Symmetries in M-theory

— Review of non-linear realizations and Kac-Moody algebras.

— Demonstration that D = 11 supergravity is a non-linear realization and the E_{11} conjecture.

— Very extended algebras

$$\mathfrak{g} \to \mathfrak{g}^{+} \to \mathfrak{g}^{++} \to \mathfrak{g}^{+++}.$$

(affine) (over) (very)

— Further evidence for $\mathfrak{g}^{+++}$ algebras.

Scalars in Supergravity Theories

The two scalars of $N = 4, D = 4$ supergravity belong to the coset $\frac{SU(1,1)}{U(1)}$ Cremmer, Scherk, Ferrara.

The non-linear realization of G with subgroup H; given $g(x^\mu) \in G$ (Callan, Coleman, Wess, Zumino) the theory is invariant under

$$g(x^\mu) \longrightarrow \underset{\substack{\in G\\ \text{rigid}}}{g_0}\, g(x^\mu) \underset{\substack{\in H\\ \text{local}}}{h(x^\mu)}$$

The Cartan forms transform as

$$\mathcal{V} \equiv g^{-1} dg \longrightarrow h^{-1} \mathcal{V} h + h^{-1} dh.$$

Let the generators of G be T^a and $h_I \in H$; then we can use the local H invariance to choose

$$g(x^\mu) = \exp \phi^a(x^\mu) T_a$$

If G involves P^μ then

$$g = e^{x^\mu P_\mu} e^{\phi^a(x^\mu) T_a}.$$

Previous Symmetries in Supergravity Theories

— $D=11$ supergravity reduced on a $n=1,..,8$ torus possess scalars which belong to a non-linear realization of E_n

($D=4$, E_7; $D=3$, E_8 and possibly $D=2$, E_9; $D=1$, E_{10})

(Cremmer Julia
de Wit Nicolai
Julia)

— IIB supergravity has $SL(2,R)$ symmetry
(Schwarz West)

— $D=11$ supergravity has an $SO(16)$ symmetry at the expense of $SO(1,10) \to SO(1,2)$
(de Wit Nicolai)

and has traces of exceptional structure (Melosch, Nicolai, Koepsell, Samtleben.)

— In $D=11$ supergravity and its reductions the gauge fields can be incorporated into the non-linear realization
(Cremmer, Julia, Lu, Pope)

— $D=11$ supergravity (gravity) near a cosmological singularity is a one dimensional mechanics confined to the Weyl chamber of E_{10} ($A_{E_{10}}$)
(Damour ...)

Symmetries in String Theory

— Evidence of E_{10} structure in threshold corrections of the heterotic string. (Harvey, Moore)

— The closed bosonic string reduced on the unique 26-dimensional Lorentzian torus has the Borcherds Fake Monster algebra as a symmetry (Moore, West)

— The action of the U-duality transformations on the moduli space of the IIA string reduced on an n-torus is the Weyl group of E_n, $n = 1, \ldots, 9, 10$?

(Elitzur, Giveon, Kutasov, Rabinovici
Obers, Pioline
Banks, Fischler, Motl)

Kac-Moody Algebras

Given a generalized Cartan matrix A_{ab} such that

(i) $A_{aa} = 2$

(ii) A_{ab} for $a \neq b$ are negative integers or zero

(iii) $A_{ab} = 0 \iff A_{ba} = 0$

and a set of Chevalley generators E_a, F_a, H_a which obey the Serre relations

$$[H_a, H_b] = 0 \; ; \; [H_a, E_b] = A_{ab} E_b$$

$$[H_a, F_b] = -A_{ab} F_b, \qquad [E_a, F_b] = \delta_{ab} H_a$$

and

$$\underbrace{[E_a, [E_a, \ldots \quad [E_a}_{1-A_{ba} \text{ factor}}, E_b] \ldots] = 0$$

plus relation for F_a }.

Then the Kac-Moody algebra is given by the multiple commutators

$$[E_{a_1}, [E_{a_2} \ldots, [E_{a_{n-1}}, E_{a_n}] \ldots]$$

plus multiple commutator of F_a's.

