CLASSICAL INTEGRABILITY IN THE BTZ BLACK HOLE

based on 1105.0480
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MOTIVATION

• The spectrum of excitations in black hole backgrounds provide useful information about the black hole.

Geodesics: classical geometry of the black hole
Quasi-normal modes: information of the relaxation times
Absorption cross-section

•In some rare examples, one has the situation where the motion of strings in the black hole background can be solved and quantized

The black hole in 2d string theory: exact CFT
The BTZ black hole with WZW term

Maldacena & Ooguri: Euclidean case
J. Troost: Minkowski case

• We will be interested in studying string propagation in the Lorentzian BTZ black hole without the WZW term

• The string world sheet in the BTZ background is integrable might be expected.

3d gravity is a Chern-Simons theory and is topological. quasi-normal modes of scalars, spinor, vectors are exactly solvable.

• This is not the situation for strings in black holes of higher dimensional Anti-de Sitter space.

Basu, et. al, Zayas, et. al

• The BTZ black hole is locally AdS_3. Strings on AdS_3 is integrable.

Integrability and the existence of infinite set of charges relies on the construction of a flat connection, a local concept.

This goes through for the BTZ case.

The construction of the charges need to be done with some care due to the quotienting.

• Our analysis follows the methods of

Kazakov, Marshakov, Minahan, Zarembo sigma model on $R \times S^3$.

Kazakov, Zarembo sigma model on $AdS_3 \times S^1$

FLAT CONNECTION AND INTEGRABILITY

• Recall the construction of the flat connection for a sigma-model propagating on a group manifold.

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau \left(\mathrm{Tr}\frac{1}{2}(g^{-1}\partial_a g g^{-1}\partial^a g)\right)$$

We can construct

$$j_\pm = g^{-1}\partial_\pm g, \qquad \sigma_\pm = \frac{1}{2}(\tau \pm \sigma)$$

The equations of motion is given by

$$\partial_+ j_- + \partial_- j_+ = 0$$

From the definition of $j_\pm$ we can show

$$\partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0$$

- Now we can construct the flat connection

$$J_{\pm}(x) = \frac{j_{\pm}}{1 \mp x}$$

which satisfies

$$\partial_{+} J_{-} - \partial_{-} J_{+} + [J_{+}, J_{-}] = 0$$

• The construction of the charges is through the monodromy matrix

$$\Omega(x) = P \exp\left(-\int_0^{2\pi} d\sigma J_\sigma\right)$$

Using the flatness of J can show

$$\partial_\tau \Omega(x) = -(J_\tau(2\pi)\Omega(x) - \Omega(x)J_\tau(0))$$

Thus if

$$J_\tau(2\pi) = J_\tau(0)$$

we have

$$\mathrm{Tr}(\Omega(x))$$

is a world sheet time independent quantity.

This forms the generating function for the infinite set of non-local charges.

•Group manifold with identification

Consider the identification

$$g \sim \tilde{A} g A$$

Then the string in general has the following boundary conditions

$$g(\tau, \sigma + 2\pi) = \tilde{A}^k g(\tau, \sigma) A^k$$

Tracing this to the current J we obtain the condition

$$J(\tau, \sigma + 2\pi) = (A^{-1})^k J(\tau, \sigma) A^k$$

• Because of this boundary condition, we need to consider the modified monodromy matrix

$$\begin{aligned} \tilde{\Omega} &= A^k \Omega, \\ \partial_\tau \tilde{\Omega} &= -(A^k J_\tau(2\pi) \Omega(x) - A^k \Omega(x) J_\tau(0)) \end{aligned}$$

Thus

$$\mathrm{Tr}\,\tilde{\Omega}(x)$$

is world sheet time independent and generates the infinite set of non-local charges.

• The BTZ black hole is a quotient of AdS_3 which is a $SL(2, R)$ group manifold.

Therefore we can construct infinite set of non-local charges for the world sheet theory.

• Recall: BTZ is a quotient of the AdS_3 hyperboloid. (Discuss the non-extremal case.)

