Sphere partition functions and the 3d superconformal *R*-charge

Daniel L. Jafferis

Harvard University

Based on: D.J. (1012.3210)

D.J., Klebanov, Pufu, Safdi (1103.1181)

6th Regional Meeting in String Theory
Milos
24 June, 2011

See also: Hama Hosomichi Lee, Martelli Sparks, Cheon Kim Kim

- Motivation: c-theorems and *R*-symmetries
- Supersymmetry on the round sphere
- Non-renormalization of Z: localization
- The exact superconformal *R*-charge
- Holographic examples

c-theorems in various dimensions

- A measure of the number of degrees of freedom in interacting field theories. It should decrease along rg flow.

- Most obvious conjecture is the thermal free energy. In 3d, the critical O(N) model is a counter-example. Also, not constant along conformal manifolds.

c-theorems in various dimensions

- In 2d, the coefficient of the trace anomaly famously has this property. RG flow is the gradient flow for this quantity.

- In 4d, $16\pi^2 \langle T^\mu{}_\mu \rangle = c(\mathrm{Weyl})^2 - 2a(\mathrm{Euler})$, and it is conjectured that *a* plays this role.

- In odd dimensions, there are no anomalies, so this has long been an open problem.

Superconformal *R*-charge

- Theories with 4 supercharges admit a U(1) *R*-symmetry, under which the susies are charged. *R*+*F* is again an *R*-symmetry for any flavor generator, *F*.

$$R = R_0 + \sum_{j=1}^{f} a_j F_j$$

- SCFTs must be *R*-symmetric, as the (now unique) *R*-charge appears in anti-commutators. The dimensions of chiral primaries are given by their *R*-charge.

- The superconformal *R* in the IR typically differs from that in the UV SCFT by mixing with abelian flavor symmetries.

4d a-maximization

- Solved using ‘t Hooft anomaly matching

- The trace anomaly, $a = \frac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R)$, in terms of the exact superconformal R.

Anselmi Freedman Grisaru Johansen

- It was shown that a is maximized as a function of a trial R-charge. Intriligator Wecht

- Gives evidence for the a-theorem.

Other proposals in 3d

- The two point function of an *R*-current is maximized for the superconformal one, since *R*-currents and flavor currents sit in different multiplets. However, it is quantum corrected and seems not to be exactly calculable in 3d.

  Barnes Gorbatov Intriligator Wright

- Myers and Sinha proposed an entanglement entropy, which reproduces *a* in 4d. It was later shown to be equivalent to the sphere partition function in 3d.

Partition functions on S^3

- Calculated by Kapustin Willett Yaakov using localization when there are no anomalous dimenions.

- This partition function of the Euclidean theory is given in classical supergravity by minus the Euclidean Einstein action of the AdS.

$$S = -\frac{1}{16\pi G_N}\int d^4x\sqrt{g}(R-2\Lambda) + S_{surf} + S_{ct} = \frac{\pi}{2G_N}$$

[Henningsson Skenderis; Emparan Johnson Myers]

- Matrix integral for the N=6 theory solved by Drukker Marino Putrov, reproducing $N^{3/2}$ behavior.

Z-minimization

- Return to Cardy's original motivation – consider the theory on S^3. Finite after power law divergences are removed.
- Susy preserving curvature couplings parameterized by an *R*-charge.
- Can be calculated exactly using supersymmetry (localization) as a function of *R*.
- Minimized by the IR *R*-charge, that uniquely corresponds to conformal coupling to curvature.

$\mathcal{N}$=2 Chern-Simons-matter theory

- Consists of a vector multiplet in the adjoint of the gauge group, and chiral multiplets in representations R_i

$$S_{CS}^{\mathcal{N}=2} = \frac{k}{4\pi}\int(A\wedge dA + \frac{2}{3}A^3 - \bar{\chi}\chi + 2D\sigma)$$

- The kinetic term for the chiral multiplets includes couplings $-\bar{\phi}_i\sigma^2\phi_i - \bar{\psi}_i\sigma\psi_i$

