

# Characterization of Rough Surfaces Using Schramm-Loewner Evolution

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# The problem:

A growing surface in 2+1 dimensions is governed by an equation of the form:

$$h_t = Dh + \zeta,$$

Where D is a differential operator,  $\zeta$  is noise.  
Can you classify the resulting surface;  
for short and/or long times ?

# The problem ...continued

Example-1; the Edwards-Wilkinson equation

$$h_t = \Delta h + \zeta,$$

Example-2; Tungsten Oxide  
deposited on glass:



# The problem ... continued

1. Fluctuations of a surface may diverge ..

$$\langle \Delta h(x) \Delta h(0) \rangle \approx x^{2\alpha}, \alpha > 0$$

for Edwards-Wilkinson in d=2,  $\alpha=0$        $\langle \Delta h^2 \rangle \approx \log(x)$

Mullins-Herring diffusion d=2,  $\alpha=1$

2. The divergence however is limited by the finite size of the system L :

$$\langle \Delta h(x) \Delta h(0) \rangle \leq L^{2\alpha}, \alpha > 0$$

# The problem . . . continued

Can define critical exponents for the surface  
(following the example of critical phenomena):

$$w = \sqrt{\langle \Delta h^2 \rangle} \quad \text{surface roughness}$$

$L$  = system size

$$w = L^\alpha f\left(\frac{t}{L^z}\right)$$

$$\begin{cases} w \approx t^\beta & t \rightarrow 0 \\ w \approx L^\alpha & t \rightarrow \infty \end{cases}$$

$$z = \frac{\alpha}{\beta}, \text{dynamical index}$$

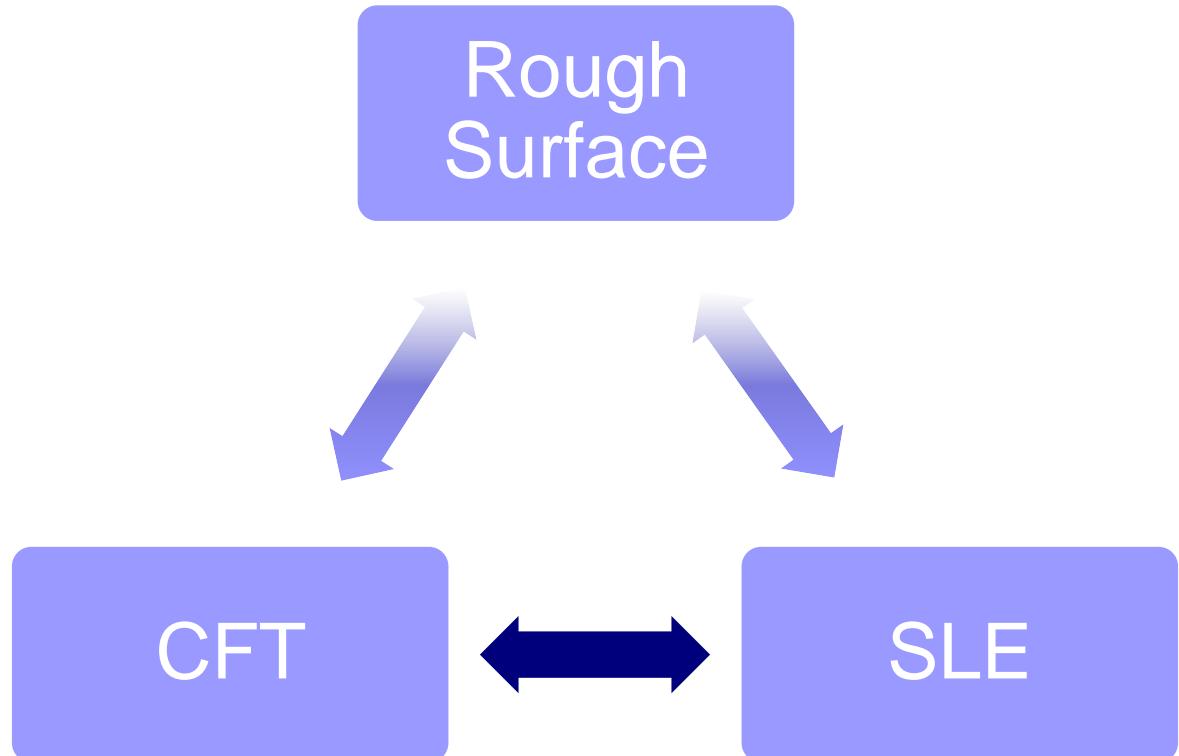
# The problem . . . continued

A vital question remains, how many independent exponents are there ?

*There are no general scaling arguments as in critical phenomena*

# Possible solution:

- The stationary distribution of the rough surface is scale invariant, so it may be conformal invariant, hence may be classified by CFT.
- Iso height lines on the rough surface, are Schramm Loewner Evolutions, so they are related to CFT's



# Overview of this talk:

- ✓ Theory of Schramm-Loewner Evolution ( $SLE_k$ ) has been developed originally as random simple curves with conformally invariant probability distribution, describing domain interfaces at criticality, for two dimensional models.
- ✓ We argue that Iso-height lines on rough surfaces (deposited or computer generated) can be regarded as  $SLE_k$
- ✓ This may be regarded as evidence of conformal invariance in systems far from equilibrium.
- ✓ This connection may be used for classification of growing surfaces in 2+1 dimensions

# Overview continued...

- ✓ We consider some examples:
  - SLE Curves in Turbulence,  $\kappa=6$ , computer simulated
  - KPZ surface,  $\kappa=8/3$ , computer simulated
    - considered together with (Ballistic Deposition model (BD), Eden model, Restricted Solid on Solid (RSOS) model, same class as the Self Avoiding Walk
    - Same universality class as BD, Eden and RSOS
  - Edwards Wilkinson surface ,  $\kappa=4$  , O(2) model, computer simulated
  - Deposited WO<sub>3</sub> surface,  $\kappa=3$ , Ising model, physically grown
  - Abelian Sandpile model and SLE( $\kappa,\rho$ ) ,  $\kappa=2$ ,  $\rho=4$

# Short Review of $SLE_k$

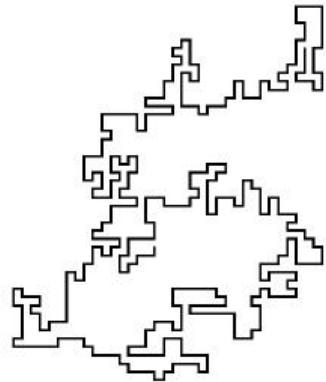
## Schramm-Loewner Evolution

- Consider the **phase boundary** in **critical** statistical models such as the Ising model, the probability distribution of such boundaries is governed by a stochastic process called the **Schramm-Loewner Evolution, or  $SLE_k$**
- The diffusivity constant  $K$ , is related to the **universality class** of the critical system.

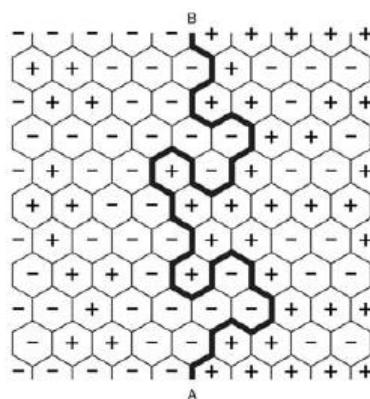
# Critical domain walls and curves

Enforce domain wall by choosing appropriate boundary conditions  
By construction domain walls are non crossing, an alternative non crossing path is SAW

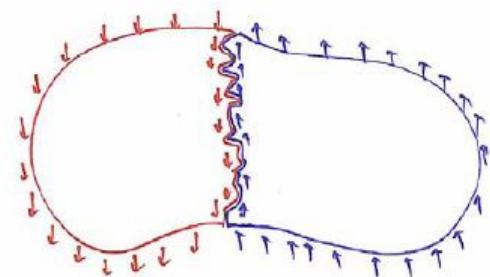
SAW random path



Percolation cluster boundary at critical concentration

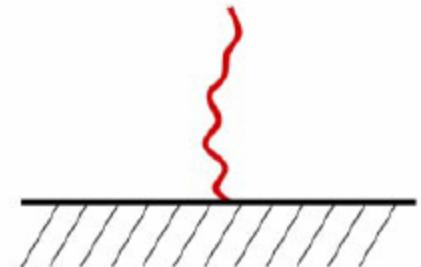


Ising domain wall at critical temperature



There is a Conformal Mapping  $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$   
 Normalized such that

$$g_t(z) \sim z + \frac{2t}{z} + \dots \quad \text{as} \quad z \rightarrow \infty$$



Evolution of  $g_t(z)$  satisfies

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

with  $a_t$  a real continuous function which  
 characterizes  $K_t$  The driving function

