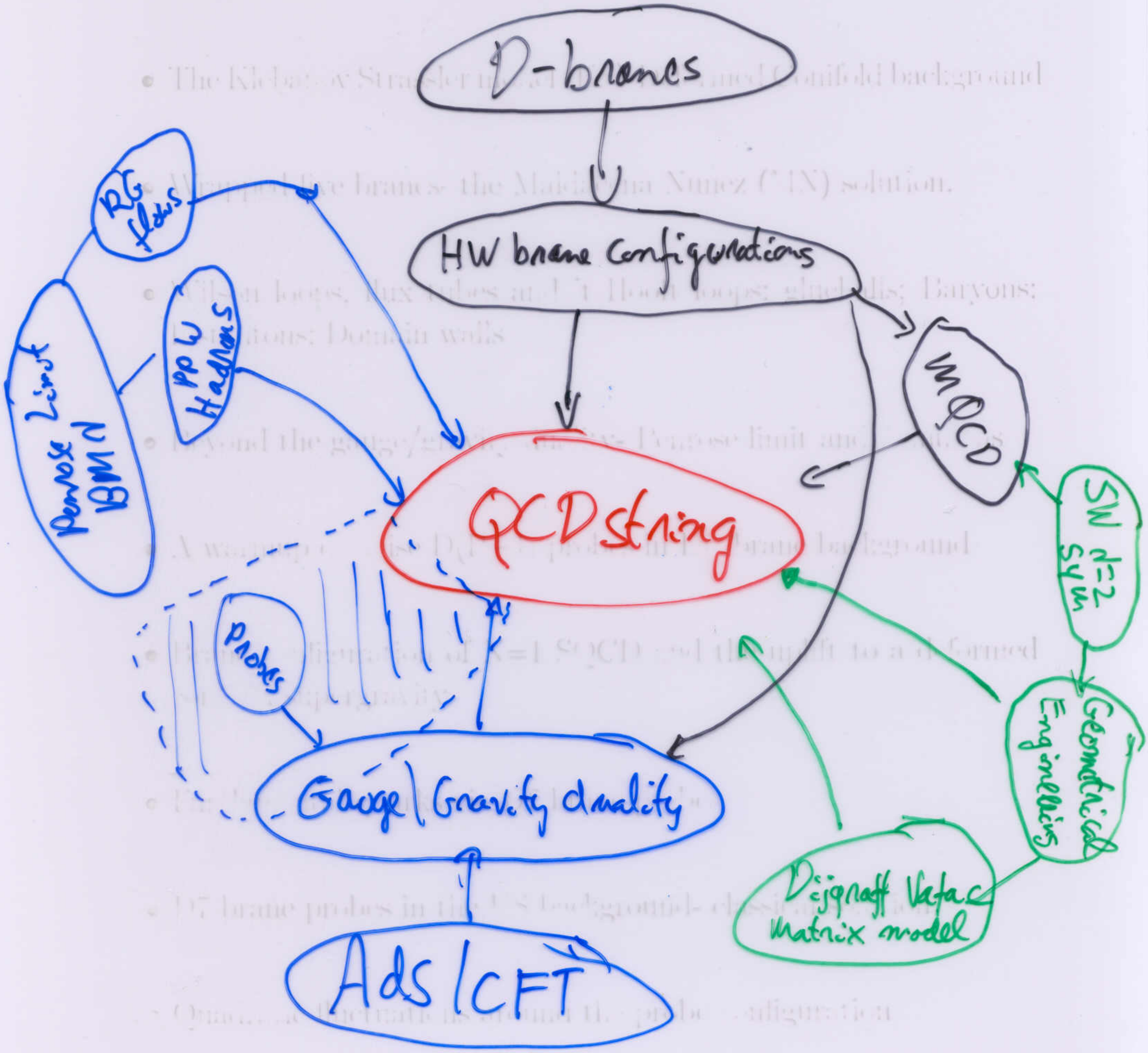
Probing Flavored Mesons of Confining Gauge Theories by Supergravity

Tadakatsu Sakai & J.S

Introduction

The interplay between string theory and gauge dynamics

the modern version



Fundamental quarks from D brane supergravity probes

- In the search for a supergravity model dual to a "realistic" strong coupling gauge dynamics one should be able to incorporate quarks in the fundamental representation of an $SU(N_c)$ gauge theory.

- Most of the known supergravity backgrounds duals of confining four dimensional gauge theories either do not incorporate quarks at all or admit quarks in the adjoint rather than the fundamental representation (the same applies for bifundamentals).

- Since the early days of strings it has been understood that fundamental quarks should correspond to open strings.

- In the modern era of closed string theory this obviously calls for D branes.

- Certain basic objects of gauge theories like baryons, instantons, monopoles, domain walls and others were shown to correspond to wrapped D brane probes.

- It is thus natural to wonder, whether one can consistently add D brane probes to supergravity backgrounds duals of confining gauge theories, which will play the role of fundamental quarks.

- In case that N_f the number of D brane probes is much smaller than N_c, one can convincingly argue that the backreaction of the probe on the bulk geometry is negligible.

- It is well known that open strings between parallel N_f D7 and N_c D3 play the role of flavored quarks in the $SU(N_c)$ gauge theory on the D3 4d world volume gauge theory.

- Karch and Katz proposed to elevate this brane configuration into a supergravity background by introducing a D7 brane probe into the $AdS_5 \times S^5$ background.

- The main idea behind our work is to introduce D7 brane probes into the Klebanov Strassler background and to extract from the spectrum of scalar and vector fluctuations of the N_f D7 branes the spectrum of the pseudo scalar and vector mesons of $SU(N_c)$ gauge dynamics with N_f flavors.

Outline

- Confinement from SUGRA
- The Klebanov Strassler model (KS)-Deformed Conifold background
- Wrapped five branes- the Maldacena Nunez (MN) solution.
- Wilson loops, flux tubes and 't Hooft loops; glueballs; Baryons; Instantons; Domain walls
- Beyond the gauge/gravity duality- Penrose limit and Annulons
- A warmup exercise D(P+4) probes in DP brane background
- Brane configuration of N=1 SQCD and the uplift to a deformed conifold supergravity
- Fundamental quarks via D7 brane probes
- D7 brane probes in the KS background- classical solution
- Quadratic fluctuations around the probe configuration

- Vector fluctuations
- Pseudo scalar fluctuations
- The spectrum of the vector and pseudo scalar mesons
- Summary and open questions

A warmup exercise D(P+4) probes in DP brane background

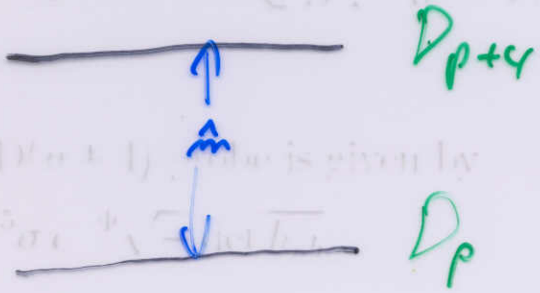
- The idea to add flavored fundamental quarks by introducing D7 brane was suggested by Karch and Katz. In Kruczenski, Mateos, Myers, Winters the spectrum of $\mathcal{N} = 2$ SYM with fundamental hypermultiplets was extracted from the supergravity of $AdS_5 \times S^5$ with D7 brane probes. It was found that for massive quarks the spectrum was discrete with a mass gap.