- There is the usual D term $\bar{\phi}_i D\phi_i$

Integrate out D, σ , and χ

$$S^{\mathcal{N}=2} = \int \frac{k}{4\pi}(A \wedge dA + \frac{2}{3}A^3) + D_\mu \bar{\phi}_i D^\mu \phi_i + i\bar{\psi}_i \gamma^\mu D_\mu \psi_i$$

$$-\frac{16\pi^2}{k^2}(\bar{\phi}_i T^a_{R_i} \phi_i)(\bar{\phi}_j T^b_{R_j} \phi_j)(\bar{\phi}_k T^a_{R_k} T^b_{R_k} \phi_k) - \frac{4\pi}{k}(\bar{\phi}_i T^a_{R_i} \phi_i)(\bar{\psi}_j T^a_{R_j} \psi_j)$$

$$-\frac{8\pi}{k}(\bar{\psi}_i T^a_{R_i} \phi_i)(\bar{\phi}_j T^a_{R_j} \psi_j).$$

Note that this action has classically marginal couplings. It is has been argued that it does not renormalize, up to shift of k, and so is a CFT.

The recipe

$$Z = \int \prod_{\text{Cartan}} \frac{d\sigma}{2\pi} \ \exp\left[\frac{i}{4\pi}\text{tr}_k \sigma^2\right] \ \text{Det}_{\text{Ad}}\left(\sinh\frac{\sigma}{2}\right)$$

$$\times \ \prod_{\substack{\text{chirals} \\ \text{in rep } R_i}} \text{Det}_{R_i}\left(e^{\ell\left(1-\Delta_i+i\frac{\sigma}{2\pi}\right)}\right)$$

$$\ell(z) = -z\log\left(1-e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\text{Li}_2(e^{2\pi i z})\right) - \frac{i\pi}{12}$$

$$\partial_z \ell(z) = -\pi z\cot(\pi z)$$

Superconformal symmetries on S^3

- The conformal group in 3d is $USp(4) = SO(3,2)$.
- In Euclidean signature, one has the real form $USp(2,2) = SO(4,1)$.
- On S^3, the $USp(2) \times USp(2) = SO(4)$ subgroup acts as rotations of the sphere.
- The $N = 2$ superconformal group is $OSp(2|4)$.
- The R-symmetry is $SO(2) = U(1)$.

Supersymmetry on the sphere

- The sphere possesses homogeneous Killing spinors, $\nabla_\mu \epsilon = \pm \frac{i}{2} \gamma_\mu \epsilon$, so one expects that supersymmetry is preserved. The associated generators square to isometries.

- It corresponds to keeping Q and S while throwing away $\bar{Q}$ and $\bar{S}$ of the superconformal algebra.

- Closely related to the 4d superconformal index on $S^3 \times R$.

OSp(2|2) × SU(2)

- The OSp(2|2) subgroup of OSp(2|2,2) does *not* contain any conformal transformations. The bosonic generators are the *R*-symmetry and $SU(2)_L$ isometries.

$$\{Q^i_A, Q^j_B\} = \delta^{ij} J_{AB} + i\epsilon_{AB}\epsilon_{ij} R$$

- Parity exchanges the two SU(2)s and is broken by this choice.

$$\delta = \tfrac{1}{\sqrt{2}}(Q^1_1 + iQ^2_1),\ \tilde{\delta} = \tfrac{1}{\sqrt{2}}(Q^1_2 - iQ^2_2)$$

$$\delta\phi = 0$$
$$\delta\bar{\phi} = \bar{\psi}\varepsilon$$
$$\delta\psi = (-i\gamma^\mu D_\mu \phi - i\sigma\phi + \frac{\Delta}{r}\phi)\varepsilon$$
$$\delta\bar{\psi} = \varepsilon\bar{F}$$
$$\delta F = \varepsilon(-i\gamma^\mu D_\mu \psi + i\sigma\psi + \frac{1}{r}(\frac{1}{2} - \Delta)\psi + i\lambda\phi)$$
$$\delta\bar{F} = 0,$$

May be obtained by writing OSp(2|2)/U(1) superfields in components. Equivalently, these are the unique transformations that satisfy the algebra.

$$\delta A_\mu = -\frac{i}{2}\lambda^\dagger \gamma_\mu \varepsilon$$
$$\delta\sigma = -\frac{1}{2}\bar{\lambda}\varepsilon$$
$$\delta\lambda = \left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D + i\gamma^\mu \partial_\mu \sigma - \frac{1}{r}\sigma\right)\varepsilon$$
$$\delta\bar{\lambda} = 0$$
$$\delta D = \left(-\frac{i}{2}(D_\mu \bar{\lambda})\gamma^\mu + \frac{1}{4r}\bar{\lambda}\right)\varepsilon.$$

Or by coupling to gravity, putting the theory on the sphere, and taking M_{Pl} to infinity. Certain background fields must be turned on to preserve supersymmetry. The fully nonlinear theory involves corrections that terminate at order $1/r^2$, together with covariantized derivatives.

