String Interactions in Gravitational Wave Backgrounds

Giuseppe D'Appollonio

LPTHE, Université Paris VI

Work done in collaboration with Elias Kiritsis

Ecole Politechnique and Crete University

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Motivations

1) Examples of non-compact WZW model

Simpler than $SL_2\mathbb{R}$, 2d black hole,
Nappi-Witten cosmological model

Retain some essential features as

infinite number of representations

spectral flow

Conformally invariant σ-models	Amati - Klimcik
Non-semisimple WZW models	Nappi - Witten
Contractions	Olive - Rabinovici - Schwimmer
Other studies	Antoniadis, Kiritsis, Kounnas, Lüst, Sfetsos, Tseytlin ...

2) Holographic description of gravitational wave backgrounds

$AdS_5 \times S^5$	pp-wave	Berenstein, Maldacena, Nastase
$\mathcal{N}=4$ SYM	$N \longrightarrow \infty$, $\frac{J^2}{N}$ fixed	

$AdS_3 \times S^3 \longrightarrow H_6$

Plan of the talk

1) σ-model point of view:

H_4 wave as the Penrose limit of $\mathbb{R} \times S^3$

algebraic point of view:

H_4 wave as a contraction of $\widehat{U(1)} \times \widehat{SU}(2)_k$

2) The structure of the WZW model based on H_4

- representations
- spectral flow
- free-field realization

3) Interactions

- 3-point couplings
- 4-point correlators

4) String amplitudes

Flat space limit

Penrose limit: the σ-model point of view

The background fields of the σ-model are

$$ds^2 = k[dr^2 + \sin^2 r d\varphi^2 + \cos^2 r d\alpha^2 - dt^2] \ , \qquad B_{\varphi\alpha} = \frac{k}{2}\cos(2r) \ .$$

We make the following change of variables

$$u = t - \alpha \quad , \quad v = \frac{t+\alpha}{\lambda^2} \quad , \quad r = \lambda\rho \ ,$$

and take the limit $\lambda \to 0$, $k \to \infty$ keeping $k\lambda^2 = 1$.

The resulting background is the Nappi-Witten gravitational wave

$$ds^2 = -2dudv - \frac{\mu^2\rho^2}{4}du^2 + d\rho^2 + \rho^2 d\varphi^2 \ , \qquad B_{\varphi u} = \frac{\mu\rho^2}{2} \ .$$

This form of the metric corresponds to the following parameterization of the H_4 group manifold

$$g = e^{\frac{u}{2}J} e^{i\frac{\bar{\zeta}}{\sqrt{2}}P^- + i\frac{\zeta}{\sqrt{2}}P^+} e^{\frac{u}{2}J - 2vK} \ .$$

Semiclassical vertex operators are given by matrix elements of group operators between states forming an irreducible representation. The generating functions is

$$\Phi^{\pm}_{p,\hat{\jmath}} = e^{\mp ipv + iju - \frac{\mu p}{2}\zeta\bar{\zeta} + i\mu p\zeta x e^{\pm\frac{i\mu u}{2}} + i\mu p\bar{\zeta}\bar{x}e^{\pm\frac{i\mu u}{2}} + \mu p x\bar{x}e^{\pm i\mu u}} \ .$$

Expanding this functions in x and $\bar{x}$ we obtain semiclassical expressions for the various states in a $V^+_{p,\hat{\jmath}}$ representation.

$$R^+_{p,\hat{\jmath};n,\bar{n}} = e^{-ipv + i[j + \frac{i\mu u}{2}(n+\bar{n})]u - \frac{\mu p}{2}\zeta\bar{\zeta}} \ \frac{(\mu p)^n}{n!} \ \zeta^{n-\bar{n}} \ L^{n-\bar{n}}_{\bar{n}}(\mu p \ \zeta \ \bar{\zeta}) \ .$$

Penrose limit: the algebraic point of view

Consider the $U(1) \times SU(2)_k$ current algebra

$$J^+(z)J^-(0) = \frac{k}{z^2} + \frac{2J^3}{z}\ ,\ \ J^3(z)J^\pm(0) = \pm\frac{J^\pm}{z}\ ,$$
$$J^0(z)J^0(0) = -\frac{k}{2z^2}\ .$$

Define the new currents as follows

$$K(z) = \frac{2i}{k}J^0(z)\ ,\ \ J(z) = i(J^0(z) - J^3(z))\ ,$$
$$P^\pm = \sqrt{\frac{2}{k}}J^\pm\ .$$

In the limit $k \to \infty$ we obtain the H_4 current algebra

$$\begin{aligned} P^+(z)P^-(w) &\sim \frac{2}{(z-w)^2} - \frac{2iK(w)}{z-w}\ , \\ J(z)P^\pm(w) &\sim \mp i\frac{P^\pm(w)}{z-w}\ , \\ J(z)K(w) &\sim \frac{1}{(z-w)^2}\ . \end{aligned}$$

The original stress-energy tensor

$$T = -\frac{1}{k}(J^0)^2 + \frac{1}{2(k+2)}(J^+J^- + J^-J^+ + 2(J^3)^2)\ ,$$

becomes

$$T = \frac{1}{2}\left[\frac{1}{2}\left(P^+P^- + P^-P^+\right) + 2JK + K^2\right]\ .$$