- Before introducing D7 branes to a confining background let us first as a warmup exercise discuss adding D(P+4) probes to DP background.

- Start with the configuration

$$\begin{array}{lccccccccccc} & 0 & 1 & \cdots & p & p+1 & \cdots & p+4 & p+5 & \cdots & 9 \\ \mathrm{D}p & \times & \times & \cdots & \times & & & & & & \\ \\ \mathrm{D}(p+4) & \times & \times & \cdots & \times & \times & \cdots & \times & & & \end{array} \tag{0.1}$$

- The Dp-branes are sitting at $x_{p+5}, \cdots, x_9 = 0$ while the D$(p+4)$-branes are at $x^2_{p+5} + \cdots + x^2_9 = \hat{m}^2$.



- The Dp-brane background in the decoupling limit reads

Itzhaki
Maldacena
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$$\boxed{\begin{aligned} ds^2 &= \alpha'\left(\frac{U^{(7-p)/2}}{\lambda^{1/2}}dx_{\parallel}^2 + \frac{\lambda^{1/2}}{U^{(7-p)/2}}dU^2 + \lambda^{1/2}U^{(p-3)/2}d\Omega_{8-p}^2\right), \\ e^{\Phi} &= (2\pi)^{p-2}g_{\mathrm{YM}}^2\left(\frac{\lambda}{U^{7-p}}\right)^{(3-p)/4}, \end{aligned}} \tag{0.2}$$

where $\lambda = g_{\mathrm{YM}}^2 N d_p$.

- We decompose the $\mathbf{S}^{8-p}$ metric as

$$d\Omega_{8-p}^2 = d\psi^2 + \sin^2\psi d\Omega_3^2 + \cos^2\psi d\Omega_{4-p}^2. \tag{0.3}$$

That is, we parameterize the transverse directions of Dp as

$$\mathbf{S}^3:\ x_{p+1}^2 + \cdots + x_{p+4}^2 = U^2\sin^2\psi, \quad \mathbf{S}^{4-p}:\ x_{p+5}^2 + \cdots + x_9^2 = U^2\cos^2\psi. = \hat{m}^2 \tag{0.4}$$

The D$(p+4)$ probe wraps the $\mathbf{S}^3$ and its transverse directions are given by ψ and $\mathbf{S}^{4-p}$. By construction, the probe brane is living at

$$\boxed{\psi = \cos^{-1}\left(\frac{\hat{m}}{U}\right), \quad \text{a single point in } \mathbf{S}^{4-p}.} \tag{0.5}$$

- Now we will check that this is a solution of the D$(p+4)$ probe action in the Dp-brane background. We use the static gauge

$$\sigma^{0,1,\cdots,p} = x_{0,1,\cdots,p}, \quad \sigma^{p+1,p+2,p+3} \in \mathbf{S}^3, \quad \sigma^{p+4} = U, \tag{0.6}$$

- The DBI action of the D$(p+4)$ probe is given by

$$\boxed{S_{\mathrm{D}(p+4)} = \mu_{p+4}\int d^{p+5}\sigma\, e^{-\Phi}\sqrt{-\det h_{ab}}}$$

$$S_{DBI} = \mu_{p+4}\, \alpha'^{(p+5)/2}\, \mathrm{vol}(\mathbf{S}^3)\mathrm{vol}(\mathbf{R}^{1,p})\, (2\pi)^{p-2} g_{\mathrm{YM}}^{-2} \int dU\, U^3 \sin^3\psi \sqrt{1+U^2\psi'^2} \tag{0.7}$$

where $\mu_p = 1/(2\pi)^p \alpha'^{(p+1)/2}$.

- The equation of motion is

$$\partial_u \left(\frac{u^5 sin^3\psi}{\sqrt{1+u^2\psi')^2}} \right) - 3u^3 sin^2\psi cos\psi \sqrt{1+u^2\psi')^2} = 0$$

One can easily check that indeed $\psi = \cos^{-1}\left(\frac{m}{U}\right)$ is a solution of this equation of motion.

The Lab:

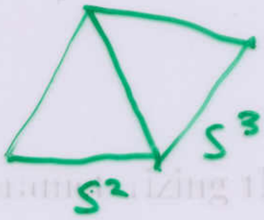
Confinement from supergravity

The Klebanov Strassler model (KS)-Deformed Conifold backgrounc

- The conifold is defined by

$$\sum_{i=1}^{4} z_i^2 = 0, \; or \quad detW = 0$$

$$W = \begin{pmatrix} \overbrace{z_1+iz_2}^{w_1} & \overbrace{z_3+iz_4}^{w_3} \\ \underbrace{z_3-iz_4}_{w_4} & \underbrace{z_1-iz_2}_{w_2} \end{pmatrix}$$



- The base of the cone is

$$T^{11} = \frac{SU(2)\times SU(2)}{U(1)}, \; topology \; of S^2 \times S^3$$

- Place $\tilde{N}$ D_3 branes at the singularity

Klebanov-Witten

Dual to

$\mathcal{N}=1$ $SU(\tilde{N})\times SU(\tilde{N})$ superconformal theory.

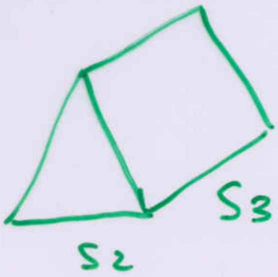
Klebanov-Tseytlin

- Add N D_5 branes wrapping the S^2,namely, N fractional D_3 branes. Dual to $\mathcal{N}=1 SU(\tilde{N}+N)\times SU(\tilde{N})$ theory with 2 chiral multiplets in $(\tilde{N}+N,\bar{\tilde{N}}) + (\overline{\tilde{N}+N},\tilde{N})$

KS

- Deform the conifold by blowing up the S^3

$$\boxed{\sum_{i=1}^{4} z_i^2 = \epsilon^2 = m^3}$$



Reminder: The deformed conifold and the KS model

- The KS background is based on adding fractional D3 branes into the deform conifold. The original KS metric

$$ds^2_{10} = h^{-1/2} dx^2_\mu + h^{1/2} dx^2_6, \tag{0.8}$$

made use of the metric of a deformed conifold of Candelas and de la Ossa.