It is uniquely determined by A_{ab} or Dynkin diagram

If A_{ab} is a positive definite matrix we get a finite dimensional semi-simple Lie algebra

Example $\quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

The Chevalley generators are

$$E_1 , E_2 , H_1 , H_2 , F_1 , F_2$$

The Serre relations are

$$[H_1, E_2] = -E_2 \quad \text{etc}$$

and

$$[E_1,[E_1,E_2]] = 0 \quad \text{etc.}$$

The resulting algebra is

$$\left.\begin{array}{ccc} E_1 , & E_2 , & [E_1, E_2] \\ (\alpha_1) & (\alpha_2) & (\alpha_1+\alpha_2) \end{array}\right\} \text{positive roots}$$

$$H_1 , H_2 \qquad \text{Cartan subalgebra}$$

$$\left.\begin{array}{ccc} F_1 , & F_2 , & [F_1, F_2] \\ (-\alpha_1) & (-\alpha_2) & (-\alpha_1-\alpha_2) \end{array}\right\} \text{negative roots}$$

ie $SU(3)$.

$D=11$ Supergravity as a Non-linear Realizat

hep-th/0005270.

field $\quad h_a{}^b, \quad A_{a_1..a_3}, \quad A_{a_1\cdots a_6}$

generator $\quad K^a{}_b, \quad R^{a_1\cdot a_3}, \quad R^{a_1\cdots a_6}$

— The $K^a{}_b$ generate $SL(11)$ or A_{10}

— The $R^{a_1..a_3}$, $R^{a_1..a_6}$ are tensors under $SL(11)$ ie

$$[K^a{}_b, R^{c_1c_2c_3}] = \delta_b^{c_1} R^{ac_2c_3} + \cdots$$

— The generators $K^a{}_b$, $R^{a_1.a_3}$, $R^{a_1..a_6}$ belong to a specific algebra, G_{11}

i.e $\quad [R^{c_1.c_3}, R^{c_4\cdot c_6}] = 2R^{c_1\cdots c_6}$

The non-linear realization is constructed from

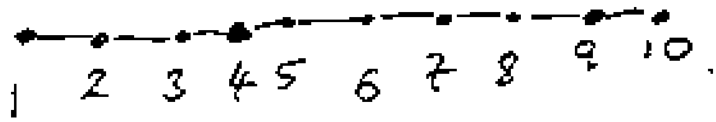
$$g = e^{x^a P_a}\, e^{h_a{}^b(x) K^a{}_b}\, e^{A^{c_1\cdot c_3} R_{c_1.c_3} + A^{c_1\cdot c_6} R_{c_1..c_6}}$$

Carrying out the simultaneous realization with the conformal group $SO(2,11)$ we find the PRECISE field equations of $D=11$ supergravity.

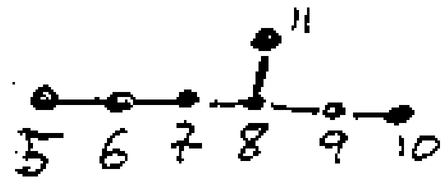
hep-th/0104081

The algebra G_{11} contains.

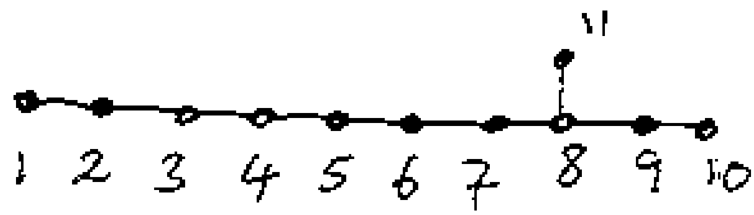
— A_{10}



and the Borel subgroup of E_7.



If the theory is invariant under a Kac-Moody algebra it must be E_{11}



G_{11} does not contain E_8. Under $SL(8)$ the adjoint of E_8 decomposes as

$$248 \to \underbrace{\underset{K^i{}_j,}{63} + \underset{D}{1}}_{GL(8)} + \underbrace{\underset{R^{x_1 \ldots x_3}}{56} + \underset{\# \epsilon_{x_1 x_2 \ldots} R^{\ldots}}{\overline{28}} + \underset{?}{8}}_{\text{positive roots}} + \underbrace{\overline{56} + 28 + \overline{8}}_{\text{negative roots}}$$

$i, j = 4, \ldots, 11$

Where is the 8?