*K. Löwner, I, Math. Ann. 89, 103, 1923*

The probability dist. of the path is conformally invariant  
 then  $a_t$  must be a 1D Brownian motion,  $B_t$ .

$$a_t = \sqrt{\kappa} B_t \quad O. Schramm, Isr. J. Math. 118, 221 (2000).$$



stochastic (schramm)-loewner evolution ( $SLE_\kappa$ )

# Conformal mapping

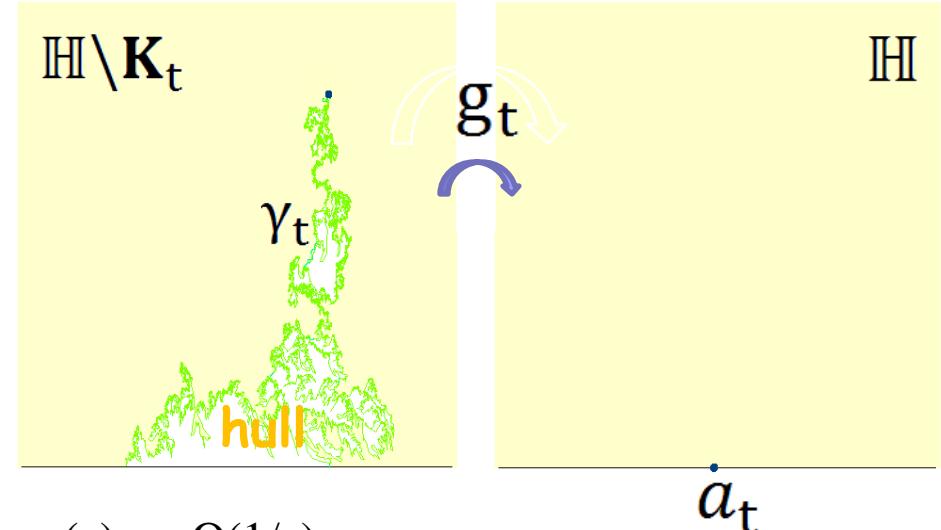
Conformal mapping is a powerful tool for characterizing shapes in 2D by means of analytic functions.

Consider a curve  $\gamma_t \in \mathbb{H}$  starting from the origin (t parameterizes the curve)

The complement of the **hull**  $K$  (the set of points which cannot be reached from infinity together with  $\gamma$ ) is simply connected, thus  $\exists$  analytic function

$$g_t: \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$$

$g_t(z)$  maps the hull  $K$  to the real axis (and the growing tip  $\tau$  to a point  $a_t \in \mathcal{R}$ )

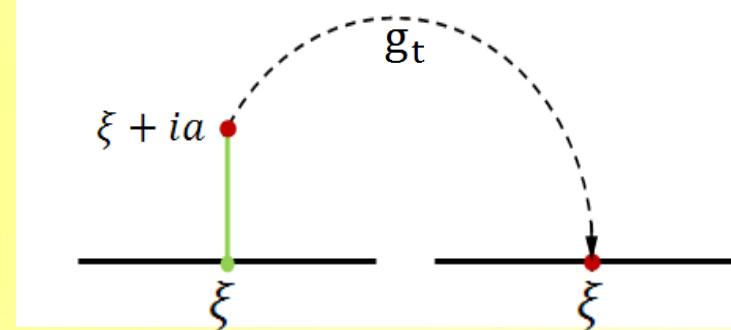


This map is unique if we fix normalization, e.g.  $g(z) \sim z + O(1/z)$  as  $z \rightarrow \infty$

## Example:

Introducing the "time"  $t = a^2/4$ , for a vertical segment starting from  $\xi \in \mathcal{R}$ :  
note  $g_t(z) \sim z + 2t/z$

$$g_t(z) = \xi + \sqrt{(z - \xi)^2 + 4t}$$



# The postulates of SLE

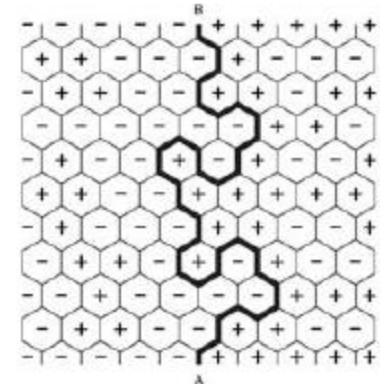
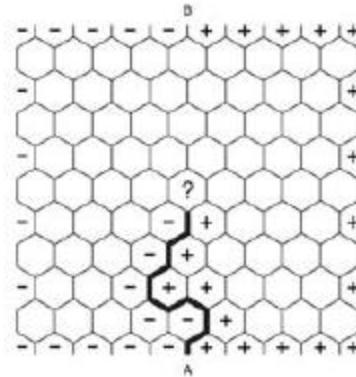
**Property 1 (Markov)** : Denote the curve by  $\gamma$ , and divide it into two disjoint parts:  $\gamma_1$  from  $r_1$  to  $\tau$ , and  $\gamma_2$  from  $\tau$  to  $r_2$ . Then the conditional measure  $\mu(\gamma_2|\gamma_1; D, r_1, r_2)$  is the same as  $\mu(\gamma_2; D \setminus \gamma_1, \tau, r_2)$ .

**Property 2 (Conformal Invariance):** Let  $\Phi$  be a conformal mapping of the interior of the domain  $D$  onto the interior of  $D'$ , so that the points  $(r_1, r_2)$  on the boundary of  $D$  are mapped to points  $(r'_1, r'_2)$  on the boundary of  $D'$ . The measure  $\mu$  on curves in  $D$  induces a measure  $*\mu$  on the image curves in  $D'$ . The conformal invariance property states that this is the same as the measure which would be obtained as the one obtained for curves from  $r'_1$  to  $r'_2$  in  $D'$ .

$$(\Phi * P)(\gamma; D, r_1, r_2) = P(\Phi(\gamma); D', r'_1, r'_2)$$

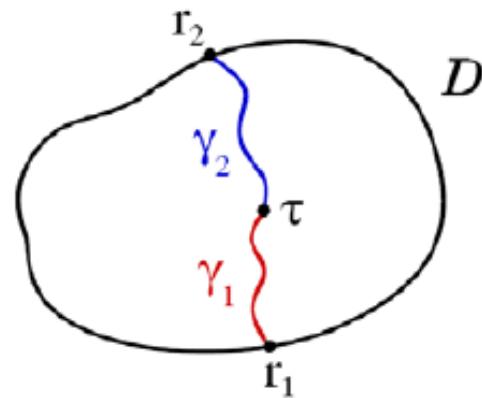
## ☀ (Domain) Markov property

Grow domain wall step by step like a random walk (Markov process)



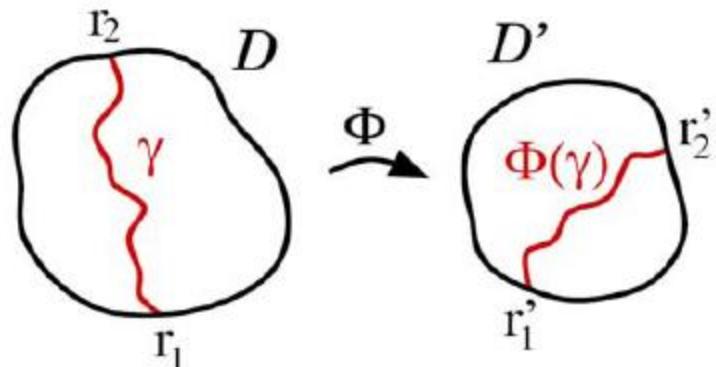
Implement Markov property in the continuum limit

$$P(\gamma_2 | \gamma_1; D, r_1, r_2) = P(\gamma_2; D \setminus \gamma_1, \tau, r_2)$$