- It turns out that for our purposes it is more convenient to use the formulation of GPSS since it admits a separation between the three cycle and two cycle of the deformed conifold. It is given by Gimon, Pando Zayas J-S, strassler

$$\begin{aligned}
\epsilon^{-4/3} ds^2_6 =& \frac{1}{4} K(\tau)\, \cosh(\tau)\, \left(d\tau^2 + (\omega^a)^2\right) \\
&+K(\tau)\, \sinh^2(\frac{\tau}{2}) \Big[\,(d\theta^2 + \sin^2\theta\, d\phi^2) - (\sin\phi\,\omega^1 + \cos\phi\,\omega^2)(d\theta) \\
&\quad - (\cos\theta\cos\phi\,\omega^1 - \cos\theta\sin\phi\,\omega^2 - \sin\theta\,\omega^3)(\sin\theta\, d\phi)\Big] \\
&+\frac{1}{4} K'(\tau)\, \sinh(\tau)[\, d\tau^2 \; + \; (\sin\theta\cos\phi\,\omega^1 + \sin\theta\sin\phi\,\omega^2 + \cos\theta\,\omega^3)^2]
\end{aligned}$$

Here

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau}, \quad h(\tau) = (g_s M\alpha')^2\, 2^{2/3}\, \epsilon^{-8/3} I(\tau), \tag{0.10}$$

$$I(\tau) = \int_\tau^\infty dx\, \frac{x\coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3}\,. \tag{0.11}$$

- For instance it is easy to verify in this formulation that for $\tau = 0$, the 6d metric reduces to $\Sigma(\omega^a)^2$, giving the $\mathbf{S}^3$ while the $\mathbf{S}^2$ shrinks to zero.

The additional fields of the background

- The dilaton is constant

$$e^{\Phi} = \textit{constant}$$

- F_5 associates with the regular D_3 branes

$F_5 = B_2 \wedge dC_2$

$$\boxed{\int_{T^{1,1}} F_5 = \tilde{N} + \textit{log correction}}$$

- and G_3 ($G_3 = H_3^{NSNS} + iF_3^{RR}$) with the fractional D_3 branes

$$\boxed{\int_{S^3} F_3 = N}$$

- The two gauge couplings of the $SU(\tilde{N}+N)$ and of the $SU(\tilde{N})$ are given by

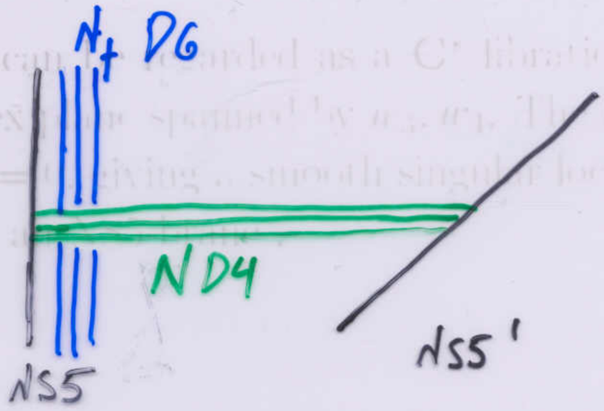
$$\boxed{\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim e^{-\phi}[(\int_{S^2} B_2) - \frac{1}{2}] \sim \tilde{N}\ln(u/u_0), \quad \frac{1}{g_1^2} + \frac{1}{g_2^2} \sim e^{-\phi} = const.}$$

This reproduces the logarithmic running of the gauge couplings expected in $\mathcal{N} = 1$ gauge theories.

Fundamental quarks from D7 probe branes

Elitzur, Giveon, Kutasov

- Recall the brane configuration that describes $N = 1$ $SU(N)$ SQCD with N_f flavors.

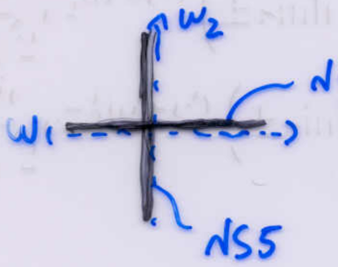


- Let us now review briefly some geometrical aspects of the deformed conifold.

- The conifold is in fact two separate cones that intersect with each other at the tips of the cones.

- The T-dual of the conifold gives us the IIA brane configuration that consists of two perpendicular NS5 branes, one at $w_1 = 0$ and the other at $w_2 = 0$.

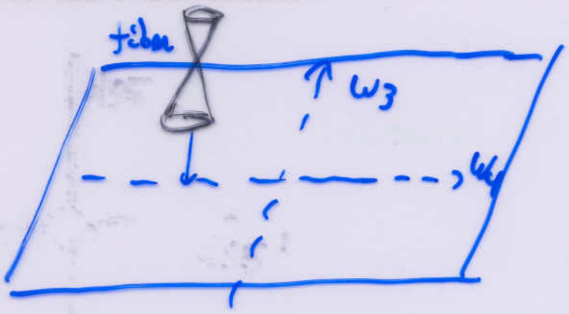
Ooguri: Vafa



- The deformed conifold is defined by

$$\boxed{w_1 w_2 - w_3 w_4 = -\frac{\epsilon^2}{2}}. \tag{0.23}$$

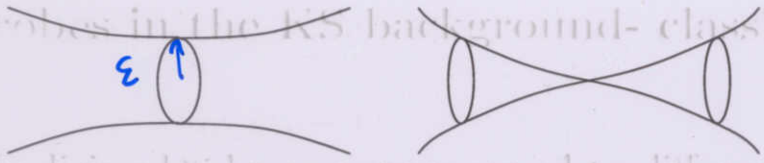
- The deformed conifold can be regarded as a $\mathbf{C}^*$ fibration over a two-dimensional complex plane spanned by w_3, w_4. The fibers get degenerate when $w_1 w_2 = 0$, giving a smooth singular locus in the base which is T-dual to an NS5-brane .



- It turns out that the condition $w_1 w_2 = 0$ can be solved as (1) $\theta = 0,\ \theta' = \pi$, (2) $\theta = \pi,\ \theta' = \pi$. Each corresponds to a cylinder that intersect with one another at a circle. To see this, note that for the two cases one finds

$$\begin{aligned} w_1 &= w_2 = 0, \\ w_3 &= \frac{i\epsilon}{\sqrt{2}} e^{\frac{i}{2}(\phi' - \psi')} \left(\pm \sinh \frac{\tau}{2} - \cosh \frac{\tau}{2} \right), \\ w_4 &= \frac{i\epsilon}{\sqrt{2}} e^{-\frac{i}{2}(\phi' - \psi')} \left(\pm \sinh \frac{\tau}{2} + \cosh \frac{\tau}{2} \right). \end{aligned} \tag{0.24}$$

- These denote two cylinders spanned by $\tau, \phi' - \psi'$ that intersect with each other at a circle at $\tau = 0$. Recall that the circle is embedded in the $\mathbf{S}^3$ and has a radius



(a) (b)

Figure 1: (a) the singular locus of a deformed conifold. (b) the singular locus of a conifold.

proportional to ϵ. This circle corresponds to a "diamond" in the T-dual picture.