The AdS_3 hyperboloid is a $SL(2,R)$

$$-u^2 - v^2 + x^2 + y^2 = -1$$
$$g = \begin{pmatrix} u+x & y+v \\ y-v & u-x \end{pmatrix}$$

The BTZ black hole is obtained by the identification

$$\begin{aligned} g &\sim \tilde{A}gA \\ \tilde{A} &= \begin{pmatrix} e^{(r_+ - r_-)\pi k} & 0 \\ 0 & e^{-(r_+ - r_-)\pi k} \end{pmatrix} \\ A &= \begin{pmatrix} e^{(r_- + r_+)\pi k} & 0 \\ 0 & e^{-(r_- + r_+)\pi k} \end{pmatrix}. \end{aligned}$$

Note that the identifications are the exponential of the σ^3 matrix.

- Recall: the parametrization for region $r > r_+$

$$u = \sqrt{A(r)}\cosh\tilde{\phi}(t,\phi), \qquad x = \sqrt{A(r)}\sinh\tilde{\phi}(t,\phi),$$
$$y = \sqrt{B(r)}\cosh\tilde{t}(t,\phi), \qquad v = \sqrt{B(r)}\sin\tilde{t}(t,\phi)$$

Where

$$A(r) = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}, \qquad B(r) = \frac{r^2 - r_+^2}{r_+^2 - r_-^2},$$
$$\tilde{t} = r_+ t - r_-\phi, \qquad \tilde{\phi} = -r_- t + r_+\phi.$$

Note

$$A(r) - B(r) = 1.$$

The $SL(2,R)$ generators are

$$t^1 = \sigma^1, \quad t^2 = i\sigma^2, \quad t^3 = \sigma^3$$

USING INTEGRABILITY TO CHARACTERIZE CLASSICAL SOLUTIONS

- We will consider the sigma model BTZ$\times S^1$

$$S = -\frac{\lambda}{2}\int d^2\sigma\left(\frac{1}{2}\mathrm{Tr}(g^{-1}\partial_a g g^{-1}\partial^a g^{-1}) + \partial_a Z\partial^a Z\right)$$

Z parametrizes the S^1.

We fix the gauge

$$Z = \frac{\hat{J}}{2\pi}\tau + \hat{m}\sigma$$

• The following global charges play an important role:

Translations along BTZ time $\rightarrow$ Energy E.

Translations along ϕ of the BTZ $\rightarrow$ Spin S

A convenient combination is

$$E + S = \frac{\lambda}{2}(r_+ - r_-)\int_0^{2\pi} d\sigma \mathrm{Tr}(\partial_\tau g g^{-1}\sigma^3),$$
$$E - S = -\frac{\lambda}{2}(r_+ + r_-)\int_0^{2\pi} d\sigma \mathrm{Tr}(g^{-1}\partial_\tau g \sigma^3).$$

Note that these charges are conserved for all the winding sectors k.

• The Virasoro constraints

$$\mathrm{Tr}(j_\pm^2) = 2\left(\frac{\hat{J}}{2\pi\lambda} \pm \hat{m}\right)$$

- The monodromy matrix $\hat{\Omega}(x)$ plays an important role.

Since it belongs to $SL(2,R)$ we can write its eigen values in the form

$$\{\exp(ip(x)), \exp(-ip(x))\}$$

$p(x)$ is called the quasi-momentum.

Thus

$$\mathrm{Tr}\,\hat{\Omega}(x) = 2\cos p(x)$$

• The strategy to determine the form of $p(x)$ is to study its analytical properties in the complex x plane

• We then use complex analysis to determine equations that constrain $p(x)$ given the charges $E, S, \hat{J}, \hat{m}$.

- We list the properties of the quasi-momemtum $p(x)$.