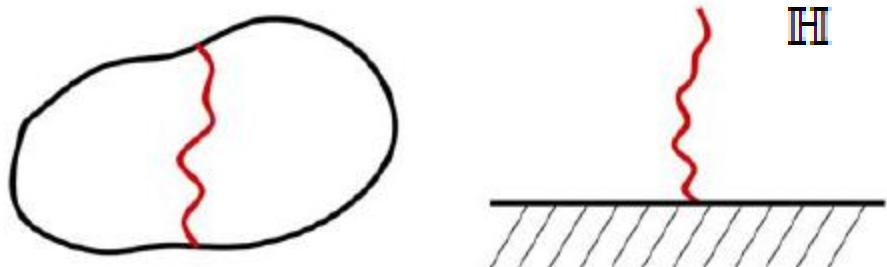


## ☀ Conformal invariance

Implement conformal invariance to ensure scaling



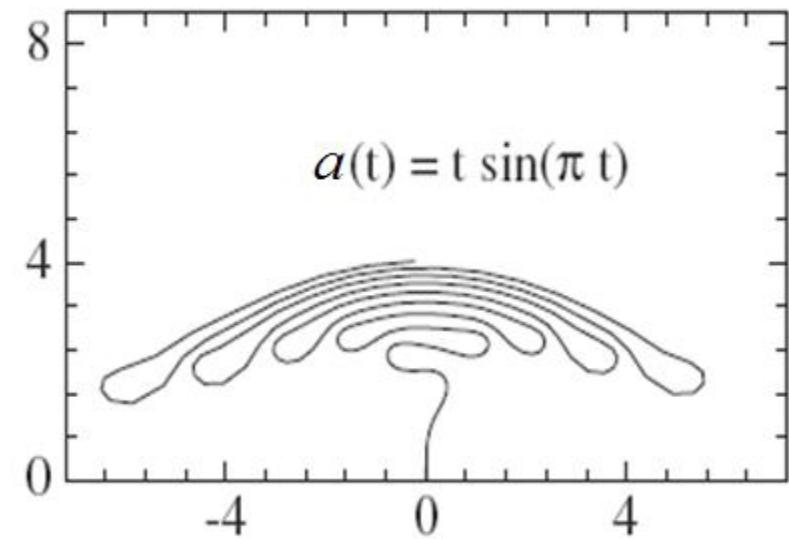
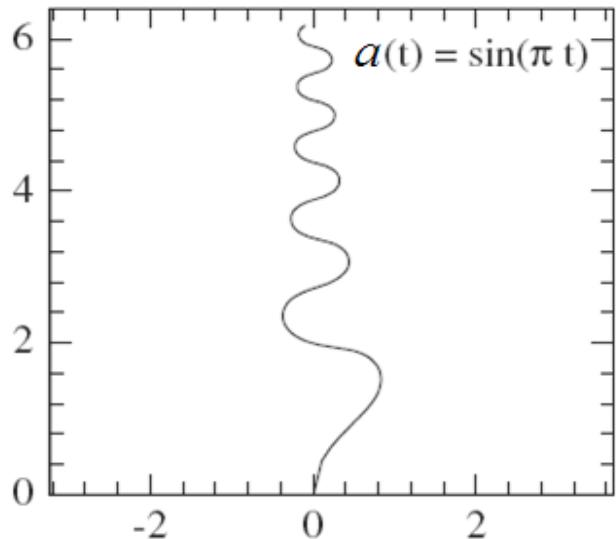
Conformal transformation to upper half plane as reference plane



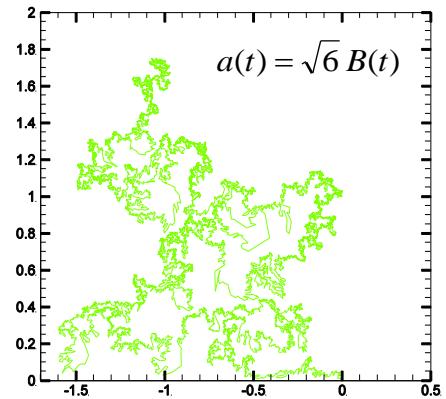
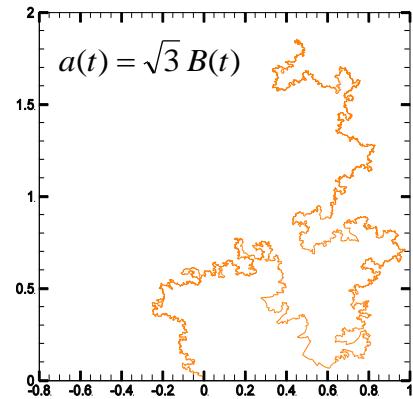
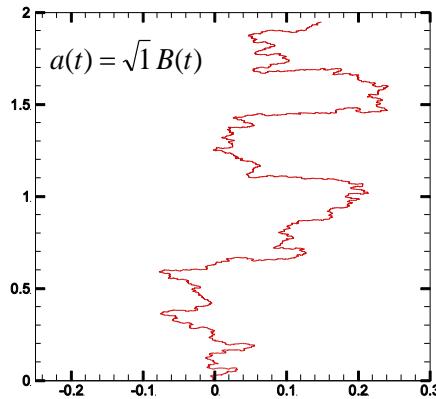
Transformation of the probability density:

$$(\Phi * P)(\gamma; D, r_1, r_2) = P(\Phi(\gamma); D', r_1', r_2')$$

## Examples of different driving functions



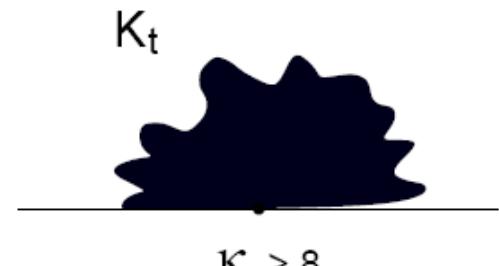
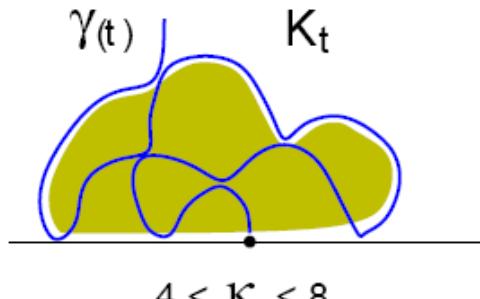
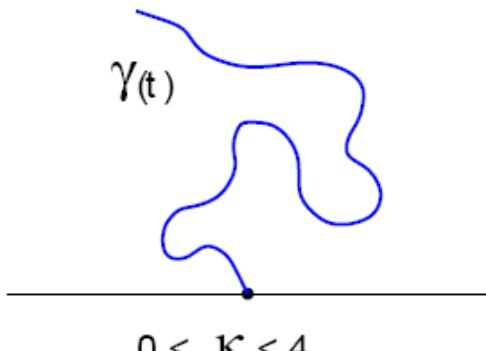
*I. A. Gruzberg and L. P. Kadanoff, J. Stat. Phys., Vol. 114, Nos. 5/6, (2004).*



diffusion coefficient  $\kappa$  parameterizes different universality classes  
of critical behavior (problems in 2d critical systems are mapped to problems in  
1d Brownian motion) (J. Cardy, SLE for theoretical physicists, Ann.Phys. 318, 81 (2005))

## The Phases of SLE :

Rohde & Schramm (2001)



The Fractal Dimension of the trace :  $D_0 = 1 + \kappa/8$  for  $2 \leq \kappa \leq 8$

Beffara (2002)

# Relation to conformal field theory

The long time probability distribution function satisfies a differential equation which is nothing but a null vector in a minimal CFT

Null vector level 2:

$$(L_{-2} - \frac{\kappa}{4} L_{-1}^2) \psi = 0$$

# SLE/CFT correspondence

Duality:

$$c_\kappa = \frac{(8 - 3\kappa)(\kappa - 6)}{2\kappa} = 1 - 3 \frac{(\kappa - 4)^2}{2\kappa}$$

$$\kappa' = \frac{16}{\kappa}$$

$$c_\kappa = c_{\kappa'}$$

The operators 'creating' n SLE traces at a boundary point are the boundary operators  $\psi_{1;n+1}$  with dimension

$$h_{1,n+1} = n(4 + 2n - \kappa)/2\kappa$$

The operators 'creating' n SLE curves at a bulk point are the bulk operators  $\phi_{0;n/2}$  with dimension