Representation theory

The H_4 algebra has three types of unitary representations

	K	J	C
$V^+_{p,\hat{\jmath}}$	p	$\{\hat{\jmath}+n\}_{n\in\mathbb{N}}$	$-2p\hat{\jmath}+p$
$V^-_{p,\hat{\jmath}}$	$-p$	$\{\hat{\jmath}-n\}_{n\in\mathbb{N}}$	$2p\hat{\jmath}+p$
$V^0_{s,\hat{\jmath}}$	0	$\{\hat{\jmath}+n\}_{n\in\mathbb{Z}}$	s^2

(1)

In terms of the representations of the original $U(1) \times SU(2)$ model, the $V^+_{p,\hat{\jmath}}$ representations result from states characterized by

$$l = \frac{1}{2}(kp - 2\hat{\jmath}) \ , \qquad m = \frac{1}{2}(kp - 2(\hat{\jmath}+n)) \ , \qquad q = \frac{k}{2}p \ .$$

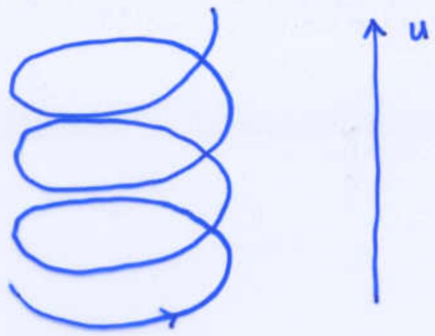
The $V^-_{p,\hat{\jmath}}$ representations result from states characterized by

$$l = \frac{1}{2}(kp + 2\hat{\jmath}) \ , \qquad m = -\frac{1}{2}(kp + 2(\hat{\jmath}-n)) \ , \qquad q = -\frac{k}{2}p \ .$$

Finally the states that form a $V^0_{s,\hat{\jmath}}$ representation correspond to states in the middle of an SU(2) representation with $q, m \sim O(1)$ as $l = \sqrt{\frac{k}{2}}s$.

States in $V^{\pm}_{p,\hat{J}} \rightarrow$ states trapped by the wave.

$$h = -p\hat{J} + \frac{p}{2}(1-p)$$

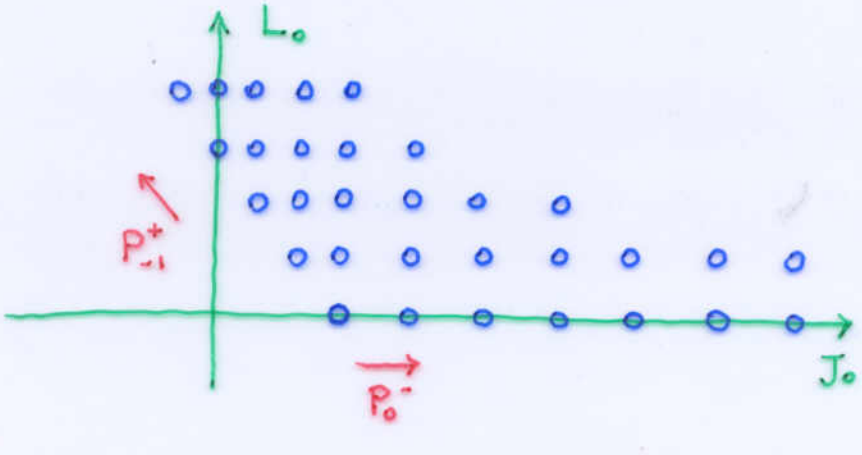


States in $V^{0}_{s,\hat{J}} \rightarrow$ free motion.

$$h = \frac{s^2}{2}$$

Highest-weight representations of the current algebra are built by acting with the negative modes of the currents J^a_{-n} on states $|R_i\rangle$ which form an unitary representations of the global H_4 and satisfy

$$J^a_n|R_i\rangle = 0 \ , \qquad n > 0 \ .$$



Unitarity constraint: $\quad |p| < 1$.

Since we are dealing with infinite dimensional representations it is useful to realize the H_4 algebra in terms of differential operators acting on auxiliary variables x and $\bar{x}$. For $V^{+}_{p,\hat{\jmath}}$ representations we have for instance

$$
\begin{aligned}
P_0^+ &= \sqrt{2}p\ x\ , \qquad P_0^- = \sqrt{2}\ \partial_x\ , \\
J_0 &= i(\hat{\jmath} + x\partial_x)\ , \qquad K_0 = ip\ .
\end{aligned}
$$

We then collect all the component fields in a single field

$$
\Phi^{+}_{p,\hat{\jmath}}(z,x) = \sum_{n=0}^{\infty} R^{+}_{p,\hat{\jmath};n}(z) \frac{(x\sqrt{p})^n}{\sqrt{n!}}\ ,\ p > 0\ .
$$

The OPE with the currents take the simple form

$$
\begin{aligned}
P^+(z)\Phi^{+}_{p,\hat{\jmath}}(w,x) &= \sqrt{2p}\ x\ \frac{\Phi^{+}_{p,\hat{\jmath}}(w,x)}{z-w}\ , \\
P^-(z)\Phi^{+}_{p,\hat{\jmath}}(w,x) &= \sqrt{2}\ \partial_x\ \frac{\Phi^{+}_{p,\hat{\jmath}}(w,x)}{z-w}\ , \\
J(z)\Phi^{+}_{p,\hat{\jmath}}(w,x) &= i(\hat{\jmath} + x\partial_x)\frac{\Phi^{+}_{p,\hat{\jmath}}(w,x)}{z-w}\ , \\
K(z)\Phi^{+}_{p,\hat{\jmath}}(w,x) &= ip\frac{\Phi^{+}_{p,\hat{\jmath}}(w,x)}{z-w}\ .
\end{aligned}
$$

These OPE are the central elements fo deriving the Ward identities and the KZ equations.