- $p(x)$ has simple poles at $x = \pm 1$

$$p(x) \sim \pi \frac{\frac{\hat{J}}{2\pi\lambda} \pm \hat{m}}{x \mp 1}$$

- $x \to \infty$

for $k = 0$

$$p(x) \sim \frac{i}{2x}\sqrt{Q_R^2}$$

with

$$\begin{aligned} Q_R^2 &= (Q_R^1)^2 - (Q_R^2)^2 + (Q_R^3)^2 \\ Q_R^i &= \int_0^{2\pi} d\sigma \,\mathrm{Tr}\left(g^{-1}\partial_\tau g t^i\right) \end{aligned}$$

for $k \neq 0$

$$p(x) \sim i\pi k(r_+ + r_-) - i\frac{1}{x}\frac{E-S}{\lambda(r_+ + r_-)}$$

- $x \to 0$

for $k = 0$

$$p(x) \to 2\pi m + i\frac{x}{2}\sqrt{Q_L^2}$$

where

$$\begin{aligned} Q_L^2 &= (Q_L^1)^2 - (Q_L^2)^2 + (Q_L^3)^2 \\ Q_L^i &= \int_0^{2\pi} d\sigma \mathrm{Tr}(t^i \partial_\tau g g^{-1}) \end{aligned}$$

for $k \neq 0$

$$p(x) \sim 2\pi m + i\pi k(r_+ - r_-) + ix\frac{E+S}{\lambda(r_+ - r_-)}$$

• Across the branch cuts we have the condition

$$p(x+i\epsilon)+p(x-i\epsilon)=2\pi n$$

where n is an integer.

This condition arises due to the uni-modular property of the monodromy matrix $\Omega(x)$.

• Note that the analytic behaviour is determined by the global charges.

These conditions are sufficient to constrain a K branch cut solutions such that the only undetermined parameters in $p(x)$ are $K-1$ numbers called filling fractions.

• It is useful to rewrite these constraints in terms of of a spectral density $\rho(x)$.

Introduce the resolvent

$$
\begin{aligned}
G(x) &= p(x) - \pi \frac{\frac{\hat{J}}{2\pi\lambda} + \hat{m}}{x-1} - \pi \frac{\frac{\hat{J}}{2\pi\lambda} - \hat{m}}{x+1}, \quad k=0 \\
G(x) &= p(x) - \pi \frac{\frac{\hat{J}}{2\pi\lambda} + \hat{m}}{x-1} - \pi \frac{\frac{\hat{J}}{2\pi\lambda} - \hat{m}}{x+1} - i\pi k(r_+ + r_-), \quad k \neq 0
\end{aligned}
$$

• Since $G(x)$ is analytic except for possible branch cuts using complex analysis it can be written as

$$G(x) = \int d\xi \frac{\rho(x)}{x-\xi}$$

where the integral is along the cuts.

In fact ρ is given by

$$\rho(x) = \frac{1}{2\pi i}\left(G(x+i\epsilon) - G(x-i\epsilon)\right)$$

• For the case $k = 0$,
the behaviour of the quasi-momentum at ∞ leads to

$$\int d\xi \rho(\xi) = -\frac{\hat{J}}{\lambda} + \frac{i}{2}\sqrt{Q_R^2}.$$

The behaviour at the origin leads to

$$\begin{aligned} -\int d\xi \frac{\rho(\xi)}{\xi} &= 2\pi(m + \hat{m}), \\ -\int d\xi \frac{\rho(\xi)}{\xi^2} &= \frac{J}{\lambda} + \frac{i}{2}\sqrt{Q_L^2}. \end{aligned}$$

The jump condition at the branch cuts leads to

$$2\int_{\xi \neq x} d\xi \frac{\rho(\xi)}{x - \xi} = -\frac{2\pi(\frac{\hat{J}}{2\pi\lambda} + \hat{m})}{x - 1} - \frac{2\pi(\frac{\hat{J}}{2\pi\lambda} - \hat{m})}{x + 1} + 2\pi n_l$$

• For the case $k \neq 0$,
the behaviour of the quasi-momentum at ∞ leads to

$$\int d\xi \rho(\xi) = -\frac{\hat{J}}{\lambda} - i\frac{E-S}{\lambda(r_+ + r_-)}$$

