^2 \times S^2$ has been constructed explicitly.

# $N = 8, SU(2)$

The heterotic horizons with $N = 8$ supersymmetries are

$$AdS_3 \times S^3 \times T^4 \ , \quad AdS_3 \times S^3 \times K_3$$

The radii of $S^3$ and $AdS_3$ are equal and the dilaton is constant.

# IIB horizons

Consider the KSEs of IIB supergravity with only 5-form flux

$$\mathcal{D}_M\epsilon = \nabla_M\epsilon + \frac{i}{48}F_{MN_1\dots N_4}\Gamma^{N_1\dots N_4}\epsilon = 0$$

- Unlike the heterotic case there is no classification of solutions to IIB KSEs
- The KSEs have been solved for backgrounds preserving one supersymmetry [Gutowski, Gran, Roest, GP]
- The backgrounds preserving 28 supersymmetries have been classified and those preserving more than 28 are maximally supersymmetric [Gutowski, Gran, Roest, GP]
- The maximally supersymmetric backgrounds have been classified [Figueroa O'Farril, GP]
- The KSEs have also been solved for backgrounds with only 5-form flux preserving 2 supersymmetries.

## IIB horizon geometry

The horizon is an 8-dimensional 2-strong Calabi-Yau with torsion [Gran, Gutowski, GP], ie it is Hermitian with hidden skew-symmetric torsion $H$ such that

$$\hat{\rho} = 0 \,, \quad d(\omega \wedge H) = \partial\bar{\partial}\omega^2 = 0 \,,$$

where $\hat{\rho}$ is the Ricci form of the compatible connection $\hat{\nabla}$ with torsion $H$.

- $H$ is hidden because it is not identified with either the RR or NSNS 3-form field strengths as they have been set to zero.
- Many solutions exist mostly constructed as torus fibrations over products of Kähler-Einstein and Calabi-Yau spaces. This gives many examples of new horizon geometries
- There are several generalizations of these geometries including imposing the k-strong condition

$$d(\omega^{k-1} \wedge H) = \partial\bar{\partial}\omega^k = 0$$

## Applications

- For the gravitational dual backgrounds $AdS_n \times_w X$, n=3,5,7, $H_{n-2} \times X$ has a 2-strong Calabi-Yau geometry with torsion. This geometry includes the well-known Sasakian-Einstein structure on $X$ for $n = 5$
- Solutions $AdS_7 \times_w X$ do not exist.
- Does any Hermitian 2n-dimensional manifold admit a k-strong structure? For $k = n - 1$, it is true as a consequence of the Gauduchon theorem. The problem for the rest of the cases is open.

## Static M-horizons

The Killing spinor equation of 11-dimensional supergravity is

$$\nabla_M \epsilon + \left( -\frac{1}{288} \Gamma_M{}^{L_1 L_2 L_3 L_4} F_{L_1 L_2 L_3 L_4} + \frac{1}{36} F_{M L_1 L_2 L_3} \Gamma^{L_1 L_2 L_3} \right) \epsilon = 0 \,,$$

- This has been solved for backgrounds preserving one supersymmetry [Gauntlett,Pakis, Gutowski], [Gillard, Gran, GP]
- The solutions preserving 30 and 31 supersymmetries are maximally supersymmetric [Gran, Gutowski, Roest, GP]
- The maximally supersymmetric backgrounds have been classified [Figueroa O'Farril, GP]

- All static M-horizons are products $AdS_2 \times X^9$, where $X^9$ has an $SU(4)$ structure [Gutowski, GP].
- The horizon sections of all electric static horizons is a circle bundle over a Kähler-Yamabe 8-dimensional manifold $B$,

$$ds^2(\mathcal{S}) = \lambda^2 + ds^2(B)$$

where

$$d\lambda = \rho(B) \ , \quad \rho(B)^2 = \text{const}$$

and $d\lambda$ is Hermitian-Einstein.
- Examples include $AdS_2 \times S^3 \times CY_6$. There are many other examples constructed as fibrations over products of Kähler-Einstein and Calabi-Yau spaces. For $AdS_2 \times S^3 \times CY_6$, $B = S^2 \times CY_6$
- $B$ is not Kähler-Einstein and $\mathcal{S}$ is not Sasakian.

## Concluding remarks

- Evidence suggest that there is a large number of black holes in heterotic, type II and 11-dimensional theories. Some of them are of Kaluza-Klein type and some others may have exotic topologies
- The horizons sections have special geometry and their existence is closely related the existence of solutions to non-linear differential equations on manifolds. In the heterotic case, the differential system is of Calabi type and includes the Monge-Ampére equation. The associated cohomological system has solutions on all del Pezzo surfaces
- IIB horizons with 5-form flux are 2-strong Calabi-Yau manifolds with torsion.
- Static M-horizons are products $AdS_2 \times X^9$, where $X^9$ has a $SU(4)$ structure. For electric horizons, $X^9$ is a fibration over a Kähler-Yamabe manifold and the length of Ricci form is constant.