Effective Actions in N=1 SUSY GT

F. Cachazo, M. Douglas, N. Seiberg, E. Witten

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1) the Chiral Ring

2) Generalization of the Konishi Anomaly

3) Chiral Ring Equations from Anomalies

4) The Effective Action and its connection to Matrix Integrals

3) Chiral Ring Equations from Anomalies

Keep S as the only off shell variable, integrating out all the other fields → consistent only if the other fields are massive, i.e. there is a superpotential:

$$W(\Phi) + \tilde{Q}\, m(\Phi) Q \quad + \ c.c$$

semiclassically:

$$\begin{cases} W'(\Phi) + \tilde{Q}\, m'(\Phi)\, Q = 0 \\ m(\Phi) Q = 0 \qquad \tilde{Q}\, m(\Phi) = 0 \end{cases}$$

Two types of solutions:

a) "Coulomb phase":

$Q = \tilde{Q} = 0$, if $W'(z) = g_n \prod_{i=1}^{n} (z - a_i)$

$$\Phi = \begin{pmatrix} \left.\begin{matrix} a_1 & & \\ & \ddots & \\ & & a_1 \end{matrix}\right\} N_1 & & 0 \\ & & \\ 0 & & \left.\begin{matrix} a_n & & \\ & \ddots & \\ & & a_n \end{matrix}\right\} N_n \end{pmatrix}$$

$\sum_{i=1}^{n} N_i = N$, $U(N)$ broken to

$$\begin{array}{cccc} U(N_1) \times & U(N_2) \times & \ldots \times & U(N_n) \\ S_1 & S_2 & & S_n \end{array}$$ with corresponding $\sum_{i=1}^{n} S_i = S$

b) Higgs phase

$Q = (h, 0, \ldots, 0)$ $\quad\quad$ $\tilde{Q} = (\tilde{h}, 0, \ldots, 0)$

$$\Phi = \begin{pmatrix} z_1 & & \\ & \left.\begin{matrix} a_1 & & \\ & \ddots & \\ & & a_1 \end{matrix}\right\} N_1 & 0 \\ & & \ddots \\ 0 & & \left.\begin{matrix} a_n & & \\ & \ddots & \\ & & a_n \end{matrix}\right\} N_n \end{pmatrix}$$

$m(z_1) = 0$

$\tilde{h} h = - \dfrac{W'(z_1)}{m'(z_1)}$

$\sum_{i=1}^{N} N_i = N-1$

- Define $T(z) \equiv \mathrm{Tr}\,\frac{1}{z-\varphi} \approx$

$$\approx \sum_{i=1}^{n} \frac{N_i}{z-a_i}$$

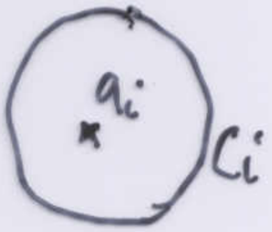
and $R(z) \equiv -\frac{1}{32\pi^2}\,\mathrm{Tr}\,\frac{W_\alpha W^\alpha}{z-\varphi} \approx$

$$\approx \sum_{i=1}^{n} \frac{S_i}{z-a_i}$$

exact definitions (gauge invariant):

$$N_i \equiv \frac{1}{2\pi i}\oint_{C_i} T(z)\,dz$$

→ is it integer?



$$S_i \equiv \frac{1}{2\pi i}\oint_{C_i} R(z)\,dz$$

→ is it a field?

for variation $\delta\Phi = \underbrace{\frac{1}{z-\Phi}}_{f(\Phi, W)}$

$$\sum_{ij}\left[W_\alpha,\left[W^\alpha,\frac{\partial f}{\partial\Phi_{ij}}\right]\right]_{ij} = \underbrace{\mathrm{Tr}\,\frac{W_\alpha W^\alpha}{z-\Phi}}_{\sim R(z)}\,\underbrace{\mathrm{Tr}\,\frac{1}{z-\Phi}}_{T(z)} + \begin{matrix}\text{single W}\\ \text{put to 0}\end{matrix}$$