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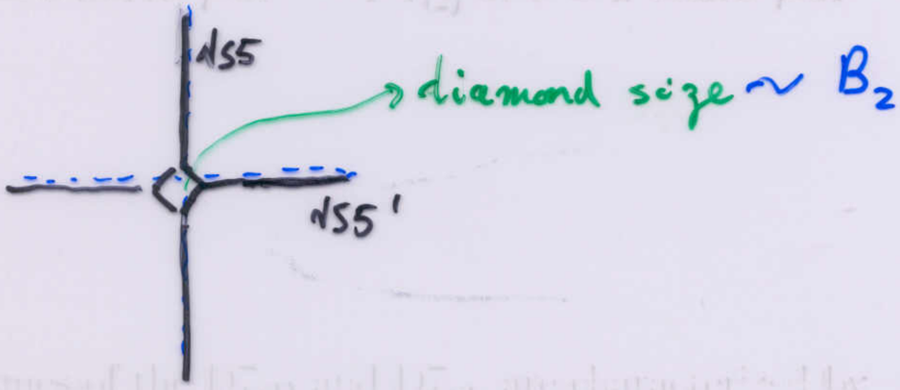


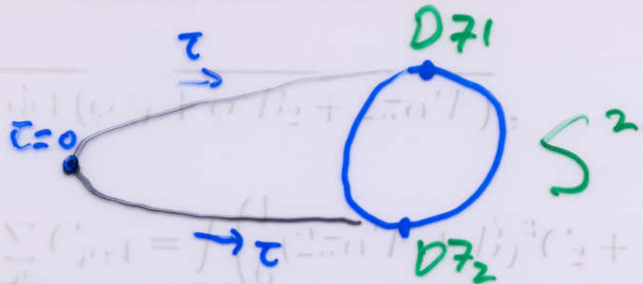
D7 brane probes in the KS background- classical solution

- Upon T-dualizing D6-branes are mapped to different D7-branes

  $D7_{(1)}$ *intersects the locus* $w_1 = 0$; $D7_{(2)}$ *intersects the locus* $w_2 = 0$

  In the deformed conifold geometry the two D7 branes are

  $$\boxed{D7_{(1)}\ at\ \theta = 0\ north\ pole \qquad D7_{(2)}\ at\ \theta = \pi\ south\ pole}$$



- The world volumes of the $\mathrm{D7}_{(1)}$ and $\mathrm{D7}_{(2)}$ are characterized by

$$w^{(1)}_{1,2} = \frac{\epsilon}{\sqrt{2}} \cos\frac{\theta'}{2} e^{\pm\frac{i}{2}(\phi'+\psi')} \left(\sinh\frac{\tau}{2} \pm \cosh\frac{\tau}{2}\right) ,$$
$$w^{(1)}_{3,4} = \frac{i\epsilon}{\sqrt{2}} \sin\frac{\theta'}{2} e^{\pm\frac{i}{2}(\phi'-\psi')} \left(\sinh\frac{\tau}{2} \mp \cosh\frac{\tau}{2}\right) , \quad (0.25)$$

and

$$w^{(2)}_{1,2} = \frac{\epsilon}{\sqrt{2}} \cos\frac{\theta'}{2} e^{\pm\frac{i}{2}(\phi'+\psi')} \left(-\sinh\frac{\tau}{2} \pm \cosh\frac{\tau}{2}\right) ,$$
$$w^{(2)}_{3,4} = \frac{i\epsilon}{\sqrt{2}} \sin\frac{\theta'}{2} e^{\pm\frac{i}{2}(\phi'-\psi')} \left(-\sinh\frac{\tau}{2} \mp \cosh\frac{\tau}{2}\right) . \quad (0.26)$$

It is easy to see that

$$\boxed{\omega_i^{(1)}(\tau = 0) = \omega_i^{(2)}(\tau = 0) \ , \quad \partial_\tau \omega_i^{(1)}(\tau = 0) = -\partial_\tau \omega_i^{(2)}(\tau = 0)} . \quad (0.27)$$

- The linear term in θ vanishs provided that the $U(1)$ gauge potential on the probe has the form

$$A = A_3(\tau)\omega^3 \ .$$

- It is then not difficult to show that the action becomes

$$\boxed{\begin{aligned} S_{\rm DBI} &= -\frac{\mu_7 R^8}{g_s}\int d^8X\sqrt{-\det(g_{(8)} + R^{-2}B_2^{(0)})} \\ &\quad\times\left[\left(1 + \frac{4}{I^{1/2}(K\sinh\tau)'}\,g^{\tau\tau}\partial_\tau\widetilde{A}_3\,\partial_\tau\widetilde{A}_3\right)\left(1 - \frac{8x}{I^{1/2}J}\widetilde{A}_3 + \frac{16}{IJ}\widetilde{A}_3^2\right)\right]^{1/2}(1 + \theta^2) \\ S_{\rm WZ} &= -\frac{\mu_7 R^8 m^4}{2^4 g_s}\int d^4x\,d\tau\,\omega^1\omega^2\omega^3\,(1 + \mathcal{O}(\theta^2)) \\ &\quad\times\left[2^{-2/3}(f+k)(f'+k')I^{-1} + (2^{5/3}(f'+k')I^{-1} + 2^3\,\partial_\tau((f+k)u))\,\widetilde{A}_3 - 2^{13/3}u\widetilde{A}_3^2\right. \end{aligned}}$$

Here $\widetilde{A}_3 = \frac{2\pi\alpha'}{R^2}A_3$ and

$$g_{(8)\alpha\beta}dX^\alpha dX^\beta = g_{IJ}dX^I dX^J + \frac{I^{1/2}}{4}\left(K\cosh\tau\left((\omega^1)^2 + (\omega^2)^2\right) + (K\sinh\tau)'(\omega^3)^2\right)$$

$$R^{-2}B_2^{(0)} = -\frac{f+k}{2^{7/3}}\,\omega^1\wedge\omega^2 \ .$$

with

$$g_{IJ}dX^I dX^J = m^2 I^{-1/2}dx_\mu^2 + \frac{I^{1/2}}{4}\,(K\sinh\tau)'\,d\tau^2 \ . \qquad (0.34)$$

Also

$$J = (K\cosh\tau)^2 + x^2 \ , \quad x = \frac{f+k}{2^{4/3}I^{1/2}} \ . \qquad (0.35)$$

These relations guarantee the smoothness at $\tau = 0$.

- Are these brane configurations solutions of the equations of motion?

- Let us now analyze the D7-probe action in the KS background. The action consists of two parts

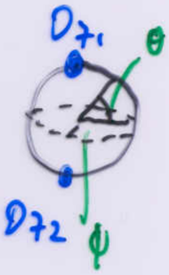
$$S = S_{\rm DBI} + S_{\rm WZ}\,, \tag{0.28}$$

where

$$S_{\rm DBI} = -\mu_7 \int d^8\sigma\, e^{-\Phi}\sqrt{-\det\left(\phi^* g + \phi^* B_2 + 2\pi\alpha' F\right)}\,, \tag{0.29}$$

$$S_{\rm WZ} = -\mu_7 \int e^{2\pi\alpha' F + B} \wedge \sum_p C_{p+1} = \int \left(\frac{1}{6}(2\pi\alpha' F + B)^3 C_2 + (2\pi\alpha' F + B) C_6\right) \tag{0.30}$$

Here $\phi^* g, \phi^* B_2$ are the pull-backs of g, B_2. F is the field strength. Φ is the dilaton and $\mu_7 = 1/(2\pi)^7 \alpha'^4$.
The two transverse coordinates of a probe D7 are taken to be θ, ϕ while upon taking the static gauge the world volume coordinates are given by $X^\alpha = (X^I, \theta', \phi', \psi')$, with $X^I = (x^\mu, \tau)$.



- What we should show is that the $D7_{(1)}$ given by $\theta = 0$, any ϕ, and the $D7_{(2)}$ given by $\theta = \pi$, any ϕ solve the equation of motion.