The behaviour at the origin leads to

$$-\int d\xi \frac{\rho(x)}{\xi} = 2\pi(\hat{m} + m - ikr_-),$$

$$-\int d\xi \frac{\rho(\xi)}{\xi^2} = \frac{\hat{J}}{\lambda} + i\frac{E+S}{\lambda(r_+ - r_-)}$$

The jump condition at the branch cuts leads to

$$2\int_{\xi\neq x} d\xi\,\frac{\rho(\xi)}{x-\xi}$$

$$= -\frac{2\pi(\frac{\hat{J}}{2\pi\lambda}+\hat{m})}{x-1} - \frac{2\pi(\frac{\hat{J}}{2\pi\lambda}-\hat{m})}{x+1} + 2\pi n_l - 2\pi ik(r_+ + r_-)$$

CLASSICAL SOLUTIONS

- To get a better feel for the construction.

To verify the properties of the quasi-momentum obtained using general considerations

We study explicit solutions.

- It is convenient to write the sigma model as

$$\begin{aligned} S &= -\frac{\lambda}{2}\int d\tau d\sigma \left[-\partial_a u \partial^a u - \partial_a v \partial^a v + \partial_a x \partial^a x + \partial_a y \partial^a y \right. \\ & \left. -\Lambda(-u^2 - v^2 + x^2 + y^2 + 1) + \partial_a Z \partial^a Z \right], \end{aligned}$$

where Λ is the Lagrange multiplier.

- The boundary conditions are

$$
\begin{aligned}
r(\tau, \sigma + 2\pi) &= r(\tau, \sigma), \\
t(\tau, \sigma + 2\pi) &= t(\tau, \sigma), \\
\phi(\tau, \sigma + 2\pi) &= \phi(\tau, \sigma) + 2\pi k
\end{aligned}
$$

- We will discuss two class of solutions:

Geodesics

Winding strings

- For geodesics we start with the ansatz

$$u+x=a(\tau)\exp(f(\tau)),\qquad u-x=a(\tau)\exp(-f(\tau)),$$
$$y+v=b(\tau)\exp(g(\tau)),\qquad y-v=b(\tau)\exp(-g(\tau)).$$

with

$$a(\tau)=\cosh\gamma(\tau),\qquad b(\tau)=\sinh\gamma(\tau).$$

Note the ansatz is independent of the the world sheet σ.

- There are two constants of motion

$$\dot{f}\cosh^2\gamma=c_1,\qquad \dot{g}\sinh^2\gamma=c_2.$$

They are related to E and S by

$$E-S=-(r_+ + r_-)2\pi\lambda(c_1+c_2),$$
$$E+S=(r_+ - r_-)2\pi\lambda(c_1-c_2).$$

- The Virasoro constraints imply either $\hat{J} = 0$ or $\hat{m} = 0$.

With $\hat{m} = 0$ they reduce to

$$(\dot{\gamma})^2 + \frac{c_1^2}{\cosh^2\gamma} - \frac{c_2^2}{\sinh^2\gamma} + (\frac{\hat{J}}{2\pi\lambda})^2 = 0.$$

- In terms of the BTZ coordinates and the conserved charges the equation is

$$r^2\dot{r}^2 = -\hat{J}^2(r^4 - Mr^2 + \frac{j^2}{4}) + (E^2 - S^2)r^2 + S^2M + ESj$$

This is the geodesic equation with mass $\hat{J}$

• The monodromy matrix and the quasi-momentum can be solved explicitly.

The monodromy matrix is

$$\Omega = \exp\left[-\frac{2\pi x}{1-x^2} g^{-1}\dot{g}\right],$$

with the quasi-momentum given by

$$p = \frac{\hat{J}}{\lambda}\frac{x}{x^2-1}$$

• The quasi-momentum thus has poles at $x = \pm 1$ without branch cuts.

The spectral density $\rho(x)$ vanishes for geodesics.