Spectral Flow

Consider representations that are highest-weight with respect to the algebra $\tilde{H}_{4,w}$ related to the original one by

$$\tilde{P}_n^{\pm} = P_{n\mp w}^{\pm} \,, \quad \tilde{K}_n = K_n - iw\delta_{n,0} \,, \quad \tilde{J}_n = J_n \,, \quad \tilde{L}_n = L_n - iwJ_n \,.$$

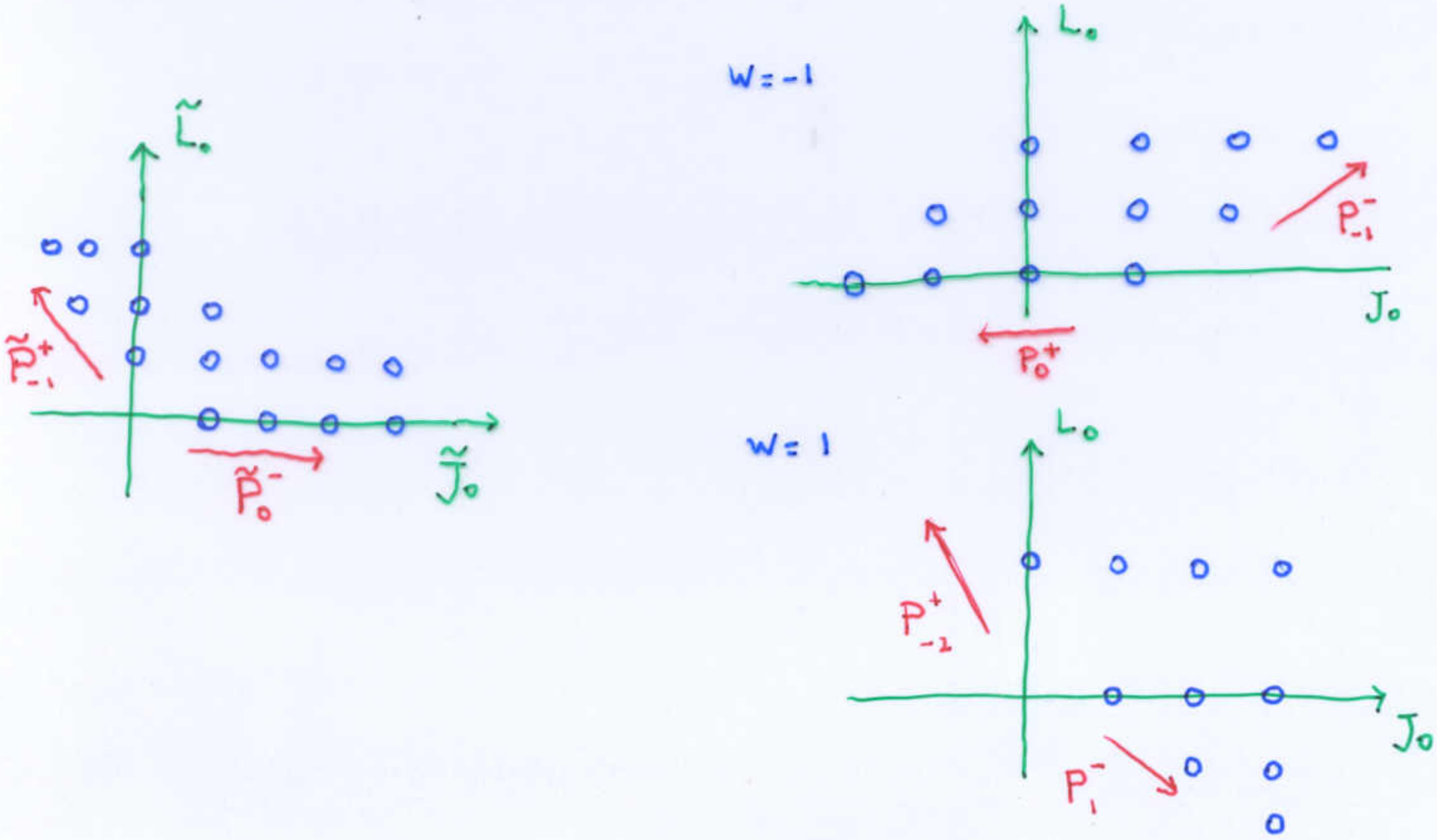
These representations are called spectral-flowed representations and we will denote them with $\Omega_w(\Phi^+_{p,\hat{\jmath}})$. (Maldacena and Ooguri, Gaberdiel)

In terms of the original modes

$$P_n^+|\psi>= 0 \,, \;\; n \geq -w \qquad P_n^-|\psi>= 0 \,, \;\; n > w \,.$$

The spectrum of L_0 is generically unbounded from below. Two exceptions

$$\Omega_{-1}(\Phi^+_{p,\hat{\jmath}}) = \Phi^-_{1-p,\hat{\jmath}} \,, \qquad \Omega_1(\Phi^-_{p,\hat{\jmath}}) = \Phi^+_{1-p,\hat{\jmath}} \,.$$



Fusion rules between spectral-flowed representations can be de-
rived using (Gaber

$$\Omega_{w_1}(\Phi_1) \otimes \Omega_{w_2}(\Phi_2) = \Omega_{w_1+w_2}(\Phi_1 \otimes \Phi_2) \; .$$

s new

From a geometric point of view, spectral flow generate
solutions of the σ-model, (Maldacena and Ooguri)

$$g(\tau,\sigma) \to e^{w(\tau+\sigma)J} g(\tau,\sigma) e^{w(\tau-\sigma)J} \; .$$

on

This corresponds to the following coordinate transformati

$$u \to u + 2w\tau \; , \qquad v \to v \; , \qquad \rho e^{i\varphi} \to \rho e^{i(\varphi + w\sigma)} \; .$$

(Kiritsis and Pioline)

called

States in the spectral flowed continuous representations are
long strings.