[Festuccia Seiberg]

Curvature couplings

- To put a non-conformal theory on the sphere, one needs to specify how to couple it to curvature.

- If the theory were conformal, those couplings could be uniquely determined by requiring Weyl invariance.

- OSp(2|2) invariance also determines the couplings uniquely, for *any* R-charge.

$$S = \int \sqrt{g}\Big(D_\mu\phi^\dagger D^\mu\phi + i\psi^\dagger D\psi + F^\dagger F + \phi^\dagger\sigma^2\phi + i\phi^\dagger D\phi - i\psi^\dagger\sigma\psi + i\phi^\dagger\lambda^\dagger\psi - i\psi^\dagger\lambda\phi$$

$$+\frac{\Delta-\frac{1}{2}}{r}\psi^\dagger\psi + \frac{2i}{r}(\Delta-\frac{1}{2})\phi^\dagger\sigma\phi + \frac{\Delta(2-\Delta)}{r^2}\phi^\dagger\phi\Big).$$

[D. Sen; Romelsberger]

From UV to IR

- Supersymmetric localization implies that the partition function is independent of the radius of the sphere, even in the non-conformal case.

- Given the *R*-charge that sits in the susy algebra, one may do the calculation on a small sphere, using the UV theory, and obtain the IR result for a large sphere.

- The difference between UV and IR theories is Q-exact, if both are coupled to curvature using the same *R*-multiplet.

Localizing the path integral

- In Euclidean path integrals, the meaning of supersymmetry is that the expectation values of Q(..) vanish.

[Witten; Duistermaat Heckman; Pestun; Kapustin Willett Yaakov]

- This can sometimes be used to show that the full partition function localizes to an integral over Q-fixed configurations. There is a 1-loop determinant from integrating out the other modes.

$$S_{loc} = \{Q, V\}, \quad [Q^2, V] = 0 \qquad\qquad Z(t) = \int \prod d\Phi \; e^{-S-tS_{loc}}$$

$$\tfrac{d}{dt} Z = -\int \prod d\Phi e^{-S-tS_{loc}} \{Q, V\} = 0$$

Gauge sector

- The unique supersymmetrization of the Yang-Mills action on the sphere is

$$\frac{1}{g_{YM}^2}\int\sqrt{g}\ \mathrm{Tr}\Big(\tfrac{1}{2}F^{\mu\nu}F_{\mu\nu}+D_\mu\sigma D^\mu\sigma+D^2 \\ +i\lambda^\dagger\nabla\lambda+i[\lambda^\dagger,\sigma]\lambda+\tfrac{2}{r}D\sigma-\tfrac{1}{2r}\lambda^\dagger\lambda+\tfrac{1}{r^2}\sigma^2\Big)$$

- It is Q-exact. There is a massless field, $\sigma = -Dr$, whose zero mode survives the localization.

- The Chern-Simons action is non-zero on space of supersymmetric configurations:

$$\frac{ik}{4\pi}\int_{S^3} 2(D\sigma) = i\pi k r^2(\sigma^2)$$

Matter sector

- A chiral multiplet has a one parameter family of supersymmetry preserving actions on the sphere.

$$S = \int \sqrt{g}\Big(D_\mu\phi^\dagger D^\mu\phi + i\psi^\dagger D\!\!\!/\,\psi + F^\dagger F + \phi^\dagger\sigma^2\phi + i\phi^\dagger D\phi - i\psi^\dagger\sigma\psi + i\phi^\dagger\lambda^\dagger\psi - i\psi^\dagger\lambda\phi$$
$$+\frac{\Delta-\frac{1}{2}}{r}\psi^\dagger\psi + \frac{2i}{r}(\Delta-\frac{1}{2})\phi^\dagger\sigma\phi + \frac{\Delta(2-\Delta)}{r^2}\phi^\dagger\phi\Big).$$

- Superpotential terms may be supersymmetrized if they do not break the R-symmetry.

- These actions are all Q-exact.

Computing the determinants

- On a tiny sphere, the theory is gaussian, except for the zero mode scalar in the vector multiplets.

- One expands the fields in angular momentum modes to determine the 1-loop determinant.