- In order to check first the $D7_{(1)}$, brane we assume $\theta = \text{const}$, $\phi = \phi(X^\alpha)$, $F \neq 0$ and expand the action around $\theta = 0$. ~~It turns out~~

- The solutions satisfies the consistency condition of the RR-flux cancellation. In the 10d formulation

$$D7_{(1)} \; has \; RR \; charge \; = 1 \qquad D7_{(2)} \; has \; RR \; charge \; = -1$$

(~~In the 5d formulation one can check that the tension vanishes at the boundaries $\theta' = 0, \pi$ of the D7 branes~~)

- To summarize, the probe D7-brane configuration looks like the following fig..

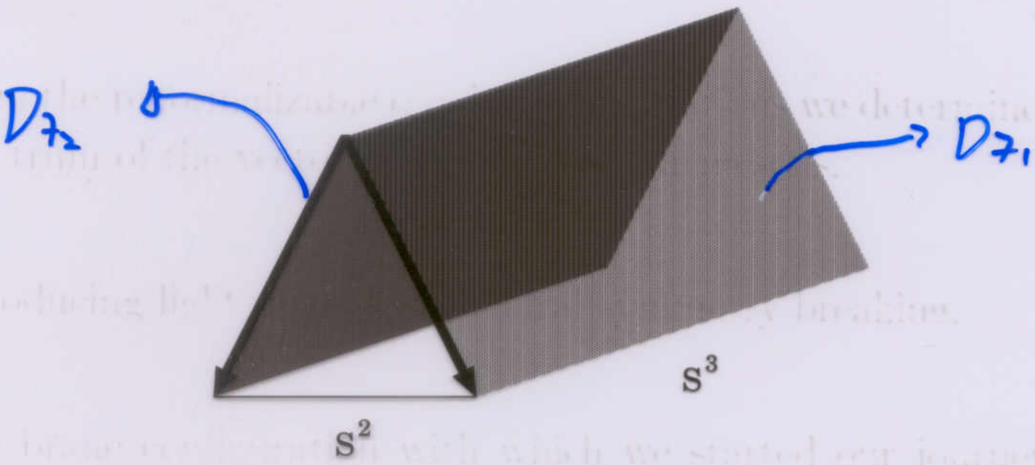


Figure 2: D7-probe configuration in a deformed conifold.The vectors denote the singular locus where the elliptic fibers get degenerate. The two shadowed surfaces that intersect with each vector are the 7-brane probes.

- What about the masses of the fundamental quarks?
Since we have not introduced an additional mass parameter they must be of the mass scale m. In the language of the T dual picture the deformation of the conifold prevents the 4-6 strings in the "diamond" construction to get masses ligher than $\sim \frac{\epsilon^{2/3}}{\alpha'}$

Quadratic fluctuations around the probe configuration

- Consider now the fluctuation around the $D7_{(1)}$-branes. Assume a configuration

$$\theta = \theta(X^I), \quad \phi = \phi(X^I) \,. \tag{0.33}$$

- The 8d scalars θ and ϕ KY reduce to the 5d scalars via a constant mode on M_3.

  The constant mode is the normalizable zero mode of the scalar harmonics as M_3 is compact.

- The 8d vector A is assumed to be general for the moment.

- Up to quadratic terms in the fluctuations, the DBI action is

$$\begin{aligned} S_{\rm DBI} = &-\frac{\mu_7 R^8 m^4}{2^4\, g_s}\int d^5X\, \omega^1\omega^2\omega^3\, (K\sinh\tau)' J^{1/2} \\ &\times\Big[1 + \frac{L}{2}\, g^{IJ}\partial_I\theta\partial_J\theta + \frac{1}{4}\left(\frac{2\pi\alpha'}{R^2}\right)^2 \left(g^{IJ}g^{KL}F_{IK}F_{JL} + u^{ab}u^{dc}F_{ac}F_{bd}\right) \\ &\quad + \frac{1}{2}\left(\frac{2\pi\alpha'}{R^2}\right)^2 g^{IJ}u^{ab}\,\partial_I A_a\,\partial_J A_b + \frac{1}{8}\left(\frac{2\pi\alpha'}{R^2}\right)^2 \frac{2^6 x^2}{IJ^2}F_{12}^2\Big] \end{aligned} \tag{0.34}$$

Here

$$L = \frac{K\cosh\tau}{4I^{1/2}J}\left[IK^2\sinh^2\tau + \frac{1}{2^{8/3}}(\tau\coth\tau - 1)^2\right] \,. \tag{0.35}$$

$a, b = 1, 2, 3$ are the indices of the orthonormal frame of $\mathbf{S}^3$,

u_{ab} is defined by the 3d part of the tensor $g_{(8)} + R^{-2}B_2^{(0)}$ and given by

$$u_{ab} = \frac{I^{1/2}}{4}\begin{pmatrix} K\cosh\tau & -x & 0 \\ +x & K\cosh\tau & 0 \\ 0 & 0 & (K\sinh\tau)' \end{pmatrix}, \qquad (0.36)$$

with

$$x = \frac{f+k}{2^{4/3}I^{1/2}}\,. \qquad (0.37)$$

- The fluctuations from the WZ term up to quadratic order are proportional to

$$\boxed{\int d^4x d\tau\, \omega^1\omega^2\omega^3\,(f'+g')h^{-1}\,(F_{12}+\theta F_{23})\ .} \qquad (0.38)$$

Here we omit the zero-th order term, namely, the contribution to the tension.

- Note that both terms are total derivatives since $\theta(X^I)$ and hence, the WZ term yields no quadratic fluctuations.

- Note that $\theta(X^I)$ is massless and has no mixing terms with any other fields. Hence $\theta(X^I)$ couples to the lowest-lying pseudo-scalar mesons and A_I to vector mesons.

- It is interesting to notice that the kinetic term of θ is always positive. This implies the stability of the probe configuration at hand.

Vector fluctuations

- Let us first study the 5d vector fluctuations.

  The action of the 5d gauge potential is

$$\boxed{S_{\mathrm{v}} \sim m^4 \int d^5X\, (K \sinh \tau)' \sqrt{J}\, g^{IJ} g^{KL} F_{IK} F_{JL}.} \tag{0.39}$$

- The equation of motion reads

$$\partial_J((K \sinh \tau)' \sqrt{J}\, g^{IJ} g^{KL} F_{IK}) = 0. \tag{0.40}$$

  For the $L = \tau$ component, this becomes

$$0 = \eta^{\mu\nu} \partial_\mu (\partial_\nu A_\tau - \partial_\tau A_\mu). \tag{0.41}$$

  It is useful to work in the gauge $A_\tau = 0$.