$$2h_{0,n/2} = [4n^2 - (\kappa - 4)^2]/16\kappa$$

The fractal dimension

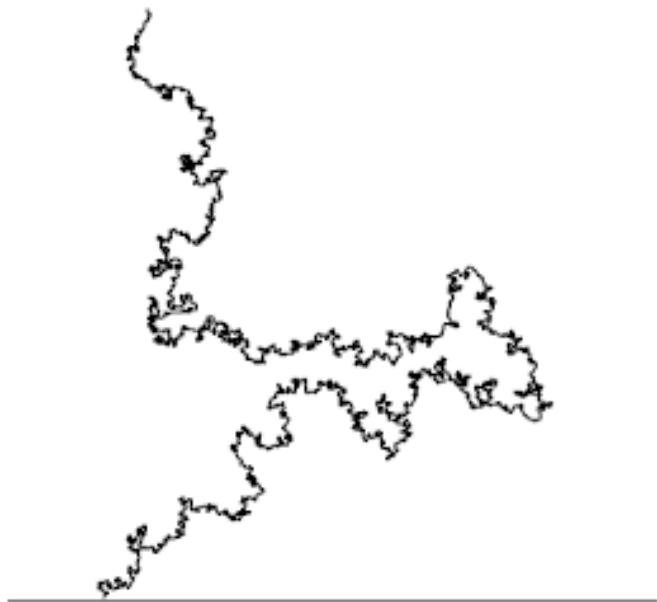
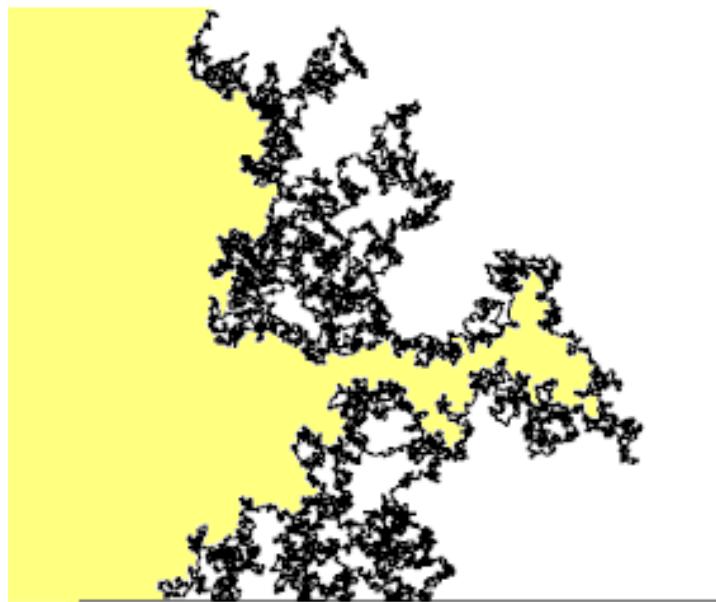
$$d_\kappa(n) = 2 - 2h_{0,n/2} = \frac{[(\kappa + 4)^2 - 4n^2]}{8\kappa} \quad \longrightarrow \quad d_\kappa(2) = 1 + \kappa/8.$$

Duality: hull boundary of  $SLE_\kappa$  is  $SLE_{\kappa'}$

$$(D_{EP} - 1)(D_H - 1) = \frac{1}{4} \quad , \kappa \geq 4$$

$$\kappa \kappa' = 16$$

B. Duplantier, Phys. Rev. Lett. 84, 1363 (2000).



# Critical exponents from SLE

$\kappa$	2	$8/3$	3	$10/3$	4	$24/5$	$16/3$	6	8
$C$	-2	0	$1/2$	$4/5$	1	$4/5$	$1/2$	0	-2
$d\kappa$	$5/4$	$4/3$	$11/8$	$17/12$	$3/2$	$8/5$	$5/3$	$7/4$	2
Model	LERW ASM	KPZ SAW	WO3 Ising spin cluster	3-Potts	EW $O(2)$	Dual 3-Potts	Ising FK cluster	Perco lation, Turbu lance	Span. Trees

# Conformal Invariance and SLE in Systems Far from Equilibrium

- Plenty of evidence for SLE in critical statistical physics models exists, but
- does SLE have any application outside of this domain, e.g. non equilibrium systems which have **scaling** behavior

# SLE Curves in Turbulence

- Using numerical experiment, it was observed that some features of  $d=2$  inverse turbulent cascade displays conformal invariance.
- It was observed that the statistics of vorticity clusters is remarkably close to that of critical percolation, one of the simplest universality classes of critical phenomena  $\kappa=6$  ,  $c=0$  .

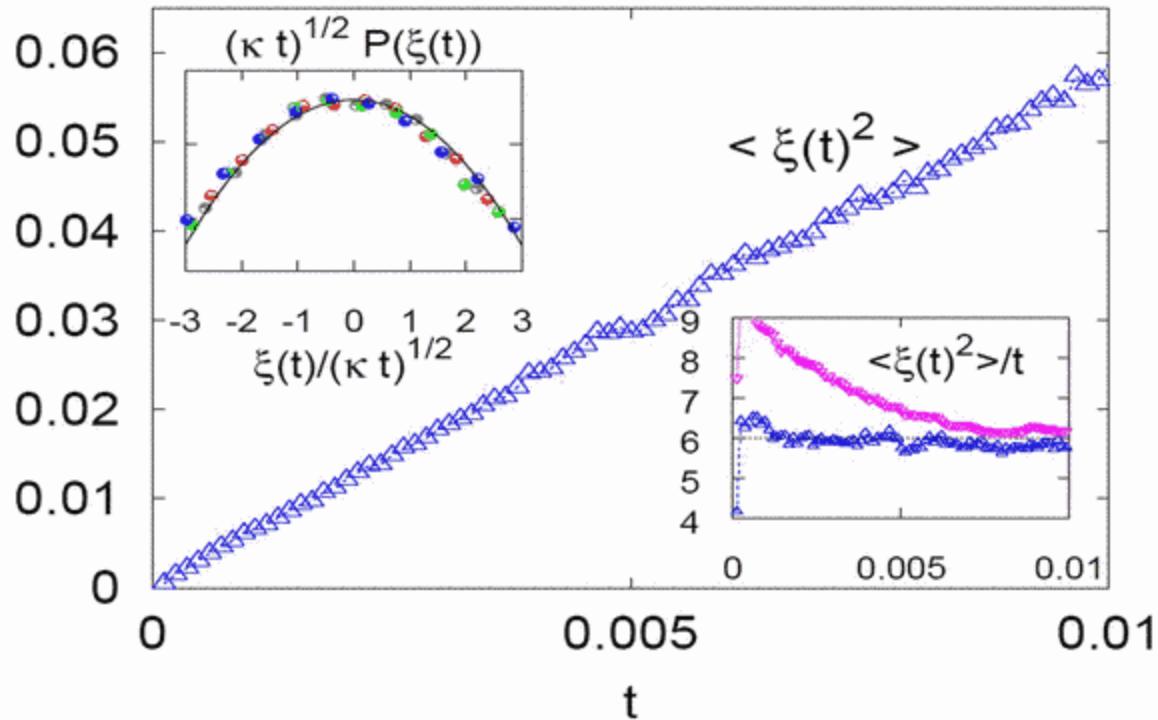


FIG. 4: The driving function is an effective diffusion process with diffusion coefficient  $\kappa = 6 \pm 0.3$ . The inverse cascade range corresponds to  $5 \cdot 10^{-5} < t < 10^{-2}$ . *Main frame*: the linear behaviour of  $\langle \xi(t)^2 \rangle$ . *Lower-right inset*: Diffusivity: blue for vorticity isolines, pink for the field with randomized phases. *Upper-left inset*: the probability density function of the rescaled driving function  $\xi(t)/\sqrt{\kappa t}$  at four different times  $t = 0.0012, 0.003, 0.006, 0.009$ ; the solid line is the Gaussian distribution  $g(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ .

Bernard et al, Nature Physics 2, p.124 (2006)

# Kardar-Parisi-Zhang Surface

- The statistics of the iso-height lines in (2+1)-dimensional Kardar-Parisi-Zhang (KPZ) model is observed to be conformally invariant and equivalent to those of self-avoiding random walks (SAW).
- Numerical evidence suggests that the iso-height lines on KPZ surface can be described by SLE<sub>8/3</sub>. This shines new light on the universality class for the KPZ dynamics in 2+1 dimensions.

# Kardar-Parisi-Zhang Equation (KPZ)

$$\frac{\partial h(\mathbf{x}, t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} |\nabla h|^2 + \eta(\mathbf{x}, t)$$

## Symmetries:

- ☼ Invariance under translation in time  $t \rightarrow t + \delta_t$
- ☼ Translation invariance along the growth direction  $h \rightarrow h + \delta_h$
- ☼ Translation invariance in the direction perpendicular to the growth direction  $\mathbf{x} \rightarrow \mathbf{x} + \delta_{\mathbf{x}}$
- ☼ Rotation and inversion symmetry about the growth direction e.g.  $\mathbf{x} \rightarrow -\mathbf{x}$