- For the vector multiplets, the result is

$$\prod_{\mathrm{roots}\,\alpha} \frac{\sinh(\alpha(\sigma/2))}{\alpha(\sigma)}$$

[Kapustin Willett Yaakov]

1-loop matter determinant

$$Z_{1-loop} = \prod_{n=1}^{\infty} \left(\frac{n+1+ir\sigma-\Delta}{n-1-ir\sigma+\Delta} \right)^n$$

Define $z = 1 - \Delta + ir\sigma$, and let $\ell(z) = \log Z_{1-loop}$

$$\ell(z) = -z\log\left(1 - e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2(e^{2\pi i z})\right) - \frac{i\pi}{12}$$

$$\partial_z \ell(z) = -\pi z \cot(\pi z)$$

The matrix integral

$$Z = \int \prod_{\text{Cartan}} \frac{d\sigma}{2\pi} \ \exp\left[\frac{i}{4\pi}\mathrm{tr}_k\sigma^2\right] \ \mathrm{Det}_{\mathrm{Ad}}\left(\sinh\frac{\sigma}{2}\right)$$

$$\times \ \prod_{\substack{\text{chirals} \\ \text{in rep } R_i}} \mathrm{Det}_{R_i}\left(e^{\ell\left(1-\Delta_i+i\frac{\sigma}{2\pi}\right)}\right)$$

Real masses

- These are background values of the real scalar in a background vector multiplet coupled to an abelian flavor symmetry.

$$\int d^4\theta \bar{Q} e^{m\theta\bar{\theta}} Q$$

- On the sphere, one needs to set $D = -\frac{\sigma}{r}$ to preserve supersymmetry.

$$\{\delta, \tilde{\delta}\} = J + \frac{1}{r}(R_{UV} + aF) - imF$$

A holomorphy

- One can check that the actions depend holomorphically on the parameters

$$z_j = a_j - irm_j$$

- Thus so does the partition function. This allows one to relate the less familiar dependence on curvature couplings to a familiar dependence on real mass deformations.

1-point functions

- Unbroken conformal invariance implies that all 1-points vanish, except for the identity operator.
- One would expect that $\frac{1}{Z}\partial_m Z = 0$, when evaluated at m=0 and the superconformal value of R.
- However, there is a subtlety – the operator defined in the UV as the real mass may contain the identity for the theory on the sphere.
- Partition function is complex due to framing of Chern-Simons theory (susy preserving UV regulator violates reflection positivity – work in progress).

Parity

- Recall that parity switches the two SU(2) isometries of S^3. Thus parity together with OSp(2|2) generates the entire superconformal group.

- The real mass is parity odd. Therefore in a parity preserving theory, its VEV must vanish.

- In a parity violating CFT, only the parity even identity operator has a VEV. Thus $\mathrm{Im}(\frac{1}{Z}\partial_m Z) = 0$

$|Z|$ extremization

- Using the holomorphy, this implies that

$$\partial_\Delta |Z| = 0$$

at the superconformal value of Δ.

- Holographic evidence and examples indicate that $|Z|$ is always minimized. Need to control 2-point functions to prove this in field theory.

AdS dual of Z

- 3d CFT describing N M2 branes on a Calabi-Yau cone is dual to $AdS_4 \times$ Sasaki-Einstein 7-manifold.

- The theory on S^3 is dual to euclidean AdS.

$$-\log\left(Z_{S^3}\right) = \frac{\pi L_{AdS}^2}{2G_N^{4d}} = N^{3/2}\sqrt{\frac{2\pi^6}{27Vol(Y)}}$$

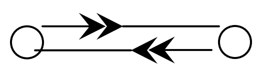
Where the metric on Y is normalized such that $R_{ij} = 6g_{ij}$

Quiver CSM theories

- $U(N)_k \times U(N)_{-k}$ CSM with a pair of bifundamental hypermultiplets

$$Z = \frac{1}{(2\pi)^{2N}} \int \prod_{i=1}^{N} d\sigma_i d\tilde{\sigma}_i \ \exp\left[\frac{ik}{4\pi}(\text{tr } \sigma^2 - \text{tr } \tilde{\sigma}^2)\right]$$

$$\times \prod_{i<j} \sinh^2\left(\frac{\sigma_i - \sigma_j}{2}\right) \sinh^2\left(\frac{\tilde{\sigma}_i - \tilde{\sigma}_j}{2}\right) \prod_{\substack{\text{chirals} \\ \text{in rep } R_i}} \text{Det}_{R_i}\left(e^{\ell\left(1-\Delta_i + i\frac{\sigma}{2\pi}\right)}\right)$$



$$W = \frac{2\pi}{k} \epsilon_{ab} \epsilon_{\dot{a}\dot{b}} (A_a B_{\dot{a}} A_b B_{\dot{b}})$$

Large N limits

- In the ‘t Hooft limit, the eigenvalues form a density. The clump has size of order 1. Thus $F \sim N^2 f(\lambda)$ to leading order, as expected from the saddle point solution to matrix models.