- We next decompose the rest components of the gauge potential in terms of the complete set of functions $\chi_n(\tau)$:

$$\boxed{A_\mu(x, \tau) = \sum_n A_\mu^{(n)}(x) \chi_n(\tau).} \tag{0.42}$$

  Substituting the decomposition into the Gauss law constraint, we obtain

$$0 = \sum_n \eta^{\mu\nu} \partial_\mu A_\nu^{(n)}(x)\, \partial_\tau \chi_n(\tau). \tag{0.43}$$

  As we will see later, the constant mode is not normalizable so that we find

$$\eta^{\mu\nu} \partial_\mu A_\nu^{(n)} = 0. \tag{0.44}$$

- Substituting the mode expansion into the action, we obtain

$$\boxed{\begin{aligned} S_{\rm v} = & \int d^4x\, d\tau\, (K\sinh\tau)'\,\sqrt{J} \\ & \sum_{n,m} \Big[I\, \eta^{\mu\nu}\eta^{\rho\sigma} F^{(n)}_{\mu\rho} F^{(m)}_{\nu\sigma} \chi_n(\tau)\chi_m(\tau) + \frac{8m^2}{(K\sinh\tau)'}\, \eta^{\mu\nu}\, A^{(n)}_\mu A^{(m)}_\nu\, \partial_\tau\chi_n(\tau)\, \partial_\tau\chi_m(\tau) \end{aligned}} \qquad (0.$$

- Now we define χ_n as the solution of the differential equation

$$-\frac{1}{\sqrt{\gamma}}\,\partial_\tau \left(\frac{\sqrt{\gamma}}{(K\sinh\tau)'\, I(\tau)}\, \partial_\tau\chi_n \right) = \lambda_n\, \chi_n(\tau), \qquad (0.46)$$

with the normalization condition given by

$$\int_0^\infty d\tau\, \sqrt{\gamma}\, \chi_n(\tau)\chi_m(\tau) = \delta_{n,m}\,. \qquad (0.47)$$

Here

$$\sqrt{\gamma} = (K\sinh\tau)'\, I(\tau)\, \sqrt{J(\tau)}\,. \qquad (0.48)$$

- It then follows that the action becomes

$$\boxed{S_{\rm v} = \sum_n \int d^4x [\eta^{\mu\nu}\eta^{\rho\sigma} F^{(n)}_{\mu\rho} F^{(n)}_{\nu\sigma} + 8m^2 \lambda_n \eta^{\mu\nu} A^{(n)}_\mu A^{(n)}_\nu].} \qquad (0.49)$$

- Thus the fields A satisfy the on-shell condition with the mass square given by

$$\boxed{M_n^2 = 4\lambda_n m^2.} \qquad (0.50)$$

- We regard this as the mass square of the vector mesons of QCD.

- The differential equation allows two independent solutions: one is normalizable and the other non-normalizable. We are interested in the normalizable solution ~~here.~~

- It turns out that the normalizable solution should behave as

$$\chi(\tau) = e^{-\frac{2\tau}{3}} f(\tau), \tag{0.51}$$

with the boundary behavior of f

$$f(\tau \to \infty) = \text{const.} \tag{0.52}$$

- It is easy to verify that f obeys the differential equation

$$\boxed{f''(\tau) + A(\tau) f'(\tau) + (B(\tau) + \lambda C(\tau)) f(\tau) = 0,} \tag{0.53}$$

with

$$\boxed{\begin{aligned} A &= \frac{1}{2}\,\partial_\tau \log J - \frac{4}{3}\,, \\ B &= -\frac{2}{3}\,\partial_\tau \log J + \frac{4}{9}\,, \\ C &= (K \sinh \tau)' \, I(\tau)\,. \end{aligned}} \tag{0.54}$$

Csaki, Ooguri, Oz Terning

- We will solve this differential equation numerically following the procedure used to find the glueball spectrum of QCD.

- We first find out the asymptotic behavior of the solution at $\tau \gg 1$ for a generic λ. Using the data as an input, the solution can be found numerically.

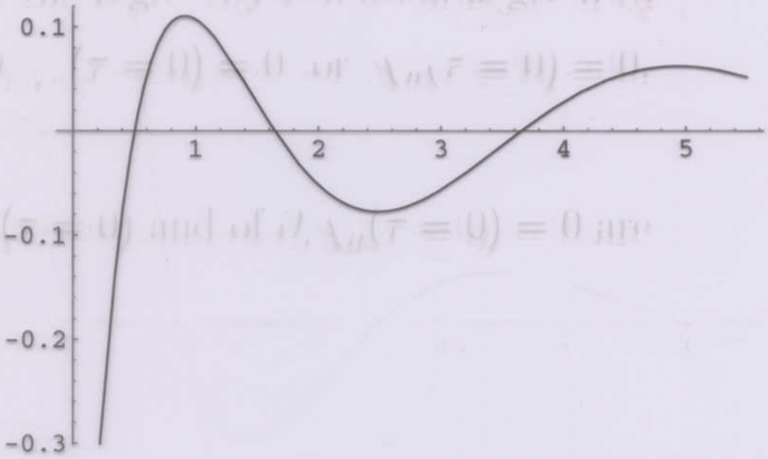


Figure 1: $\partial_\tau \chi(\tau = 10^{-7})$ as a function of λ

- By imposing a regularity condition at $\tau = 0$ to be discussed in a moment, only solutions with appropriate values of λ are allowed.

- In order to obtain the asymptotic solution, we notice that for large τ

$e^{-2n\tau}$

$$A(\tau) = \sum_{n=0,1,\cdots} a_n(\tau)e^{-2n\tau},\ B(\tau) = \sum_{n=0,1,\cdots} b_n(\tau)e^{-2n\tau},\ C(\tau) = e^{-\frac{2\tau}{3}} \sum_{n=0,1,\cdots} c_n(\tau)$$

where

$$\begin{aligned} &a_0 = -\frac{2}{3},\quad a_1 = \frac{50}{3},\quad a_2 = \frac{8\tau}{3},\quad \cdots \\ &b_0 = 0,\quad b_1 = -\frac{100}{9},\quad b_2 = -\frac{16\tau}{9},\quad \cdots \\ &c_0 = \tau,\quad c_1 = \frac{4\tau^2}{3},\quad c_2 = \frac{32\tau^3}{9},\quad \cdots \end{aligned} \tag{0.56}$$

- Expanding $f(\tau)$ as follows

$$f(\tau) = \sum_{n=0,1,\cdots} f_n(\tau)e^{-\frac{2n\tau}{3}}, \tag{0.57}$$

it follows from (0.53) that the coefficients f_n obey the recursion relation

$$\boxed{\begin{aligned} f_n'' - \frac{4n}{3}f_n' + \frac{4n^2}{9}f_n + \sum_{m=0}^{[\frac{n}{3}]} \left[a_m\left(f_{n-3m}' - \frac{2}{3}(n-3m)\,f_{n-3m}\right) + b_m f_{n-3m}\right] \\ + \lambda \sum_{m=0}^{[\frac{n-1}{3}]} c_m f_{n-3m-1} = 0\,. \end{aligned}} \tag{0.58}$$

By setting $f_0 = 1$, the solution is given by

$$\boxed{f_0 = 1,\quad f_1 = -\frac{9\lambda\tau}{8},\quad f_2 = \frac{27\lambda^2\tau^2}{64},\quad f_3 = \frac{25}{12} - \frac{81\lambda^3\tau^3}{1024},\quad \cdots} \tag{0.59}$$