The spectrum of our model is then given by

$$\Omega_{\pm w}(\Phi^{\pm}_{p,\hat{\jmath}}) \text{ with } p < 1 \text{ and } w \in \mathbb{N},$$

and by

$$\Omega_w(\Phi^0_{S,\hat{\jmath}}) \text{ with } \hat{\jmath} \in [-1/2, 1/2) \text{ and } w \in \mathbb{Z}.$$

Free field resolution

The H_4 current algebra can be represented in terms of free fields (Kiritsis and Kounnas)

$$\begin{array}{rclcl} J & = & \partial v \ , & & K = \partial u \ , \\ P^{+} & = & ie^{-iu}\partial y \ , & & P^{-} = ie^{iu}\partial \tilde{y} \ . \end{array}$$

In this formalism, primary fields for the $V^{\pm}$ representations correspond to twist fields $H^{\mp}_{p}(z)$ characterized by the OPEs

$$\partial y(z) H^{-}_{p}(w) \sim (z-w)^{-p} \ , \qquad \partial \tilde{y}(z) H^{-}_{p}(w) \sim (z-w)^{-1+p} \ ,$$

$$\partial y(z) H^{+}_{p}(w) \sim (z-w)^{-1+p} \ , \qquad \partial \tilde{y}(z) H^{+}_{p}(w) \sim (z-w)^{-p} \ .$$

The ground state of a $V^{\pm}$ representation is then given by

$$R^{\pm}_{p,\hat{\jmath};0}(z) = e^{i\hat{\jmath}u(z) \pm ipv(z)} H^{\mp}_{p}(z) \ ,$$

and the other states are obtained through the action of $P^{\mp}_{0}$.

Simple description of spectral flow by w units:

multiplication by e^{iwv}

$$\Omega_{w}(R^{+}_{p,\hat{\jmath};0}) = e^{i\hat{\jmath}\ u + i(p+w)v} H^{-}_{p} \ , \qquad \Omega_{-w}(R^{-}_{p,\hat{\jmath};0}) = e^{i\hat{\jmath}\ u - i(p+w)v} H^{+}_{p} \ .$$

OPE

We now discuss the OPE between the local conformal primary fields of the H_4 algebra. The two-point function is

$$< \Phi^{+}_{p_1,\hat{\jmath}_1} \Phi^{-}_{p_2,\hat{\jmath}_2} >= \delta(p_1 - p_2)\delta(\hat{\jmath}_1 + \hat{\jmath}_2)\frac{e^{-p_1(x_1x_2+\bar{x}_1\bar{x}_2)}}{|z_{12}|^{4h}} ,$$

The x and z dependence of the three-point functions is fixed by the conformal and H_4 Ward identities,

$$< \Phi^{a}_{q_1} \Phi^{b}_{q_2} \Phi^{c}_{q_3} >= \frac{\mathcal{C}_{abc}(q_1, q_2, q_3) D_{abc}(x_1, x_2, x_3; \bar{x}_1, \bar{x}_2, \bar{x}_3)}{|z_{12}|^{2(h_1+h_2-h_3)}|z_{13}|^{2(h_1+h_3-h_2)}|z_{23}|^{2(h_2+h_3-h_1)}} .$$

up to the structure constants $\mathcal{C}_{abc}$.

Consider as an example the following OPE

$$[\Phi^{+}_{p_1,\hat{\jmath}_1}] \otimes [\Phi^{+}_{p_2,\hat{\jmath}_2}] = \sum_{n=0}^{\infty}[\Phi^{+}_{p_1+p_2,\hat{\jmath}_1+\hat{\jmath}_2+n}] .$$

The D function in this case is given by

$$\begin{aligned} & D_{++-}(x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3) \\ = & \left|e^{-x_3(p_1x_1+p_2x_2)}(x_2 - x_1)^{-L}\right|^2 \delta(p_3 - p_1 - p_2)\delta_{\mathbb{N}}(-L) , \end{aligned}$$

where $L = \hat{\jmath}_1 + \hat{\jmath}_2 + \hat{\jmath}_3$ and $\delta_{\mathbb{N}}(a) \equiv \sum_{n=0}^{\infty} \delta(a - n)$.

Three-point couplings

- Couplings between states with $p \neq 0$. Let us start with a coupling of the form $< + + - >$

$$\mathcal{C}_{++}{}^{+}(q_1, q_2, q_3) = \frac{1}{\Gamma(1 + \hat{\jmath}_3 - \hat{\jmath}_1 - \hat{\jmath}_2)} \left[\frac{\gamma(p_3)}{\gamma(p_1)\gamma(p_2)} \right]^{\frac{1}{2} + \hat{\jmath}_3 - \hat{\jmath}_1 - \hat{\jmath}_2} .$$

  Here

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$$

  and moreover $p_3 = p_1 + p_2$ and $\hat{\jmath}_3 = \hat{\jmath}_1 + \hat{\jmath}_2 + n$, $n \in \mathbb{N}$.