## The absent Symmetry:

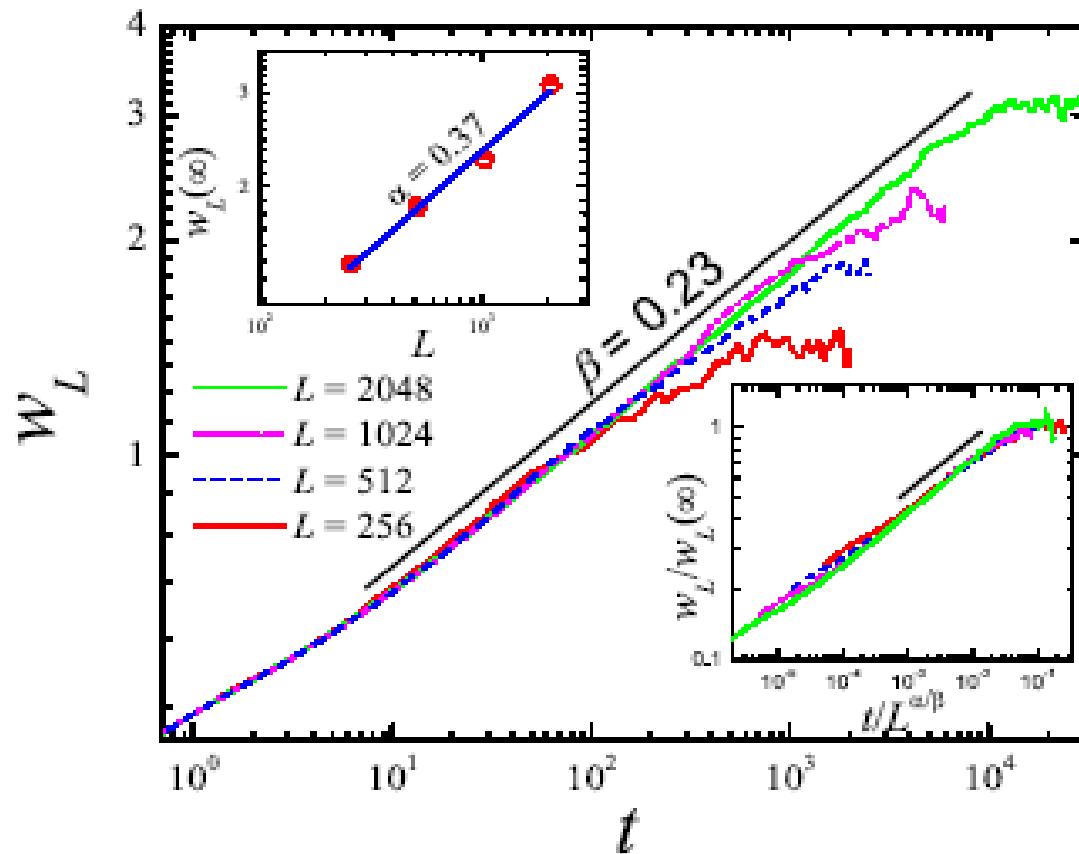
- ☼ Up/down symmetry  $h \rightarrow -h$

**Exact exponents for  $d=1$ :**  $\alpha=0.37$ ,  $\beta=1/3$

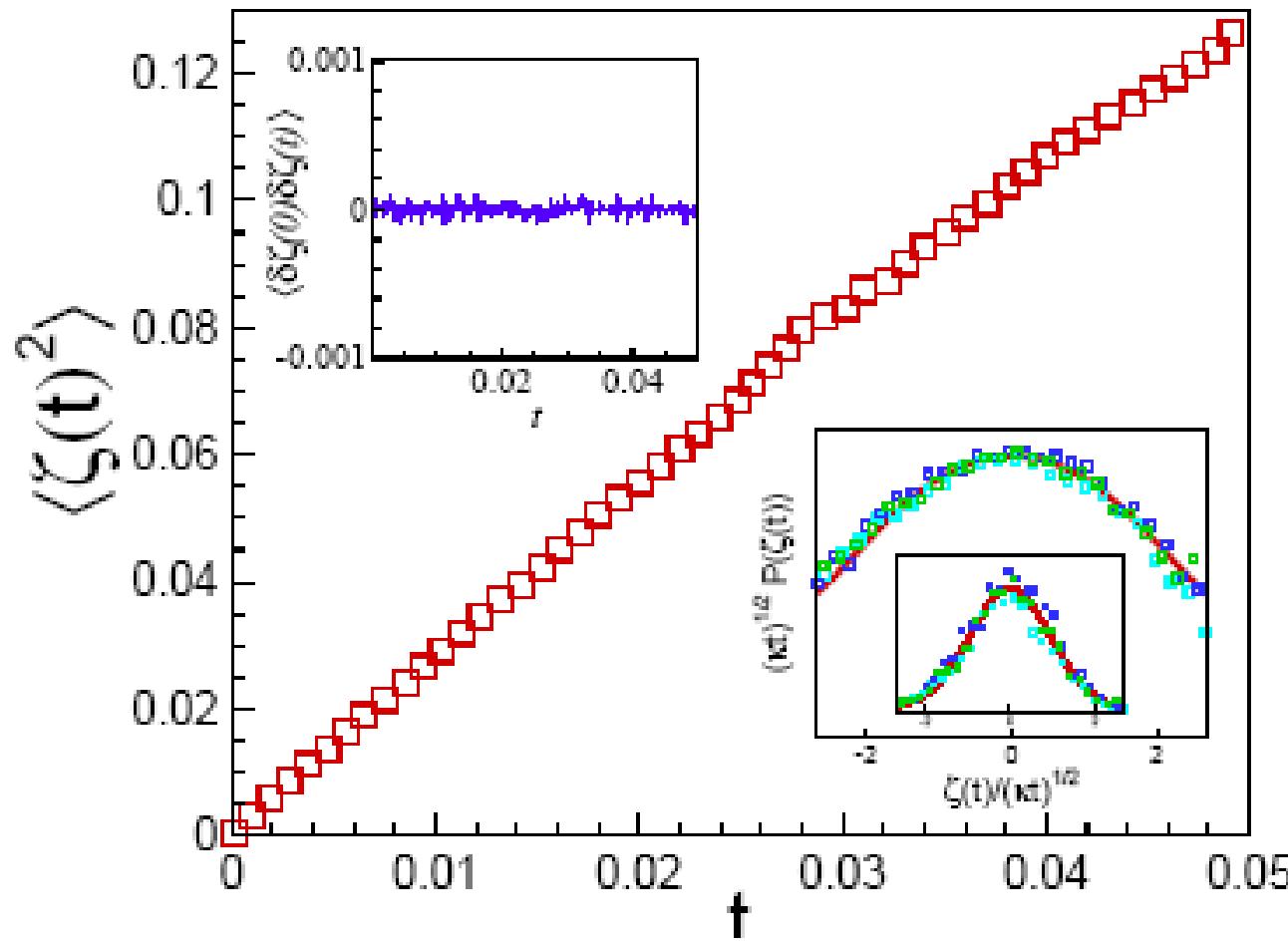
**Approximate exponents for  $d=2$ :**  $\alpha=1/2$ ,  $\beta=0.23$

# Scaling properties of KPZ d=2

## numerical experiment



# Diffusitivity constant $k=8/3$ (0.1)



# SLE on BD, Eden and RSOS

There is also evidence that the iso-height lines in the ballistic deposition (BD), Eden and restricted solid-on-solid (RSOS) models have conformally invariant properties all in the same universality class as the self-avoiding random walk (SAW), equivalently SLE8/3. This leads to the conclusion that all these discrete growth models fall into the same universality class as KPZ in 2d.

A.A. Saberi, H. Dashti-Naserabadi, and S. Rouhani  
Phys. Rev. E 82, 020101(R) (2010)

$w = \sqrt{\langle \Delta h^2 \rangle}$  is surface roughness

$L$  = system size

$$w = L^\alpha f\left(\frac{t}{L^z}\right)$$

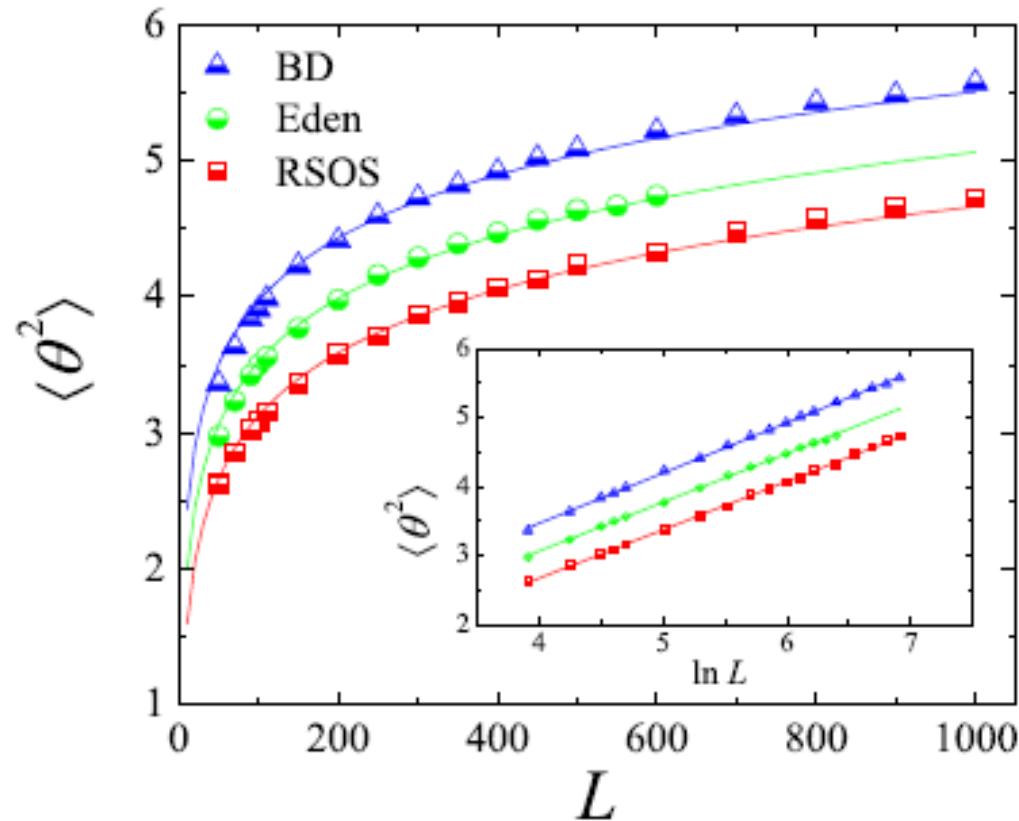
$$\begin{cases} w \approx t^\beta & t \rightarrow 0 \\ w \approx L^\alpha & t \rightarrow \infty \end{cases}$$