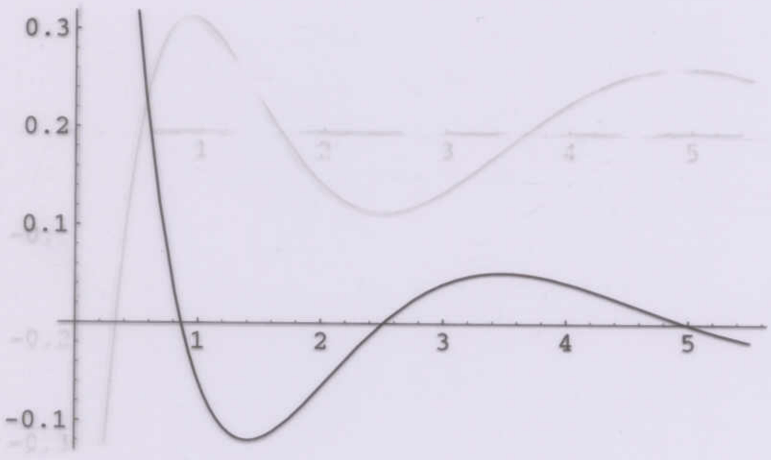


Figure 2: $\chi(\tau = 10^{-7})$ as a function of λ

- Now let us discuss what is the regulatory condition to be imposed at $\tau = 0$.

  We denote by $\chi_n^{(1)}(\tau)$, $\chi_n^{(2)}(\tau)$ ~~by~~ the solutions on $D7_{(1)}$ and $D7_{(2)}$, respectively. Since both the solutions obey the same differential equation with the same asymptotic behavior, the two solutions are related as

$$\boxed{\chi_n^{(1)}(\tau) = \pm\chi_n^{(2)}(\tau).} \tag{0.60}$$

  This shows that the regularity condition is given by

$$\boxed{\partial_\tau \chi_n(\tau = 0) = 0 \ \text{ or } \ \chi_n(\tau = 0) = 0.} \tag{0.61}$$

- The plots of $\chi_n(\tau = 0)$ and of $\partial_\tau \chi_n(\tau = 0) = 0$ are

- Thus the allowed values of the eigenvalue λ are

$$\lambda = 0.50,\ 0.86,\ 1.64,\ 2.50,\ 3.66,\ 4.98,\ \cdots \tag{0.62}$$

  which give us the mass spectrum of vector mesons

$$\boxed{\frac{M^2}{m^2} = 2.00,\ 3.44,\ 6.56,\ 10.0,\ 14.6,\ 19.9,\ \cdots.} \tag{0.63}$$

  Thus we see that the spectrum of vector mesons is discrete with a mass gap which is of the order of

$$\boxed{m = m_{gb} = \frac{e^{2/3}}{2^{1/3}(g_s M \alpha')}}$$

Pseudo scalar fluctuations

- Let us now compute the scalar meson spectrum by analyzing the fluctuation of θ. We first analyze the 5d scalars that emerge from the 8d scalars

- As shown in the DBI action , the fluctuation of θ on both of the probe D7-branes is governed by

$$\boxed{S_{\rm sc} \sim m^2 \int d^4x d\tau (K\sinh\tau)' \sqrt{J}\, \frac{L}{2} g^{IJ} \partial_I \theta\, \partial_J \theta\ .} \tag{0.64}$$

- We decompose $\theta(X^I)$ in terms of the complete set of appropriate functions $\xi_n(\tau)$.

$$\theta(x^\mu, \tau) = \sum_n \theta^{(n)}(x)\, \xi_n(\tau), \tag{0.65}$$

that are solution of

$$-\frac{1}{\sqrt{\rho}} \partial_\tau \left(\frac{\sqrt{\rho}}{(K\sinh\tau)' I} \partial_\tau \xi \right) = \alpha_n \xi_n. \tag{0.66}$$

with the normalization condition given by

$$\int_0^\infty d\tau \sqrt{\rho}\, \xi_n(\tau)\, \xi_m(\tau) = \delta_{nm}, \tag{0.67}$$

where

$$\sqrt{\rho} = (K\sinh\tau)' \sqrt{IJ}\, L, \tag{0.68}$$

- Now the action becomes

$$\boxed{S_{\rm sc} = \frac{1}{2} \int d^4x \sum_n [\eta^{\mu\nu} \partial_\mu \theta^{(n)} \partial_\nu \theta^{(n)} + 4\alpha_n m^2 \theta^{(n)} \theta^{(n)}].} \tag{0.69}$$

Thus we obtain the scalar mesons with the mass square

$$\boxed{M_n^2 = 4\alpha_n m^2.} \tag{0.70}$$

- In a similar manner to the vector case we impose an asymptotic behavior ξ

$$\xi(\tau) = e^{-\frac{4\tau}{3}} g(\tau) \ , \tag{0.71}$$

with

$$g(\tau \to \infty) = \text{const.} \tag{0.72}$$

so that the solutions are normalizable

- We find that g obeys the differential equation

$$g''(\tau) + D(\tau)g'(\tau) + (E(\tau) + \alpha C(\tau))g(\tau) = 0, \tag{0.73}$$

with

$$\begin{aligned} D &= \frac{1}{2}\partial_\tau \log\left(\frac{\sqrt{\rho}}{(K \sinh \tau)' I}\right) - \frac{8}{3} \ , \\ E &= -\frac{4}{3}\left(D + \frac{4}{3}\right) \ . \end{aligned} \tag{0.74}$$

- As before, we first solve the asymptotic behavior of g for a generic α. For that, we need the asymptotic behavior of the coefficients D, E:

$$D = \sum_{n=0,1,\cdots} e^{-2n\tau} d_n \ , \quad E = \sum_{n=0,1,\cdots} e^{-2n\tau} e_n \ , \tag{0.75}$$

where

$$d_0 = -\frac{4}{3}\,, \quad d_1 = \frac{8\tau}{3}\,, \quad d_2 = \frac{32\tau^2}{3}\,, \cdots$$
$$e_0 = 0 \quad e_1 = -\frac{32\tau}{9}\,, \quad e_2 = -\frac{128\tau^2}{9}\,, \cdots \tag{0.76}$$

- Expanding $g(\tau)$ as

$$g(\tau) = \sum_{n=0,1,\cdots} g_n(\tau) e^{-\frac{2n\tau}{3}}, \tag{0.77}$$

it follows from the differential equations that the coefficients g_n obey the recursion relation

$$g_n'' - \frac{4n}{3} g_n' + \frac{4n^2}{9} g_n + \sum_{m=0}^{[\frac{n}{3}]} \left[d_m \left(g_{n-3m}' - \frac{2}{3}(n-3m)\, g_{n-3m} \right) + e_m f_{n-3m} \right]$$
$$+ \alpha \sum_{m=0}^{[\frac{n-1}{3}]} c_m g_{n-3m-1} = 0\,. \tag{0.78}$$

By setting $g_0 = 1$, the solution is given by

$$g_0 = 1, \quad g_1 = -\frac{3\alpha\tau}{4}, \quad g_2 = \frac{27\alpha^2\tau^2}{128}, \quad g_3 = \frac{8\tau}{15} - \frac{81\alpha^3\tau^3}{2560}\,, \quad \cdots \tag{0.79}$$

Using this, we solve the differential equations numerically. As before, we have to impose the regularity condition at $\tau = 0$:

$$\boxed{\xi(\tau = 0) = 0\,, \quad \text{or} \quad \partial_\tau \xi(\tau = 0) = 0\,.} \tag{0.80}$$

It turns out that now the values of α that follow from both conditions are identical since θ has to vanish at $\tau = 0$.