We can modify the G_{11} algebra by adding a new generator $R^{a_1 \ldots a_8, b}$

- gives Borel subalgebra of E_8

•

$$\underbrace{\underset{A_{c_1 \cdot c_3}}{R^{c_1 \cdot c_3}} \quad \underset{A_{c_1 \ldots c_6}}{R^{c_1 \ldots c_6}}}_{\text{dual formulation of 3-form}} \quad ; \quad \underbrace{\underset{h_a{}^b}{K^a{}_b}, \quad \underset{h_{a_1 \ldots a_8, b}}{R^{a_1 \ldots a_8, b}}}_{\text{dual formulation of gravity.}}$$

- leads precisely to the TA algebra.

We need a formulation of gravity based on $e_\mu{}^a = (e^h)_\mu{}^a$, $h_{a_1 \ldots a_8, b}$.

A set of equations equivalent to Einstein's is given by

$$\epsilon^{\mu\nu\tau_1 \ldots \tau_q} Y_{\tau_1 \ldots \tau_q, d} = 2(\omega_{d,}{}^{\mu\nu} - e_d{}^\nu \omega_{,c}{}^{c\mu} + e_d{}^\mu \omega^c{}_{,c}{}^\nu)$$

$$\epsilon^{\mu\tau_1 \ldots \tau_q} \partial_{\tau_1} Y_{\tau_2 \ldots \tau_q, d} = \text{terms bilinear in } Y$$

At the linearized level

$$Y_{\tau_1 \ldots \tau_q, d} = \partial_{[\tau_1} h_{\tau_2 \ldots \tau_q], d}$$

as required.

The Closed Bosonic String

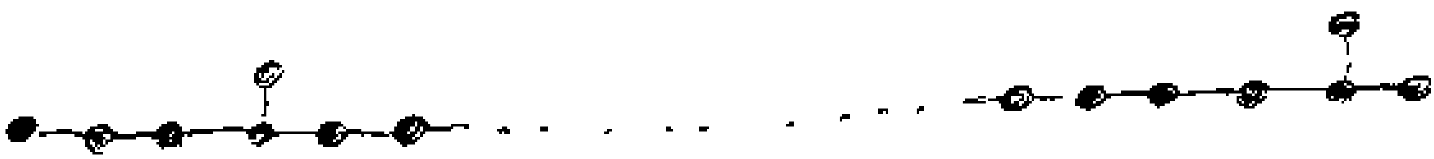
The effective action can be constructed as a non-linear realization for the algebra with generators

$$K^a{}_b, \quad R \quad R_{a_1\ldots a_{24}}; \quad R_{a_1 a_2}, \quad R_{a_1\ldots a_{22}}$$

$$h_a{}^b; \quad \phi \quad \phi_{a_1\ldots a_{24}}; \quad B_{a_1 a_2} \quad B_{a_1\ldots a_{22}}$$

gravity   the spin 0   the 2 form

The corresponding Kac-Moody algebra has rank 27 with Dynkin diagram



Predicts symmetry of reduced theory on an n torus to be

$$\frac{O(n,n)}{O(n)\otimes O(n)} \quad \text{for } n = 3,\ldots,22$$

$$\frac{O(24,24)}{O(24)\times O(24)} \quad \text{for } n = 23$$

Gravity in D dimensions

Lambert-West
hep-th/0107209.