Model: Exponent	BD	Eden	RSOS	KPZ
$\alpha$	0.28	0.36	0.39	0.37
$z$	1.70	1.65	1.58	1.61
$\beta$	0.15	0.20	0.24	0.23

$$\beta = \frac{\alpha}{z}$$

$$\alpha + z = 2$$

# Winding angle distribution



$$\langle \theta^2 \rangle = a + \frac{\kappa}{4} \ln(L)$$

# Edwards-Wilkinson Surface

- In the absence of the nonlinear term in the KPZ equation, the diffusivity  $\kappa$  changes from  $8/3$  to  $4$ , indicating that the iso-height lines of the Edwards-Wilkinson (EW) surface are also conformally invariant, and belong to the universality class of the domain walls in the  $O(2)$  spin model.

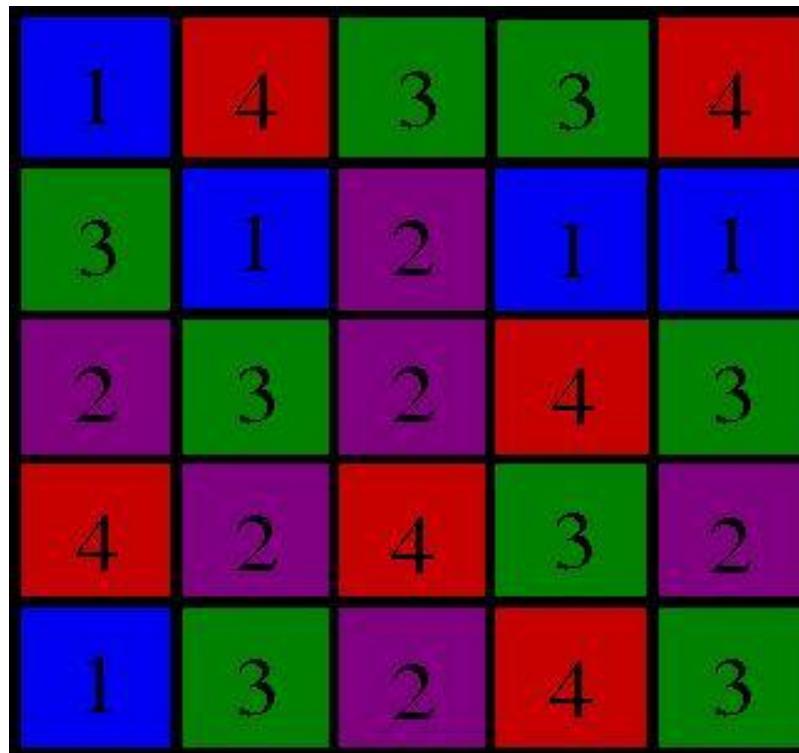
$$\kappa = 3.7 \pm 0.2$$

*Proof :* O. Schramm, S. Sheffield, J. Math (2006)

# Observation of SLE( $\kappa, \rho$ ) on the Abelian Sandpile Model

There is evidence that the avalanche frontier in ASM is SLE<sub>2</sub>  
A. A. Saberi, S. Moghimi-Araghi, H. Dashti-Naserabadi, and S. Rouhani

**Phys. Rev. E 79, 031121 (2009)**



# SLE( $\kappa, \rho$ )

- A more successful method of numerical estimates may be derived from (hydrodynamically normalized) SLE( $\kappa, \kappa - 6$ ) which is the measure of the continuum limit of critical curves going from the boundary to itself. Its driving function is:

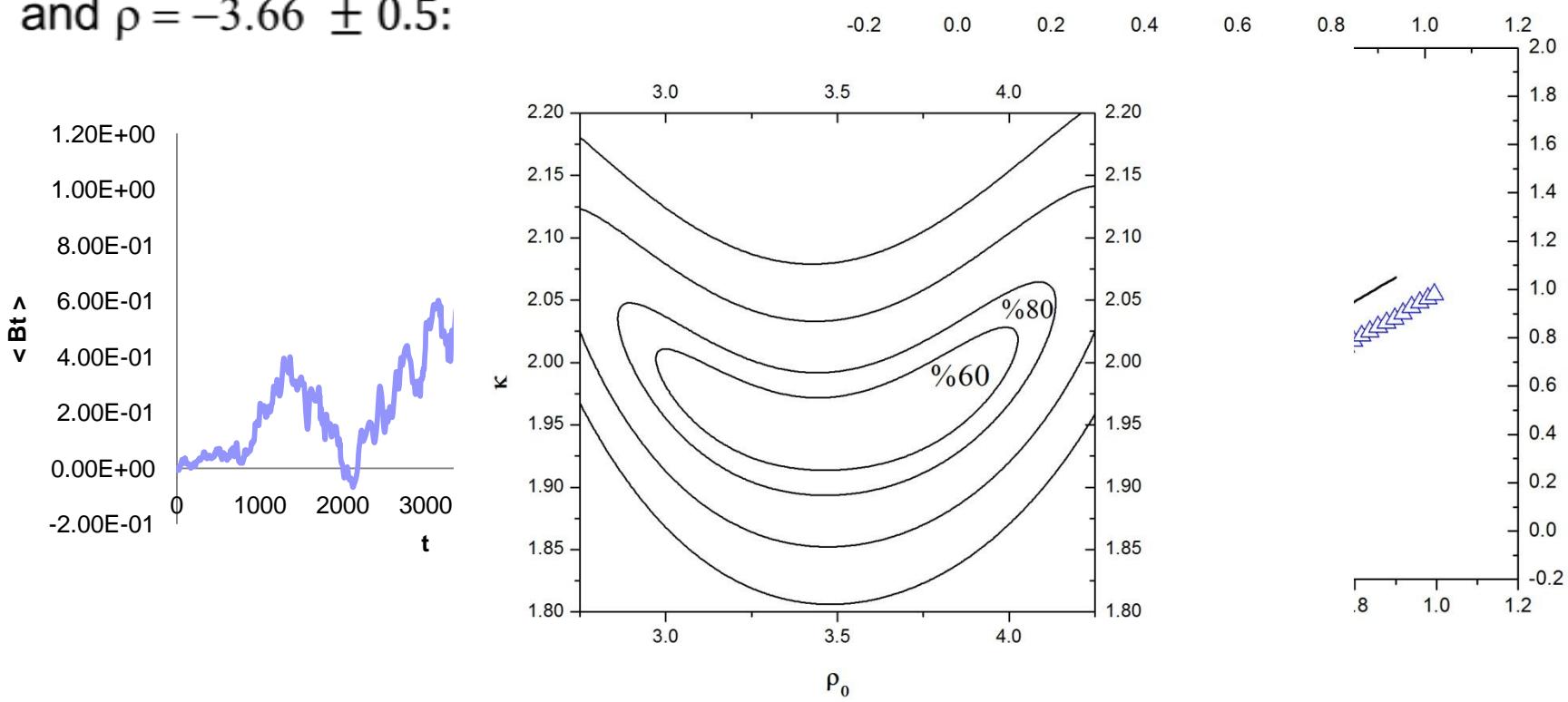
$$\begin{cases} d\xi_t = \sqrt{\kappa} dB_t + \frac{\rho_c}{\xi_t - \eta_t} dt \\ d\eta_t = \frac{2}{\eta_t - \xi_t} dt \end{cases} \quad \rho_c = \kappa - 6$$

In the discrete set up we have :  $\delta\xi_n = \sqrt{\kappa}\delta B_n + \frac{\kappa - 6}{\xi_n - \eta_n}\delta t_n$

→ 
$$\frac{\xi_n - \sum_{i=1}^n \left[ \frac{(\kappa-6)\delta t_i}{\xi_i - \eta_i} \right]}{\sqrt{\kappa}} = B_n$$

# Observation of SLE( $\kappa, \rho$ ) on Abelian Sandpile Model

$\kappa$  and  $\rho$  should be fixed in such a way that the resulting function  $B_t$  be a Brownian motion. Suppose  $\rho$  as a free parameter, so for  $\kappa = 1.95 \pm 0.07$  and  $\rho = -3.66 \pm 0.5$ :