- For large N at fixed k, there is still a density, but the clump has size of order $\sqrt{N}$. Requires cancellation of long range forces.

Matrix models for $\mathcal{N}$=2 quivers

- The saddle point equations are given by the vanishing of the forces:

$$F_i^{(a)} = F_{i,\text{ext}}^{(a)} + F_{i,\text{self}}^{(a)} + \sum_b F_{i,\text{inter}}^{(a,b)} + \sum_b F_{i,\text{inter}}^{(b,a)}$$

$$F_{i,\text{ext}}^{(a)} = \frac{ik_a}{2\pi}\lambda_i^{(a)}$$

$$F_{i,\text{self}}^{(a)} = \sum_{j\neq i} \coth \frac{\lambda_i^{(a)}-\lambda_j^{(a)}}{2}$$

$$F_{i,\text{inter}}^{(a,b)} = \sum_j \left[\frac{\Delta_{(a,b)}-1}{2} - i\frac{\lambda_i^{(a)}-\lambda_j^{(b)}}{4\pi}\right] \coth \left[\frac{\lambda_i^{(a)}-\lambda_j^{(b)}}{2} - i\pi\left(1-\Delta_{(a,b)}\right)\right]$$

Ansatz

- Want a clump of size strictly between O(1) and O(N) - long range forces must then cancel.

$$\lambda_i^{(a)} = N^{1/2} x_i + i y_{a,i} + o(N^0)$$

- Use an eigenvalue density, ϱ(x), for the universal x components, and functions $y_a(x)$.

$$\frac{k_a}{2\pi} N^{3/2} \int dx\, \rho(x) x y_a(x) + \Delta_m^{(a)} N^{3/2} \int dx\, \rho(x) x$$

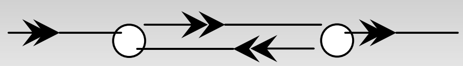
$$-N^{3/2} \frac{2-\Delta^+_{(a,b)}}{2} \int dx\, \rho(x)^2 \left[\left(y_a - y_b + \pi \Delta^-_{(a,b)} \right)^2 - \frac{1}{3}\pi^2 \Delta^+_{(a,b)} \left(4 - \Delta^+_{(a,b)} \right) \right]$$

$$N^{3/2} \int dx\, \rho(x) x \left(\frac{1-\Delta_a}{2} - \frac{1}{4\pi} y_a(x) \right)$$

- Algebraic in ϱ!

[Herzog Klebanov Pufu Tesileanu; Santamaria Marino Putrov]

An example



$$\mathcal{W}_{fl} = p_1 A_1 q_1 + p_2 A_2 q_2$$

- Describes M2 branes on a CY cone. 1-loop quantum corrections are crucial to finding the moduli space. At level 0, gives $AdS_4 \times Q^{111}$.

$$F = \frac{4\sqrt{2}\pi N^{3/2}}{3} \frac{\hat{\Delta}(\hat{\Delta}+k+1)}{\sqrt{(k+1)^2(k-1)-4(k+1)\hat{\Delta}-2\hat{\Delta}^2}}$$

- To leading order in N, independent of the fundamental flavor R-charge.

Volume minimization

- In Sasaki-Einstein geometry, the Reeb vector is paired with the radial direction in the Kahler form on the CY cone.
- For toric SE, it is part of the $U(1)^4$ isometry.
- The volume can be computed as a function of this embedding (in general, a Sasakian manifold with Kahler cone). It is minimized by the SE one.
- The whole function matches the field theory Z!

Summary

- Explained 3d N=2 R-symmetric theories on the sphere.
- Computed the IR partition function exactly in the UV theory as a function of R-charge parameterized curvature couplings.
- $|Z|$ is minimized by the IR superconformal R.
- Looked at examples with M-theory AdS_4 duals.