  The others are given by similar expression, for instance when we have one Φ^+ and one Φ^- operator with $p_1 > p_2$ the coupling is

$$\mathcal{C}_{+-}{}^{+}(q_1, q_2, q_3) = \frac{1}{\Gamma(1 - \hat{\jmath}_3 + \hat{\jmath}_1 + \hat{\jmath}_2)} \left[\frac{\gamma(p_1)}{\gamma(p_2)\gamma(p_3)} \right]^{\frac{1}{2} - \hat{\jmath}_3 + \hat{\jmath}_1 + \hat{\jmath}_2}$$

  where $p_3 = p_1 - p_2$ and $\hat{\jmath}_3 = \hat{\jmath}_1 + \hat{\jmath}_2 - n$, $n \dot{\in} \mathbb{N}$.

- Couplings between two states with $p \neq 0$ and a state with $p = 0$

$$\mathcal{C}_{+-0}(p, \hat{\jmath}_1; p, \hat{\jmath}_2; s, \hat{\jmath}_3) = e^{\frac{s^2}{2}[\psi(p) + \psi(1-p) - 2\psi(1)]}$$

  where $\psi(x) = \frac{d \ln \Gamma(x)}{dx}$ is the digamma function.

- Couplings between states with $p = 0$ are the same as in flat-space.

Z equation is a consequence of the existence of the null

$$\ldots - \frac{1}{2}(P^-_{-1}P^+_0 + P^+_{-1}P^-_0) - J_{-1}K_0 - K_{-1}J_0 - K_{-1}K_0\Big] |V\rangle$$

highest-weight representation V of the affine algebra.

case this equation becomes a partial differential equation form

$$\sum_{j=1,j\neq i}^{4} \frac{1}{z_{ij}} \left[\frac{1}{2}(D_i^+ D_j^- + D_j^- D_i^+) + D_i^J D_j^K + D_i^K D_i^J + D_i^K D_j^K\right] A \,.$$

the global conformal and H_4 symmetry writing

$$A(z_i, \bar{z}_i, x_i, \bar{x}_i) = \prod_{i<j}^{4} |z_{ij}|^{2\left(\frac{h}{3} - h_i - h_j\right)} K(x_i, \bar{x}_i) \mathcal{A}(z, \bar{z}, x, \bar{x}) \ .$$

$= \sum_{i=1}^4 h_i$ and the cross-ratios are

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}} \ , \qquad z = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} \ .$$

The KZ
vector

$\Big[L_-$

in any

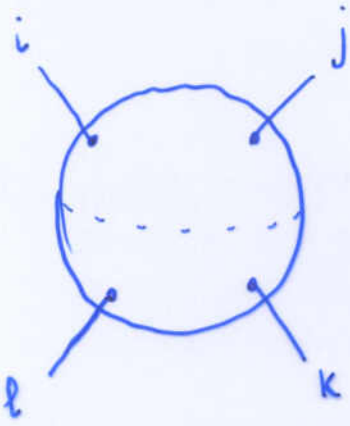
In our
of the

$\partial_{z_i} A =$

We fix

Here h

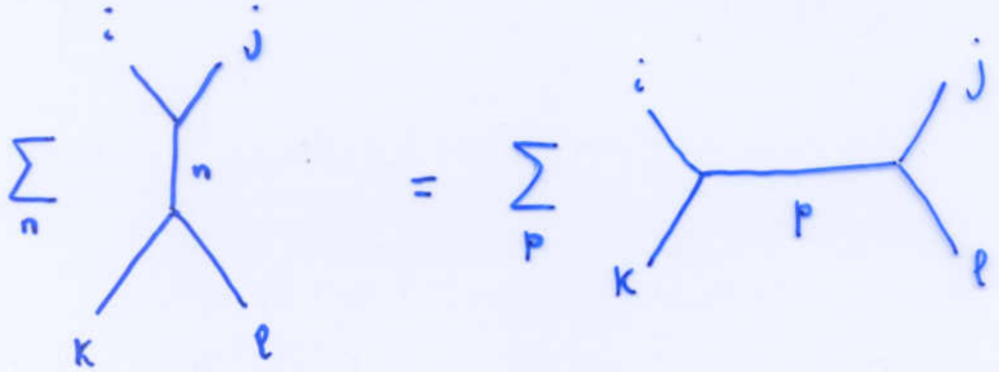
Using the operator algebra, we decompose each four-point function as a sum over intermediate representations of the affine algebra.



$$= \sum_n C_{ijn}\, C_{\ell k n}\, |\mathcal{F}_n(z)|^2$$

The functions that appear in this decomposition are called conformal blocks.

We can choose to decompose the four-point functions in different ways and all of them must agree due to the associativity of the operator algebra.



Four-point function → monodromy invariant combination of the conformal blocks

$< \Phi^{+}_{p_1,\hat{\jmath}_1} \Phi^{+}_{p_2,\hat{\jmath}_2} \Phi^{+}_{p_3,\hat{\jmath}_3} \Phi^{-}_{p_4,\hat{\jmath}_4} >$

Momentum conservation requires

$$p_1 + p_2 + p_3 = p_4 \ .$$

From the global H_4 symmetry constraints we obtain

$$K(x_i, \bar{x}_i) = \left| e^{-x_4(p_1x_1+p_2x_2+p_3x_3)}(x_3 - x_1)^{-L} \right|^2 \ ,$$

where $L = \hat{\jmath}_1 + \hat{\jmath}_2 + \hat{\jmath}_3 + \hat{\jmath}_4$ and

$$x = \frac{x_2 - x_1}{x_3 - x_1} \ .$$

The decomposition in conformal blocks in this case is

$$\mathcal{A}(z, \bar{z}, x, \bar{x}) \sim \sum_{i=0}^{|L|} \mathcal{F}_n(z, x) \bar{\mathcal{F}}_n(\bar{z}, \bar{x}) \ ,$$

We have a finite number of conformal blocks. Moreover states with $p = 0$ can not flow in the intermediate channels.