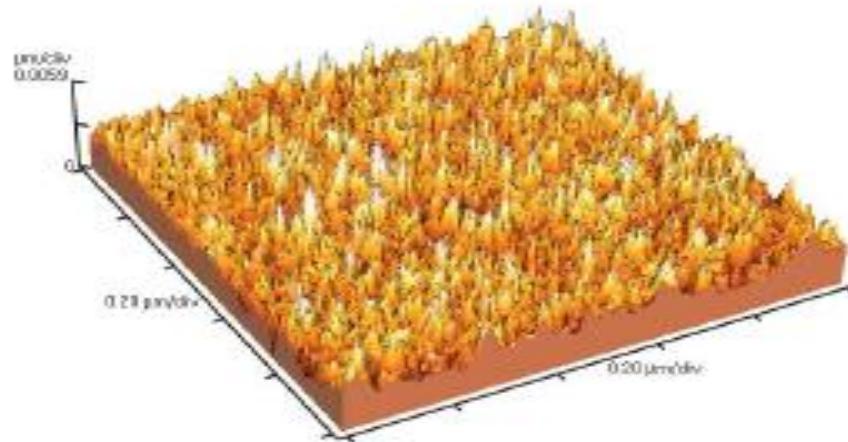


# Tungsten Oxide Surface ( $\text{WO}_3$ )

- The random surfaces generated by the deposition of the  $\text{WO}_3$  on glass substrates can be subjected to the same analysis after a digitization by AFM.
- These iso-height lines are also conformally invariant with the same statistics of domain walls in the critical Ising model and belong to the family of Schramm-Loewner evolution with  $\kappa = 3$ .
- This is the only physical observation of SLE curves.

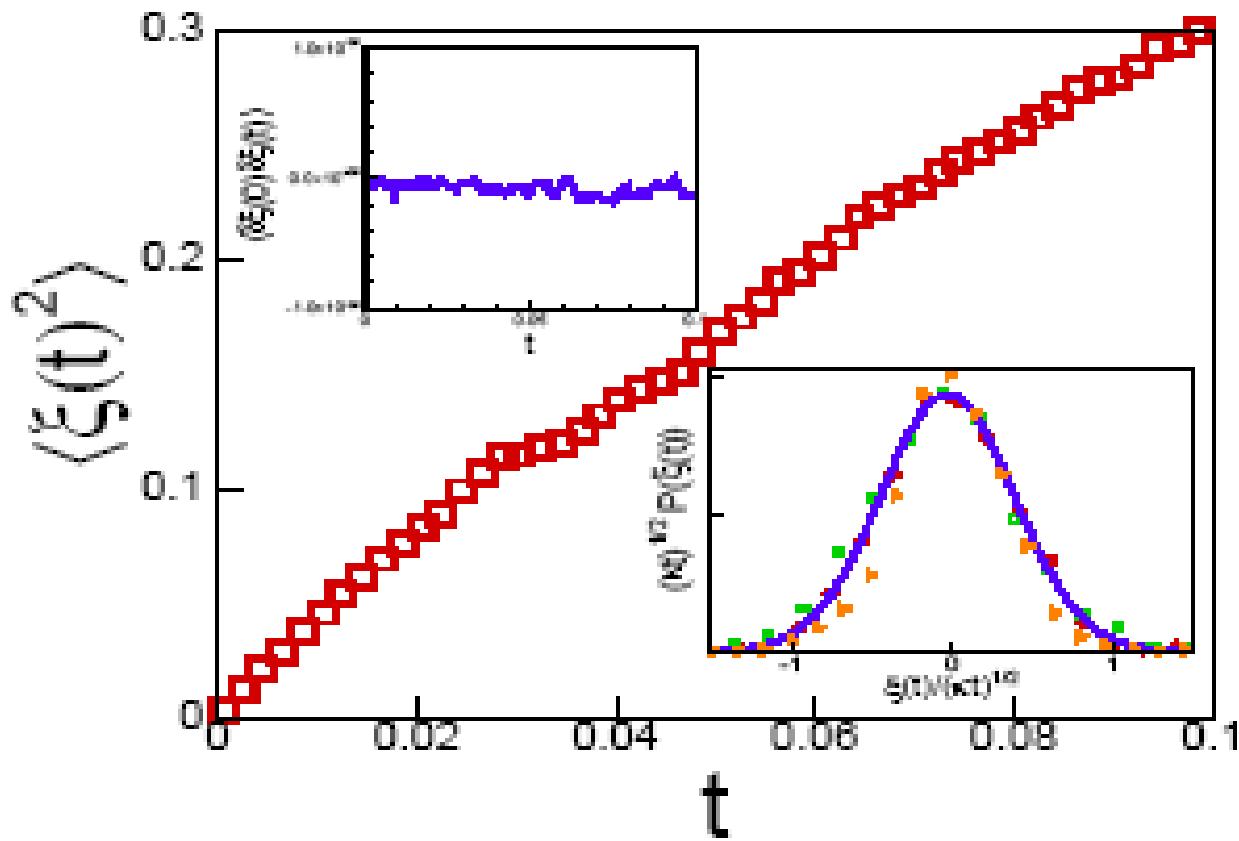
# Tungsten Oxide Surface

- In the same universality class of critical Ising model



2-A. A. Saberi, M. A. Rajabpour, and S. Rouhani  
Phys. Rev. Lett. 100, 044504 (2008)

# Diffusitivity Constant $k = 3.0$ (0.2).



- WO<sub>3</sub> is the Only physical observation of SLE

# Speculation....

- ✓ Don't cut the loops take them as whole:

**Get a measure for random curves in 2d**

- ✓ The only successful example is the coulomb gas :

Oded Schramm, Scott Sheffield, arXiv:math/0605337v3

- ✓ **Self Avoiding Polygons in 2d lattice**

“Polygons, Polyominoes and Polycubes” Anthony J. Guttman, Springer, 2009

# String-gauge duality (speculative)

- AdS/CFT correspondence holds that dual to a CFT, there exists a gravity theory in higher dimension.
- What does this imply for the rough surface/SLE/CFT connection ?

*Best chance: Coulomb Gas :: Edwards-Wilkinson..*

THANK YOU !

Publications:

A. A. Saberi, M. A. Rajabpour, and S. Rouhani

Phys. Rev. Lett. 100, 044504 (2008)

Conformal Curves on the WO<sub>3</sub> Surface

A. A. Saberi, Niry MD, Fazeli SM, Rahimi Tabar MR, and S. Rouhani.

Physical Review E, vol. 77, 051607 (2008)

Conformal invariance of isoheight lines in a two-dimensional Kardar-Parisi-Zhang surface.

A. A. Saberi, S. Moghimi-Araghi, H. Dashti-Naserabadi, and S. Rouhani

Phys. Rev. E 79, 031121 (2009)

Direct evidence for conformal invariance of avalanche frontiers in sandpile models

A. A. Saberi and S. Rouhani

Phys. Rev. E 79, 036102 (2009)

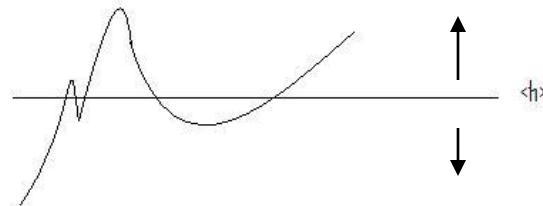
Scaling of clusters and winding-angle statistics of isoheight lines in two-dimensional Kardar-Parisi-Zhang surfaces

A. A. Saberi, H. Dashti-Naserabadi, and S. Rouhani

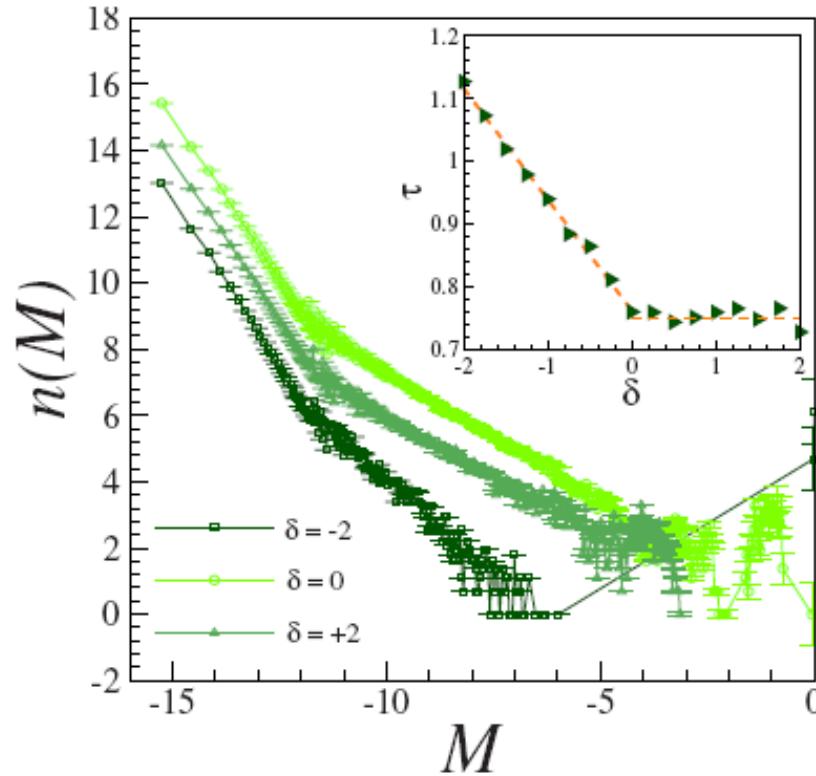
Phys. Rev. E 82, 020101(R) (2010)