- For winding strings we start with the anstaz

$$u + x = a(\tau) e^{(f(\tau)+\nu_1\sigma)}, \qquad u - x = a(\tau) e^{-(f(\tau)+\nu_1\sigma)},$$
$$y + v = b(\tau) e^{(g(\tau)+\nu_2\sigma)}, \qquad y - v = b(\tau) e^{-(g(\tau)+\nu_2\sigma)}.$$

where again

$$a(\tau) = \cosh \gamma(\tau), \qquad b(\tau) = \sinh \gamma(\tau)$$

From the anstaz the BTZ coordinates are given by

$$t = \frac{r_+(g(\tau)+\nu_2\sigma) + r_-(f(\tau)+\nu_1\sigma)}{r_+^2 - r_-^2},$$
$$\phi = \frac{r_-(g(\tau)+\nu_2\sigma) + r_+(f(\tau)+\nu_1\sigma)}{r_+^2 - r_-^2}.$$

The periodicity conditions

$$t(\tau, \sigma + 2\pi) = t(\tau, \sigma), \qquad \phi(\tau, \sigma + 2\pi) = \phi + 2\pi k$$

results in

$$\nu_1 = r_+ k, \qquad \nu_2 = -r_- k.$$

We have the constants of motion

$$\dot{f} \cosh^2 \gamma = c_1, \qquad \dot{g} \sinh^2 \gamma = c_2.$$

• The Virasoro constraints are

$$(\dot\gamma)^2 + \frac{c_1^2}{\cosh^2\gamma} - \frac{c_2^2}{\sinh^2\gamma} + \nu_1^2 \cosh^2\gamma - \nu_2^2 \sinh^2\gamma + (\hat J^2 + \hat m^2) = 0,$$

$$c_1\nu_1 + c_2\nu_2 + \hat J \hat m = 0.$$

• The general solution can be found in terms of Jacobi elliptic functions.

The mondromy matrix and the quasi-momentum can be evaluated explicity for this solution.

- The quasi-momentum is given by

$$\cos p(x) = \cosh \pi \sqrt{\tilde{D}}$$

with

$$\tilde{D} = \left\{ (\nu_2 - \nu_1)^2 - \frac{4}{(1-x^2)^2} \left(\frac{\hat{J}^2}{4\pi^2\lambda^2} + m^2 + 2x \frac{\hat{J}\hat{m}}{2\pi\lambda} \right) + \frac{4}{1-x^2} \left(\frac{c_2^2}{\sinh^2 \gamma_0} - \frac{c_1^2}{\cosh^2 \gamma_0} \right) + 4(\nu_2 - \nu_1)\tilde{A} \right\}^{1/2}$$

- From this explicit formula for $p(x)$

one can determine the behaviour at $x \to \pm 1, \infty, 0$ and verify the behaviour obtained from general considerations.

- For this solution the spectral density $\rho(x)$ has 2-branch cuts.

- We have evaluated the first non-local charge for this solution.

- The behaviour of the spectral density enables a classification of the solutions.

RELATION WITH THE $SL(2,R)$ SPIN CHAIN

• We start with the following Bethe equations for a twisted version of the $SL(2,R)$ spin chain.

$$\left(\frac{x(u_k+\frac{i}{2})}{x(u_k-\frac{i}{2})}\right)^L \exp(2i\pi k(\tilde{r}_+ + \tilde{r}_-)) = \prod_{j=1, j\neq k}^{M} \frac{u_k-u_j-i}{u_k-u_j+i},$$

with

$$\begin{aligned} x(u) &= \frac{u}{2}+\frac{u}{2}\sqrt{1-\frac{2g^2}{u^2}}, \\ \frac{x(u_k+i/2)}{x(u_k-i/2)} &= \exp ip_k \end{aligned}$$

The cyclicity constraint is given by

$$\prod_{k=1}^{M}\left(\frac{x(u_k+\frac{i}{2})}{x(u_k-\frac{i}{2})}\right)=\exp(2\pi i k\tilde{r}_-).$$

The energy of the chain is given by

$$D=2g^2\sum_{k=1}^{M}\left(\frac{i}{x(u_k+\frac{i}{2})}-\frac{i}{x(u_k-\frac{i}{2})}\right).$$

- Let us now take the thermodynamic limit :

$L \to \infty$, scale $u_k \to L u_k$.