The KZ equation reads

$$\begin{aligned} \partial_z F_n &= \frac{1}{z}\left[-(p_1x + p_2x(1-x))\partial_x + Lp_2x\right] F_n \\ &- \frac{1}{1-z}\left[(1-x)(p_2x + p_3)\partial_x + Lp_2(1-x)\right] F_n \ . \end{aligned}$$

The conformal blocks are

$$F_n(z,x) = f^n(z,x)(g(z,x))^{|L|-n} \ , \qquad n = 0,...,|L| \ ,$$

and the four-point function

$$\mathcal{A}(z,\bar{z},x,\bar{x}) \sim \left(C_{12}|f(z,x)|^2 + C_{34}|g(z,x)|^2\right)^{|L|} \ ,$$

where

$$C_{12} = \frac{\gamma(p_1+p_2)}{\gamma(p_1)\gamma(p_2)} \ , \qquad C_{34} = \frac{\gamma(p_4)}{\gamma(p_3)\gamma(p_4-p_3)} \ .$$

The functions f and g are linear combinations of hypergeometric functions

$$f(z,x) = \frac{z^{1-p_1-p_2}p_3}{1-p_1-p_2}\varphi_0 - xz^{-p_1-p_2}\varphi_1 \ , \qquad g(z,x) = \gamma_0 - \frac{xp_2}{p_1+p_2}\gamma_1 \ ,$$

for instance

$$\begin{aligned} \varphi_0 &= F(1-p_1, 1+p_3, 2-p_1-p_2, z) \ , \\ \varphi_1 &= F(1-p_1, p_3, 1-p_1-p_2, z) \ . \end{aligned}$$

The four-point function can be expressed as a sum over all conformal blocks with the corresponding three-point couplings

$$\mathcal{A}(z,\bar{z},x,\bar{x}) = \sum_{n=0}^{|L|} \mathcal{C}_{++-}(q_1,q_2,n)\mathcal{C}_{+-+}(q_3,q_4,|L|-n)|\mathcal{F}_n(z,x)|^2 \ .$$

$< \Phi^{+}_{p_1,\hat{\jmath}_1} \Phi^{-}_{p_2,\hat{\jmath}_2} \Phi^{+}_{p_3,\hat{\jmath}_3} \Phi^{-}_{p_4,\hat{\jmath}_4} >$

Momentum conservation requires

$$p_1 + p_3 = p_2 + p_4 \ .$$

The function K is

$$K(x_i, \bar{x}_i) = \left| e^{-p_2 x_1 x_2 - p_3 x_3 x_4 - (p_1 - p_2) x_1 x_4} (x_1 - x_3)^{-L} e^{-\frac{x}{4}(p_1 - 2p_2 - p_3)} \right|^2 \ ,$$

and $x = (x_1 - x_3)(x_2 - x_4)$.

The KZ equation is

$$\begin{aligned} z(1-z)\partial_z F_n &= \left[x\partial_x^2 + (ax + 1 - L)\,\partial_x + \frac{x}{4}(a^2 - b^2) + \rho_{12} \right] F_n \\ &+ \ z\left[-2ax\partial_x + \frac{x}{4}(b^2 - c^2) - \rho_{12} - \rho_{14} \right] F_n \ , \end{aligned}$$

where

$$2a = p_1 + p_3 \ , \quad b = p_1 - p_2 \ , \quad c = p_2 - p_3 \ .$$

In this case the correlator factorizes on an infinite number of conformal blocks. When $p_1 \neq p_2$ it is an infinite sum

$$\mathcal{A}(z, \bar{z}, x, \bar{x}) = \sum_{n=0}^{\infty} \mathcal{C}_{+--}(q_1, q_2, n) \mathcal{C}_{+-+}(q_3, q_4, n - L) |\mathcal{F}_n(z, x)|^2 \ .$$

When $p_1 = p_2$ we have a continuum of intermediate states

$$\mathcal{A}(z, \bar{z}, x, \bar{x}) = \int_0^{\infty} ds s \mathcal{C}^2_{+-0}(p, s) |\mathcal{F}_s(z, x)|^2 \ .$$

Moreover when $p_1 + p_3 \geq 1$ we can see explicitely that the correlator factorizes on spectral flowed representations $\rightarrow$ they are necessary for the consistency of the model.

When $p_1 \neq p_2$ the conformal blocks are

$$F_n(z,x) = \nu_n \frac{e^{xg_1(z)}}{(f_1(z))^{1-L}} L_n^{|L|}(x\gamma_\psi(z))\psi(z)^n , \qquad n \in \mathbb{N} ,$$

where $L_n^{|L|}$ is the n-th generalized Laguerre polynomial and all the other functions involved can be expressed in terms of hypergeometric functions.

For instance

$$\psi(z) = \frac{f_2(z)}{f_1(z)} , \qquad \gamma_\psi(z) = -z(1-z)\partial \ln \psi ,$$

and $f_1(z) = F(p_3, 1-p_1, 1-p_1+p_2, z)$.