# Where to cut ?

- The natural level is mean height  $\langle h \rangle$  what happens as we offset the cut by  $\delta$  ?



$$n(M) \sim M^{-\tau}$$

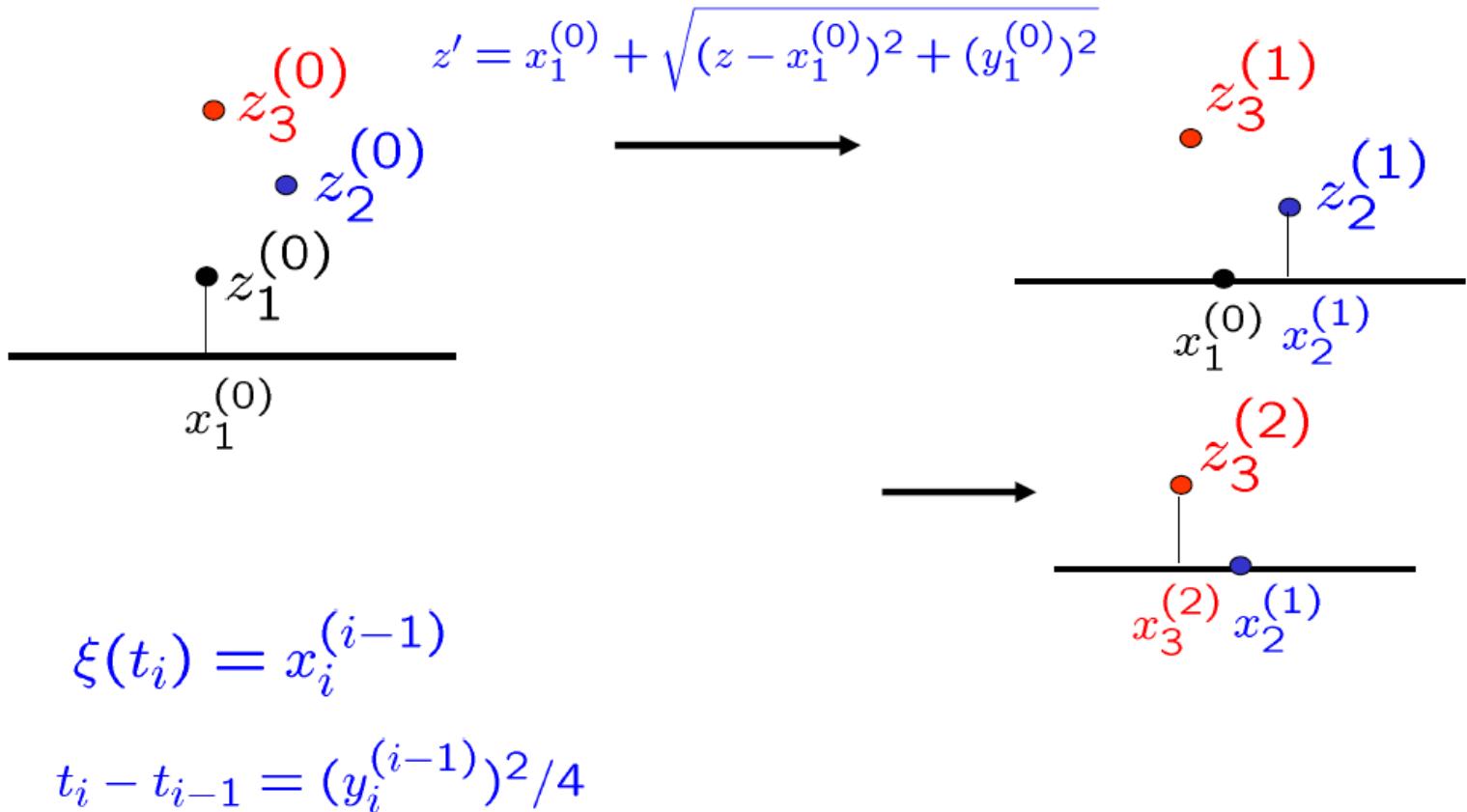


log-log plot of the number of clusters of area between  $M$  and  $1.05M$ , at three different cuts with  $\delta = -2$ , 0, and  $+2$ . Inset: the exponents for the island distributions as a function of  $\delta$ . The errors are less than 0.05 for all exponents. The slope of the dashed lines is  $-1.770.03$ , the best fitted for  $\delta < 0$ , and  $\delta = 0$ , drawn for comparison for  $\delta > 0$  at  $\tau = 0.75$ .

# Discretize a path and find the deriving function

Using the successive discrete,  
conformal slit maps - based on the  
piecewise constant approximation of  
SLE :  
swallow one segment of the curve at  
each time step

# iterated slit map



Introducing the “time”  $t=a^2/4$ ,  $g_t(z) \sim z + 2t/z$ ,  
for a vertical segment :

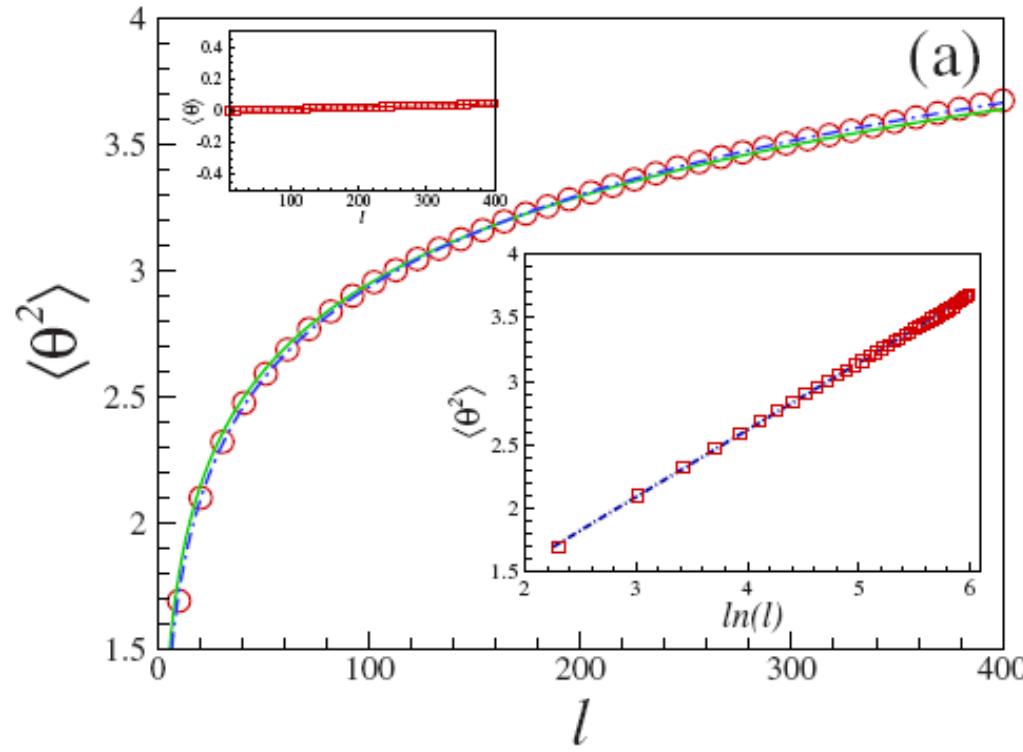
$$g_t(z) = \xi + \sqrt{(z - \xi)^2 + 4t}$$

# Winding angle

$$\langle \theta^2 \rangle \approx \frac{\kappa}{4} \ln(L)$$



L is the distance between P and Q



**Winding angle statistics** of isoheight lines of **2D-KPZ surface** simulated on square lattice of size  $2048^2$ . Main:logarithmic behavior of the variance of the winding angle as a function of distance  $L$ . Upper-left inset: the mean winding angle as a function of  $L$ . Lower-right inset: semi logarithmic behavior of the winding angle. Dotted-dashed lines show the best fit according to  $\kappa = 2.760$ . The solid line in the main frame shows the fit with  $\kappa = 8/3$  for comparison with SAW.