Conjectured that gravity has a Kac-Moody symmetry

Consistent with E_{11}

Very Extended Algebras

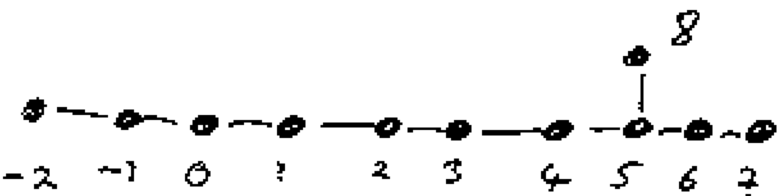
Gaberdiel, Olive, West hep-th/0205068

Consider a finite dimensional semi-simple Lie algebra g with simple roots α_a, $a = 1, \dots, r$ spanning a lattice Λ_g

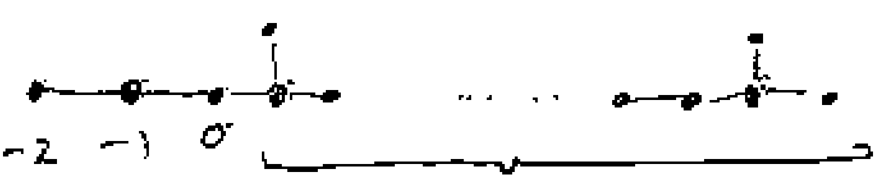
extension	rank	root added	Lattice	det
affine, g^+	$r+1$	$\alpha_0 = k - \theta$	$\Lambda_g \oplus \Pi^{(1,1)}$, $x \cdot k = 0$	0
over extended, g^{++}	$r+2$	$\alpha_{-1} = -(k+\bar{k})$	$\Lambda_g \oplus \Pi^{(1,1)}$	$-\det A_g$
very extended, g^{+++}	$r+3$	$\alpha_{-2} = k - (\ell + \bar{\ell})$	$\Lambda_g \oplus \Pi^{(1,1)} \oplus \Pi^{(1,1)}$, $(\ell - \bar{\ell}) \cdot x = 0$	$-2\det A_g$

$k^2 = \bar{k}^2 = 0$, $k \cdot \bar{k} = 1$, $\ell^2 = \bar{\ell}^2 = 0$, $\ell \cdot \bar{\ell} = 1$

Example



$e_{11} = e_8^{+++}$



d_n

if $n = 24$, k_{27}, $= d_{24}^{+++}$

Non-linear Realization of Cartan subalgebra of E_8^{+++}

Englert, Houart, West, Taormina to appear soon

A group element of the Cartan subalgebra of E_8^{+++} is

$$g = e^{q^a(x) H_a}$$

Now $GL(11) \subset E_8^{+++}$ and

$$H_a = r_a{}^b K^b{}_b$$

and so

$$g = e^{p^a(x) K^a{}_a}$$

where $p^T = q^T r$.

The non-linear realization of $IGL(11)$ is

$$g = e^{x^a P_a} e^{h_a{}^b(x) K^a{}_b}$$

is "gravity". Hence

$$e_\mu{}^a = (e^h)_\mu{}^a = e^{p^a} \delta_\mu{}^a$$

Hence we have just the diagonal components of the metric.

Weyl Group of E_8^{+++}

generated by reflections in planes $\perp$ to simple roots α_a i.e.

$$S_a(\alpha_b) = \alpha_b - (\alpha_a, \alpha_b)\alpha_a \equiv (S_a)_b{}^c \alpha_c$$

and

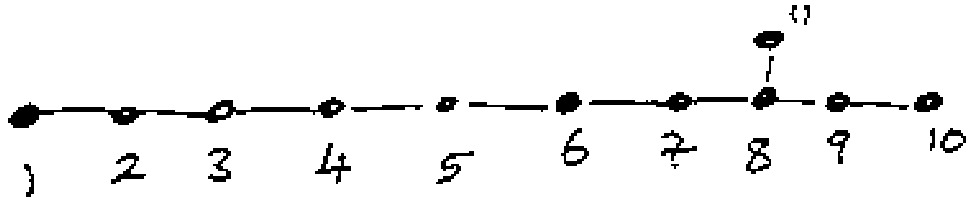
$$S_a(H_b) = (S_a)_b{}^c H_c \quad \text{or} \quad q'^T = q^T S$$

On the $p^a = \frac{1}{2} \ln g_{aa}$.

$$p'^T = p^T t \quad \text{where} \quad t = r^{-1} s r$$

Hence we find the effect of Weyl transformations on the diagonal components of the metric

In any application of M-theory in which only the diagonal components of the metric play the crucial role we can test the Weyl group of E_8^{+++}.