The full correlator is given by

$$\mathcal{A}(z,\bar{z},x,\bar{x}) \sim \frac{1}{S^{1+|L|}} \left| e^{xq(z)-xz(1-z)\partial \ln S} \right|^2 \left(\frac{u}{2}\right)^{-|L|} I_{|L|}(u) .$$

where

$$S = |f_1|^2 - r|f_2|^2 , \qquad u = \frac{2\sqrt{r}|xz(1-z)W(f_1,f_2)|}{S} .$$

When $p_1 = p_2$ the conformal blocks have a similar structure

$$F_s(z,x) = \frac{e^{xg_1(z)}}{(c_1(z))^{1-L}} e^{\frac{s^2}{2}\rho(z)} (xz(1-z)\partial\rho)^{\frac{L}{2}} J_{|L|}(v) .$$

Fusion and braiding

We can factorize the four-point functions around $z = 0$, 1 and $\infty \rightarrow$ three sets of conformal blocks.

Linear transformations between the different basis: braiding and fusion matrices.

Previous discussions for non-compact models (Liouville, $SL_2(\mathbb{R})$)
Teschner, Ponsot

The change of basis between blocks corresponding to $V^{\pm}$ states can be written as $(u = 1 - z)$

$$F_n(z,x) = \sum_{m=0}^{\infty} c^L_{nm} G_m(u,x) \ ,$$

where

$$c^L_{nm} = \frac{\Gamma(m+n+|L|+1)}{m!\Gamma(m+|L|+1)} \frac{1}{r_1^{n+m+|L|+1}} \left(\frac{r_2}{p_1 - p_2}\right)^n [(p_3 - p_2)s_1]^m F(-m,-n,-m-n-|L|;\theta) \ ,$$

and

$\sin(\pi p_4) \sin(\pi p_2)$

$\sin(\pi p_1) \sin(\pi p_3)$

cks corresponding to
$\neq 0$ states

$x)$.

oup.

Similarly we can change basis from the blo
$p = 0$ states to blocks corresponding to $p \neq$

$$F_s(u,x) = \sum_{m=0}^{\infty} c^L_m(s) F_m(z,$$

Relations with the quantum Heisenberg gr

Null vectors

The representations $\Phi^{\pm}_{1,\hat{\jmath}}$ contain a null vector at level one

$$\psi_{-1}(z,x) = P^{-}_{-1}(x)\Phi^{-}_{1,\hat{\jmath}}(z,x)\ , \qquad \psi_{1}(z,x) = P^{+}_{-1}(x)\Phi^{+}_{1,\hat{\jmath}}(z,x)\ .$$

Correlators involving $\Phi^{\pm}_{1,\hat{\jmath}}$ satisfy additional differential equations.

We can use these correlators to compute three-point couplings involving spectral flowed states.

We define the operator generating spectral flow by one unit as follows

$$\Sigma^{\pm}(z,\bar{z}) = \lim_{p\to 1} \frac{1}{\sqrt{\gamma(p)}} \Phi^{\pm}_{p,0}(z,\bar{z},0,0)\ .$$

Consider $\langle \Phi^{+}_{p_1}\Phi^{+}_{p_2}\Phi^{+}_{p_3}\Phi^{-}_{1}\rangle$. From this correlator we can extract

$$\langle \Phi^{+}_{p_1,\hat{\jmath}_1}\Phi^{+}_{p_2,\hat{\jmath}_2}\Omega_{-1}(\Phi^{-}_{p_3,\hat{\jmath}_3})\rangle = \frac{1}{|L|!}\left(\frac{\gamma(p_1+p_2)}{\gamma(p_1)\gamma(p_2)}\right)^{\frac{1}{2}+|L|}$$

$$\frac{|z_{12}z_{32}z_{13}^{-1} - x|^{2|L|}|x_3 - x_1|^{2|L|}}{|z_{12}|^{2(h_1+h_2-h_3)}|z_{13}|^{2(h_1+h_3-h_2)}|z_{23}|^{2(h_2+h_3-h_1)}}\ .$$

Here $h_3 = \hat{\jmath}_3(1-p_3) + \frac{p_3}{2}(1-p_3)$, the conformal dimension of the ground states in $\Phi^{-}_{1-p_3,\hat{\jmath}_3}$ and the constant appearing in the second line is $\mathcal{C}_{++-}(p_1,p_2,p_1+p_2)$ as expected.

From $\langle\Phi^+_{p_1}\Phi^-_{p_2}\Phi^+_{p_3}\Phi^-_1\rangle$ we obtain

$$\langle\Phi^+_{p_1,\hat{\jmath}_1}\Omega_{-1}(\Phi^-_{p_2,\hat{\jmath}_2})\Phi^+_{p_3,\hat{\jmath}_3}\rangle = \frac{1}{L!}\left(\frac{\gamma(p_1)}{\gamma(p_2)\gamma(p_1-p_2)}\right)^{\frac{1}{2}+L}$$
$$\frac{\left|x_2^L e^{-p_2x_1x_2+p_2x_2(x_1-x_3)z_{12}z_{32}z_{13}^{-1}}\right|^2}{|z_{12}|^{2(h_1+h_2-h_3)}|z_{13}|^{2(h_1+h_3-h_2)}|z_{23}|^{2(h_2+h_3-h_1)}} ,$$

where

$$h_2 = \hat{\jmath}_2(1+p_2) + \frac{p_2}{2}(1-p_2) - L ,$$

is the dimension of a state in the representation $\Omega_{-1}(\Phi^-_{p_2,\hat{\jmath}_2})$ obtained by acting L times with P_1^+ on the ground state.