Introduce a density function for the magnons

$$\int du \tilde{\rho}(u) = \frac{M}{L}.$$

We redefine

$$u = x + \frac{g'^2}{x}, \qquad \text{where} \quad g' = \frac{g}{L}.$$

- Then the
  - The normalization of the density
  - The Bethe equations
  - The cyclicity constraint
  - The equation for the energy

are given by

$$
\begin{aligned}
\int dx\left(1-\frac{g'^2}{x^2}\right)\tilde{\rho}(u(x)) &= \frac{M}{L}. \\
2\int_{y\neq x} dy\,\frac{\tilde{\rho}(u(x))}{(x-y)} &= -\frac{1}{x}-2\pi k(\tilde{r}_+ +\tilde{r}_-)+2\pi n \\
\int \frac{\tilde{\rho}(x)}{x} &= -2\pi(\hat{m}-k\tilde{r}_+), \\
\frac{D}{L} &= 2g'^2\int dx\,\frac{\tilde{\rho}(x)}{x^2}.
\end{aligned}
$$

• At the leading order in $\frac{g'}{L}$, these equations are same as the four equations which constrain the spectral density $\rho(x)$ of the BTZ sigma model upon the identification.

$$g' = \frac{g}{L} = \frac{\lambda}{2\hat{J}}, \qquad \tilde{\rho}(u(x)) = \rho(x),$$

$$ir_+ \to \tilde{r}_+, \qquad ir_- \to \tilde{r}_-, \qquad \hat{m} + m \to \tilde{m}$$

$$\frac{1}{4\hat{J}}\left(\frac{E-S}{(\tilde{r}_+ + \tilde{r}_-)} - \frac{E+S}{(\tilde{r}_+ - \tilde{r}_-)}\right) = \frac{M}{L},$$

$$\frac{D}{L} = -1 + \frac{E+S}{\hat{J}(\tilde{r}_+ - \tilde{r}_-)}.$$

BMN AND MAGNON LIKE STATES

• We can use the equations constraining the equations for the spectral density to obtain dispersion relations of states analogous to the ones studied in the case of the sigma model on $R \times S^3$

- For a delta function distribution of the spectral density

$$\rho(x) = \sum_s S_s \delta(x - x_s)$$

One can solve the integral equations and obtain a dispersion relation similar to the plane wave.

$$-i\frac{\hat{J}}{\lambda} + \frac{E+S}{\lambda(r_+ - r_-)} = \sum_s \frac{\tilde{S}_s \hat{J}}{2\pi\lambda} \left(\sqrt{1 + \frac{4\pi^2\lambda^2}{\hat{J}^2}(n_s + ik(r_+ - r_-))^2} - 1 \right)$$

with

$$\sum_s (n_s + ik(r_+ - r_-))\tilde{S}_s = 2\pi k r_-$$

$$\hat{J} \sum_s \tilde{S}_s = \frac{E-S}{r_+ + r_-} + \frac{E+S}{r_+ - r_-}$$

• For a distribution of spectral density which is constant between two points in the complex plane we obtain the dispersion relation

$$Q_+ - 2\hat{J}i = \sqrt{Q_-^2 - 16\lambda^2 \sin^2 \frac{p}{2}},$$

with

$$Q_+ = \frac{E+S}{r_+ - r_-} + \frac{E-S}{r_+ + r_-}, \qquad Q_- = \frac{E+S}{r_+ - r_-} - \frac{E-S}{r_+ + r_-}.$$

This resembles the magnon dispersion relation but with complex momentum.

CONCLUSIONS

• We have seen that the classical sigma sigma model on the BTZ black hole background is integrable.

• We have seen that the integrability can be used to study and organize the classical solutions of the sigma model.

• It will be interesting if one can use integrability to find the complete allowed spectrum of strings around the BTZ black hole.

A step in this direction is to investigate more classical solutions.

• Finding this spectrum will have implications for the dual conformal field theory corresponding to the BTZ background.