In correlators $\langle R(z)\; T(z)\rangle$ factorize, i.e. anomaly equation

$$\left\langle \mathrm{Tr}\,\frac{W'(\Phi)}{z-\Phi}\right\rangle + \left\langle \tilde{Q}\,\frac{m'(\Phi)}{z-\Phi}\,Q\right\rangle =$$

$$2\,\langle R(z)\rangle\langle T(z)\rangle$$

- Similarly for

$$\delta\Phi = \underbrace{-\frac{1}{32\pi^2}\,\frac{W_\alpha W^\alpha}{z-\Phi}}_{f(\Phi, W)}$$

$$\langle -\frac{1}{32\pi^2} \mathrm{Tr} \frac{W'(\Phi) W_\alpha W^\alpha}{z-\Phi} \rangle = \langle R(z) \rangle^2$$

- and from

$$\delta Q = \frac{1}{z-\Phi} Q :$$

$$\langle \tilde{Q} \frac{m(\Phi)}{z-\Phi} Q \rangle = \langle R(z) \rangle$$

4) Solution of the Anomaly Equations and connection to Matrix Integrals

$$\langle R(z)\rangle^2 = -\frac{1}{32\pi^2}\,\mathrm{Tr}\,\frac{W'(\varphi)\,W_\alpha W^\alpha}{z-\varphi} =$$

$$= -\frac{1}{32\pi^2}\,W'(z)\,\mathrm{Tr}\,\frac{W_\alpha W^\alpha}{z-\varphi} +$$

$$+ \frac{1}{32\pi^2}\,\mathrm{Tr}\left(\underbrace{\frac{W'(z)-W'(\varphi)}{z-\varphi}}_{\text{n-1 degree polynomial in } z \ \equiv \frac{1}{4}f(z)} W_\alpha W^\alpha\right)$$

$$R(z)^2 = W'(z)\,R(z) + \frac{1}{4}f(z)$$

i.e. $$R(z) = \frac{1}{2}\left[W'(z) - \sqrt{(W'(z))^2 + f}\,\right]$$

n-parameters of $f \iff n : S_i = \frac{1}{2\pi i}\oint_{C_i} R(z)$

poles in $z \longrightarrow$ cuts

Equation for T solvable in terms of n additional parameters $\Longleftrightarrow N_i$

Therefore all the operators of the chiral ring : $\mathrm{Tr}(\Phi^p) \leftrightarrow T(z)$

and $\mathrm{Tr}(W_\alpha W^\alpha \Phi^p) \longleftrightarrow R(z)$

are expressed as functions of S_i, N_i

- Connection to Matrix Model

$$\int d^{\mathcal{N}^2} M \; \exp\left(-\frac{\mathcal{N}}{g_m} \mathrm{tr}\, W(M)\right)$$

Identity from change of variable:

$$M \to M + \epsilon \frac{1}{z-M}$$

$$\text{Jacobian} \to \left(\mathrm{Tr}\left(\frac{1}{z-M}\right)\right)^2$$

$+$ variation exp $= 0$

$$\left(\frac{g_m}{N}\right)^2 \left\langle \left(\mathrm{Tr}\,\frac{1}{z-M}\right)^2 \right\rangle = \frac{g_m}{N}\left\langle \mathrm{Tr}\,\frac{W'(M)}{z-M}\right\rangle$$

't Hooft limit ($N \to \infty$)

there is factorization and

$R_m(z) \equiv \frac{g_m}{N}\left\langle \mathrm{Tr}\left(\frac{1}{z-M}\right)\right\rangle$ satisfies

$$R_m(z)^2 = W'(z)\,R_m(z) + \frac{1}{4} f_m(z)$$

there is ambiguity in the limit:
putting N_i eigenvalues around a_i

f_m corresponds to the parameters

$$g_m \frac{N_i}{N} = \frac{1}{2\pi i}\oint R_m(z)\,dz$$

Correspondence:

Matrix	SUSY GT
$\mathrm{Tr}(M^p)$ numbers	numbers ← Chiral Ring
factorization: large N	Chiral Ring
$g_m \frac{N_i}{N}$	S_i
?	N_i
$\mathrm{Tr}\, \frac{1}{z-M}$	$-\frac{1}{32\pi^2} \mathrm{Tr}\, \frac{W_\alpha W^\alpha}{z-\Phi}$
?	$\mathrm{Tr}\, \frac{1}{z-\Phi}$

The Generating Functional

if $W(\Phi)$ and $W(M)$ correspond to the same $W(z) = \sum_{k=0}^{n} \frac{g_k z^{k+1}}{k+1}$

from $R_m(z)$ we obtain:

$$\langle \mathrm{Tr} \frac{M^{k+1}}{k+1} \rangle = \frac{\partial \tilde{F}_m}{\partial g_k}$$

from $R(z)$ we obtain

$$\langle \mathrm{Tr} \frac{W_\alpha W^\alpha}{k+1} \Phi^k \rangle \approx \sum_{i=1}^{n} \frac{S_i a_i^{k+1}}{k+1}$$

but $\frac{\partial F}{\partial g_k} = \langle \mathrm{Tr} \frac{\Phi^k}{k+1} \rangle \approx \sum_{i=1}^{n} N_i \frac{a_i^{k+1}}{k+1}$

therefore:

$$\underline{\frac{\partial F}{\partial g_k} = \sum_i N_i \frac{\partial}{\partial S_i} \frac{\partial F_m}{\partial g_k}}$$

1) The Chiral Ring

N=1 SUSY GT with U(N) gauge group

Gauge multiplet: $V = -\theta\sigma^\mu\bar\theta A_\mu - i\bar\theta\bar\theta\theta\cdot\lambda + \dots$

(W-Z gauge) in adjoint

$$W_\alpha = -\frac{1}{4}\bar D\bar D\, e^{-V} D_\alpha e^{V} = -i\lambda_\alpha + \dots$$

W_α chiral : $\bar D W_\alpha = 0$

$$W_\alpha W^\alpha = -\lambda_\alpha\lambda^\alpha + \dots + \theta\theta\Big(-2i\lambda\sigma^\mu\partial_\mu\bar\lambda$$

$$-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}F_{\mu\nu}\tilde F^{\mu\nu} + \dots\Big)$$

$$S = -\frac{1}{32\pi^2}\underset{\substack{\uparrow\\ \text{fund}}}{\mathrm{Tr}}\left(W_\alpha W^\alpha\right), \quad \bar D S = 0$$

$\dot\alpha$

$\dot\alpha$

Integrate $\frac{\partial \mathcal{F}}{\partial g_k}$:

"initial condition" $\mathcal{F}\big|_{g_k=0}$

$\mathcal{F}\big|_{g_k=0} = \log \Lambda \; S$ + Veneziano-Yankielowicz

$\tilde{\mathcal{F}}_m\big|_{g_k=0} =$?

The lagrangian : $\mathcal{L} = \int d^2\theta \left(2 i \tau_B S + c.c\right)$

$$\tau_B = \frac{\tilde{\Theta}_B}{2\pi} + i\,\frac{4\pi}{g_B^2} \to \Lambda = \mu \exp(i\pi\tau)$$

matter Φ = adjoint $Q, \tilde{Q}$ = fund, $\overline{\text{fund}}$

$$\Phi = \varphi + \sqrt{2}\,\theta \cdot \psi + \ldots$$

Interactions:

- gauge : $\bar{\Phi} \exp \operatorname{ad} V\, \Phi + \bar{Q} \exp V\, Q$
- superpotential : $\operatorname{Tr} W_1(\Phi) +$

$$+ \tilde{Q}\, W_2(\Phi)\, Q + \ldots$$

<u>Chiral ring</u> : set of gauge invariant, chiral operators $\mathcal{O}_i$:

as superfields $\bar{D}\, \mathcal{O}_i = 0$

$$\updownarrow$$

$$[\bar{Q}, \mathcal{O}^{\text{lowest}}\} = 0$$

Properties of the chiral ring:

a) the correlator of a commutator $[\bar{Q}^{\dot\alpha}, X]$ with chiral operators is 0:

$$\langle 0| [\bar{Q}^{\dot\alpha}, X]\, \theta_1 \dots \theta_n |0\rangle = 0$$

b) the correlator of (lowest components) chiral operators is x-independent

$$\frac{\partial}{\partial x^\mu}\theta = [\{\bar{Q}^{\dot\alpha}, Q^\alpha\}(\sigma_\mu)_{\dot\alpha\alpha}, \theta] =$$

$$= (\sigma_\mu)_{\dot\alpha\alpha}\{\bar{Q}^{\dot\alpha}, [Q^\alpha, \theta]\}$$

c) due to clustering, the correlators factorize:

$$\langle \theta_1(x_1) \dots \theta_n(x_n) \rangle = \langle \theta_1 \rangle \langle \theta_2 \rangle \dots \langle \theta_n \rangle$$

⇒ effective action of chiral operators is a function of numbers ⇒ no info about spectrum, S-matrix, ...

The list of independent chiral operators:

(modulo commutators)

a) $\mathrm{Tr}\left(\Phi^p\right)$ $p=1,2,\dots$ with possible relations due to the eq of motion (deformed by quantum effects)

b) For $\mathcal{O} \in$ adjoint, chiral:

$$\left[\bar{Q}^{\dot\alpha},\ \underset{\substack{\uparrow \\ \text{cov derivative}}}{D_\mu} \mathcal{O}\right] = \sigma_\mu^{\dot\alpha\alpha}\left[W_\alpha, \mathcal{O}\right]$$

therefore $\left[W_\alpha, \mathcal{O}\right] \approx 0$ i.e.

$\left\{W_\alpha, W_\beta\right\} \approx 0$ i.e. only $W_1 W_2$ independent

and order Φ, W is not important:

$$\mathrm{Tr}\left(\Phi^p\right),\ \mathrm{Tr}\left(\Phi^p W_\alpha\right),\ \mathrm{Tr}\left(\Phi^p W_\alpha W^\alpha\right)$$

c) $W_\alpha Q$ is not in the ring:

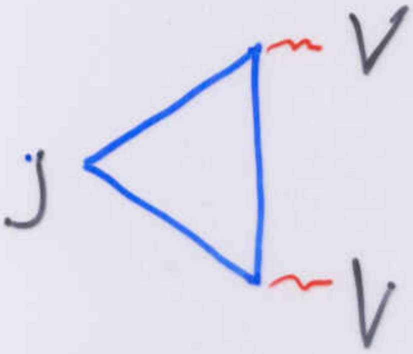
$Q, \tilde{Q}$ adds $\tilde{Q} \cdot \Phi^p Q$

2) The Konishi Anomaly Generalized

KoAn = AVV supersymmetrized

global U(1) symmetry of a chiral superfield $\Phi \to \Phi + \epsilon \Phi$ (commutes w. SUSY)

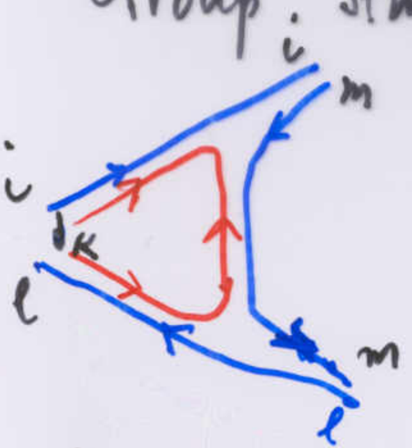
current $J = \bar{\Phi}\, e^{\mathrm{ad}\, V} \Phi$



$$= \frac{1}{32\pi^2} \frac{D^2 W_\alpha W^\alpha}{\Box}$$ nonlocal W-on shell

$$\bar{D}^2 J = \frac{1}{32\pi^2} \bar{D}^2 \frac{D^2 W_\alpha W^\alpha}{\Box} = \frac{1}{32\pi^2} W_\alpha W^\alpha \text{ - local}$$