This three-point coupling coincides with $\mathcal{C}_{+--}(p_1, 1-p_3, p_2)$ since it can be written as

$$\langle\Phi^+_{p_1,\hat{\jmath}_1}(z_1,x_1)\Omega_{-1}(\Phi^-_{p_2,\hat{\jmath}_2})(z_2,x_2)\Omega_1(\Phi^-_{1-p_3,\hat{\jmath}_3})(z_3,x_3)\rangle ,$$

and then related to a three-point function between highest-weight states of the form $< + - - >$.

String amplitudes

We can combine the Nappi-Witten gravitational wave with some internal CFT in order to get a critical string theory background $\mathcal{C} = \mathcal{C}_{H_4} \times \mathcal{C}_{int} \times \mathcal{C}_{gh}$.

Simplest choice $\mathcal{C}_{int} = \mathbb{R}^{22}$.

Internal part of a vertex operator: $e^{i\vec{p}\vec{X}}$, $h = \frac{\vec{p}^2}{2}$.

The four-point string amplitudes are given by the CFT four-point correlators integrated on the world-sheet.

The string amplitude can then be written in general as

$$\mathcal{A}_{string} = \int d^2z|z|^{2\sigma_{12}-\frac{4}{3}}|1-z|^{2\sigma_{14}-\frac{4}{3}}K(x_i,\bar{x}_i)\mathcal{A}(z,\bar{z};x,\bar{x}) \ .$$

In flat space

$$\mathcal{A}_{string} \sim \int d^2z|z|^{\frac{\alpha'}{2}(p_1+p_2)^2-4}|1-z|^{\frac{\alpha'}{2}(p_2+p_3)^2-4} \ .$$

The amplitude has a pole whenever

$$\alpha'(p_1+p_2)^2 = 4(1-N) \ , \qquad N \in \mathbb{N} \ ,$$

and therefore the poles in the amplitude, due to the propagation of on-shell states in the intermediate channel, precisely match the spectrum of the bosonic string.

A similar discussion applies in our case: the amplitude has a pole when the intermediate state is on shell, with the dispersion relation implied by the wave background

$$h_{12} - n(p_1 + p_2) = 1 - N \ , \qquad n = 0, \dots |L| \ , \qquad N \in \mathbb{N} \ ,$$

where

$$h_{12} = -(p_1+p_2)(\hat{\jmath}_1+\hat{\jmath}_2)+\frac{\mu}{2}(p_1+p_2)(1-\mu(p_1+p_2))+\frac{(\vec{p}_1+\vec{p}_2)^2}{2} \ .$$

When $p_1 = p_2 = p$ and $p_3 = p_4 = l$, the amplitude factorize on the continuum and can be written as

$$\mathcal{A}_{string} \sim \int d^2z|z|^{2(h_{12}-2)} \int_0^\infty dss \ C_{+-0}(p,s)C_{+-0}(l,s)|z|^{s^2}|x|^k \ ,$$

where now $h_{12} = \frac{(\vec{p}_1+\vec{p}_2)^2}{2}$.

The integrals can be expressed in terms of the Exponential Integral function. When $L = n = 0$ for instance we have

$$\mathcal{A}_{string} \sim \int_0^\sigma dr \frac{1}{r^\delta \ln r} \ ,$$

with $\delta = 3 - 2h_{12}$.

The integral is convergent for $\delta < 1$ and in the limit $\delta \to 1^-$ the amplitude behaves as

$$\mathcal{A}_{string} \sim \ln{(h_{12} - 1)} \ .$$

There is a logarithmic branch cut starting from $h_{12} = 1$.

Flat space limit

Reintroduce the parameter μ in the metric performing a boost $u \to \mu u$, $v \to \frac{v}{\mu}$

$$ds^2 = -2dudv - \frac{\mu^2 r^2}{4} du^2 + dx_T^2 \ .$$

There are two interesting limits to consider, $\mu \to 0$ and $\mu \to \infty$.

In both cases one recovers string theory in flat space, even though the states that survive in the two limits are very different.

Our model contains the following states:

- Short strings $\Omega_{\pm w}(\Phi^{\pm}_{p,\hat{\jmath}})$ have

$$h = \mp \left(p + \frac{w}{\mu}\right) \hat{\jmath} + \frac{\mu p}{2}(1 - \mu p) \ ,$$

  with $p \in (0, \frac{1}{\mu})$ and $w \in \mathbb{N}$.

- Long strings have $\Omega_w(\Phi^0_{s,\hat{\jmath}})$

$$h = -\frac{w\hat{\jmath}}{\mu} + \frac{s^2}{2} \ ,$$

  with $\hat{\jmath} \in [-\mu/2, \mu/2)$ and $w \in \mathbb{Z}$.

The limit $\mu \to 0$ can be thought as a contraction of the H_4 algebra.

The highest weight representations reconstruct the flat space spectrum: the potential flattens and the confined states describe larger and larger orbits until they become free.

We scale the quantum numbers as follows

$$\hat{\jmath} = p^- \mp \frac{s^2}{2p} , \qquad n = m + \frac{s^2}{2p\mu} ,$$

respectively for $V^{\pm}_{p,\hat{\jmath}}$ representations.

Consider now the case $\mu \to \infty$.

States in spectral flowed continuous representations have $p = w/\mu$, which becomes a continuous variable in the limit, and $\hat{\jmath} \in \mathbb{R}$.

All operators with $p \notin \mathbb{Z}$ behave as if $\mu p \to 1$ and decouple: they are so strongly trapped by the potential that disappear from the spectrum.

Long string states which did not feel the potential remain free.

Compare with the small and large radius limit of a compactified boson.