— S_a for $a = 1, \dots, 10$ takes $p_a \Longleftrightarrow p_{a+1}$

ie interchanges neighbouring components of the metric.

— S_{11}

$$p'_a = p_a + \frac{1}{3}(p_9 + p_{10} + p_{11}) \qquad a = 1, \dots, 8$$

$$p'_a = p_a - \frac{2}{3}(p_9 + p_{10} + p_{11}) \qquad a = 9, 10, 11$$

Consider M - theory compactified on a torus. R_a

- S_a, $a = 1, .., 10$ ~ $R_a \leftrightarrow R_{a+1}$
part of "T - duality" ie $O(d,d;\mathbb{Z})$.

- S_{11} is a double T - duality

$$R_9 \to \frac{l_s}{R_9}, \quad R_{10} \to \frac{l_s}{R_{10}} \; ; \; g_s \to \frac{g_s l_s^2}{R_9 R_{10}}$$

where $p_a = \ln \frac{R_a}{l_p}$

The Weyl transformation of $E_8^{+++} = E_{11}$ include the U - dualities of IIA string.

E_{11} and Kasner Solutions

A solution of M-theory is

$$ds^2 = -e^{2\hat{p}_0 u}(du)^2 + \sum_{j=1}^{D-1} e^{2\hat{p}_j u}(dx^j)^2$$

Kasner

provided

$$\sum_j \hat{p}_j = \hat{p}_0 \quad , \quad \sum_j \hat{p}_j^2 = \hat{p}_0^2 .$$

The Weyl transformations of E_8^{+++} act on this solution :-

$$S_a \quad , \quad a = 2, .., 11 \quad \rightsquigarrow \quad \hat{p}^a \leftrightarrow \hat{p}^{a+1}$$

$S_1 \rightsquigarrow \hat{p}_0 \leftrightarrow \hat{p}_1$ takes the above solution to

$$ds^2 = -e^{2\hat{p}_0 v}(dt)^2 + e^{2\hat{p}_1 v}(dv)^2 + \sum_{a=2} e^{2\hat{p}^a v}(dx^a)^2$$

provided

$$\hat{p}_0 + \sum_{j=2} \hat{p}_j = \hat{p}_1 \quad ; \quad \hat{p}_0^2 + \sum_{j=2}(\hat{p}_j)^2 = \hat{p}_1^2 .$$

The solutions form a representation of the Weyl group of E_8^{+++}.

D^{+++}_{n-3}

— $e^{q^a H_a} = e^{p^a K^a{}_a} \, e^{\phi \cdot R}$ $(D = n-1$

corresponds to gravity coupled to dilaton and 2-form gauge field, ($n = 27$ it is closed bosonic string).

— S_a, $a = 1, 2, \dots, n-2$ $\sim$ $p^a \leftrightarrow p^{a+1}$

(part of T-duality)

— S_{n-1} is a double T-duality

— S_n an S-duality $g_s' = \frac{1}{g_s}$, ...

— $n-1 \leftrightarrow n-2$ automorphism is a single T-duality

— Kasner like solutions carry a representation of Weyl plus automorphisms of D^{+++}_{n-3}

Conclusion

We have argued that "D = 11 supergravity" is invariant under $E_8^{+++} = E_{11}$. A similar calculation with IIA and IIB supergravity also leads to E_{11}.

— The "effective action for the closed bosonic string has D_{24}^{+++} (gravity $\leftrightarrow A_{D-3}^{+++}$)

— Is the non-linear realization of E_8^{+++} (D_{24}^{+++}) dynamically the same as supergravity or is it a new theory?

— Presented new evidence for g^{+++}.

— Kasner like solutions carry a representation of the Weyl + automorphism of g^{+++}.

E_8^{+++} $\longleftrightarrow$ "M-theory"

D_{D-2}^{+++} $\longleftrightarrow$ "bosonic string in D-dimensions

A_{D-3}^{+++} $\longleftrightarrow$ "gravity"

G^{+++} $\longleftrightarrow$ known theory.

Supergravity $\longrightarrow$ string theory $\longrightarrow$ M theory

$\longrightarrow$? - theory $\longrightarrow$? - theory
(closed bosonic string) (contains as limits all the above theories)