Group structure for $\bar{\Phi}_{ij} \left(e^V \Phi\right)_{kl}$:



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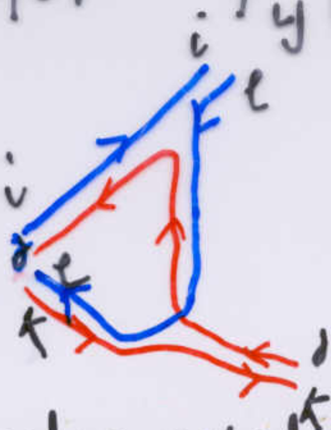


vector current

+ j↔l, i↔k

$$\frac{1}{32\pi^2}\left[(W_\alpha W^\alpha)_{il}\,\delta_{jk} + (W_\alpha W^\alpha)_{jk}\,\delta_{ij} - 2(W_\alpha)_{il}(W^\alpha)_{jk}\right]$$

after trace $\frac{N}{16\pi^2}\mathrm{Tr}(W_\alpha W^\alpha) - \frac{1}{16\pi^2}\mathrm{Tr}W_\alpha\,\mathrm{Tr}W^\alpha$

- the form of the rhs is unique by chirality + quantum numbers or W-Z condition (both perturbative)
- the coefficient is not renormalized (Adler-Bardeen) by ren group ($\theta\bar\theta$ term) or holomorphic no renormalization (both perturbative)

Generalization

if $\delta\Phi = \epsilon\Phi$ is not a symmetry

$\bar{D}^2 J = \mathrm{Tr}\left(\Phi\frac{\partial W}{\partial\Phi}\right) +$ anomaly

Consider general variation.

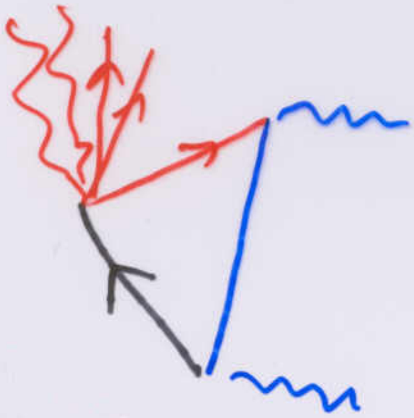
$\delta\Phi = f(\Phi, W_\alpha)$ - with the same chirality and gauge transformation

the current

$$J_f \equiv \mathrm{Tr}\left(\bar{\Phi}\, e^{\mathrm{ad}V} \underbrace{f(\Phi, W_\alpha)}_{\text{no short distance sing}}\right)$$

J_f =

anomalous contribution from



i.e

$$\frac{\partial f}{\partial \Phi_{ij}} \times \left(\bar{\Phi}_{kl}\ \Phi_{ij}\ \text{triangle}\right)$$

summing over kl . ij .

$$\bar{D}^2 J_f = \mathrm{Tr}\left(f(\Phi, W_\alpha)\frac{\partial W_1}{\partial \Phi}\right) +$$

$$+\frac{1}{32\pi^2}\sum_{ij}\left[W_\alpha,\left[W^\alpha,\frac{\partial f}{\partial \Phi_{ij}}\right]\right]_{ji}$$

- form unique by holomorphicity + quantum numbers (perturbatively)

- no renormalization

In the ring (correlators with $\mathcal{O}_i$),

$$\left\langle \mathrm{Tr}\left(f(\Phi, W_\alpha)\frac{\partial W_1}{\partial \Phi}\right) \mathcal{O}_1 \ldots \right\rangle$$

$$+\frac{1}{32\pi^2}\left\langle \sum_{ij}\left[W_\alpha,\left[W^\alpha,\frac{\partial f}{\partial \Phi_{ij}}\right]\right]_{ji} \mathcal{O}_1 \ldots \right\rangle = 0$$