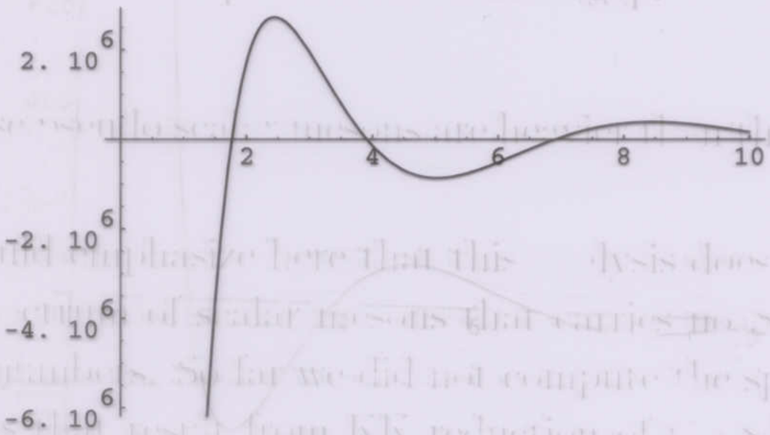


Figure 3: $\partial_\tau \xi(\tau = 0.0001)$ as a function of α

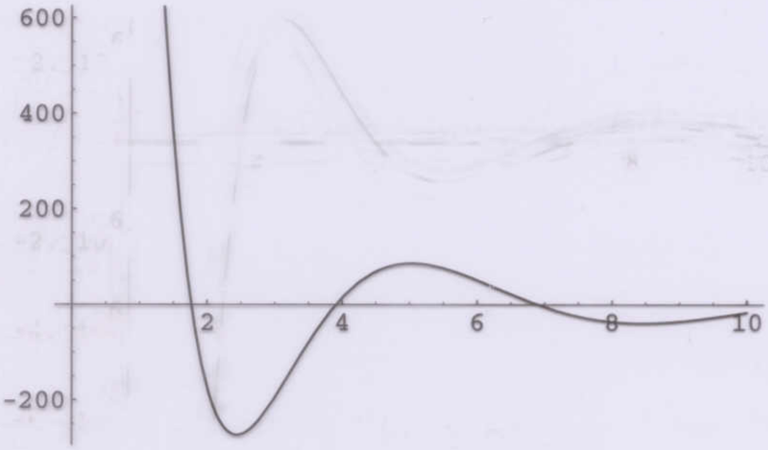


Figure 4: $\xi(\tau = 0.0001)$ as a function of α

- We obtain from this the following eigenvalues

$$\alpha = 1.77,\ 3.91,\ 6.90,\ \cdots \tag{0.81}$$

which give us the mass spectrum of scalar mesons

$$\boxed{\frac{M^2}{m^2} = 7.08,\ 15.6,\ 27.6,\ \cdots.} \tag{0.82}$$

- Again we found a discrete spectrum we a mass gap.

- Notice that these pseudo scalar mesons are heavier than the vector mesons.

  However, it should emphasize here that this analysis does not exhaust all the spectrum of scalar mesons that carries no $SU(2) \times U(1)$ quantum numbers. So far we did not compute the spectrum of the 5d scalars that result from KK reduction of the 8d vector by a vector harmonics.

  We first recall that the lowest-lying vector harmonics on $\mathbf{S}^3$ transform under the isometry $SU(2) \times SU(2)$ as $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$. M_3 at hand is topologically $\mathbf{S}^3$ and admits as the isometry the subgroup $SU(2) \times U(1)$.

  The lowest-lying vector harmonics then splits into some irreducible representation of $SU(2) \times U(1)$. In particular, one of them is the singlet.

  $\frac{M^2}{m^2} = 5.68,\ 6.52,\ 12.0,\ 15.2 \ldots$

- It is easy to check that in general these scalar fields are massive $(M^2 > 0)$ because of the F^2 terms as well as a non-vanishing eigenvalue of the vector harmonics.

- **Summary of the vector and pseudo scalar mesons**

- Since the fluctuations belong to the adjoint representation of $U(N_f)$, so do the dual meson states.

- Recall also that no state in the adjoint of $U(N_f)$ is charged under the baryon number $U_B(1)$, where $U(N_f) = SU(N_f) \times U_B(1)$.

- There exist two kinds of mesons

  pseudo-scalar and vector mesons.

- We analyzed the vector mesons that follow from the fluctuations of the 8d gauge fields and the pseudo scalars that follow from the fluctuations of the 8d scalars.

- In addition there are pseudo scalar mesons from the 8d vectors. Most of them carry non-trivial $SU(2) \times U(1)$ charges.There is no counterpart of these quantum numbers in the dual QCD. but the 8d vectors yield also singlets.

- As far as our concern is in quadratic fluctuations, we can concentrate on the case $N_f = 1$, since to this order any non-abelian interaction is irrelevant and consequently a field in the adjoint representation of $U(N_f)$ reduces to N_f^2 free fields.

- The non-vector modes are in fact pseudo scalars rather than scalars. Recall that the 8d super Yang-Mills(SYM) theory on a D7 comes from dimensional reduction of 10d SYM. In ten dimensions, the parity transformation for the 10d vector field is defined as

$$\mathcal{P}A(t, \vec{x}_9)\mathcal{P}^{-1} = -A(t, -\vec{x}_9) \; . \tag{0.83}$$

Upon dimensional reduction to 8d, we find that the scalar fields transform as a pseudo-scalar

$$\mathcal{P}\phi(t, \vec{x}_7)\mathcal{P}^{-1} = -\phi(t, -\vec{x}_7) \; . \tag{0.84}$$

- The mass scale of the mesons is the same scale as that of the glueballs

$$\boxed{M_m^2 \sim M_{gb}^2 \sim \frac{\epsilon^{4/3}}{2^{2/3}(g_s M \alpha')^2}}$$

- Recall that the mass of the fundamental quarks $\sim \frac{\epsilon^{2/3}}{\alpha'}$.

Summary and open questions

- We have added D7 brane probes to the KS model to describe massive flavored fundamental quarks

- For $N >> N_f$ backreaction can be neglected.

- We made use of the geometry of defored conifold to determine the probes that are solutions of the equations of motion.

- From the renormalizable quadratic fluctuations we determined the spectrum of the vector and pseudo scalar mesons.

Open questions:

- Introducing light quarks with chiral symmetry breaking.

- The brane configuration with which we started our journey has $\mathcal{N} = 1$ supersymmetry. It is not clear whether ~~the~~ after the introduction of the D7 branes supersymmetry is still preserved.

- It will be interesting to determine the Wilson loop in this context and to realize the proces of breaking of the string due to the dynamical quarks.

- Constructing baryons using wrapped D3 brane connected with N strings to the D7 brane probes.