STRINGS AND WAVES

J. Figueroa - O' Farrill + GP

hep-th/0105308 hep-th/0211089

M. Blau, J. Figueroa- O' Farrill, C. Hull +GP

hep-th/020181 hep-th/0110242

M. Blau, J. Figueroa - O' Farrill + GP

hep-th/0202111

J. Russo, A. Tseytlin + GP

hep-th/0211289

M. Blau, M. O' Loughlin, A. Tseytlin + GP.

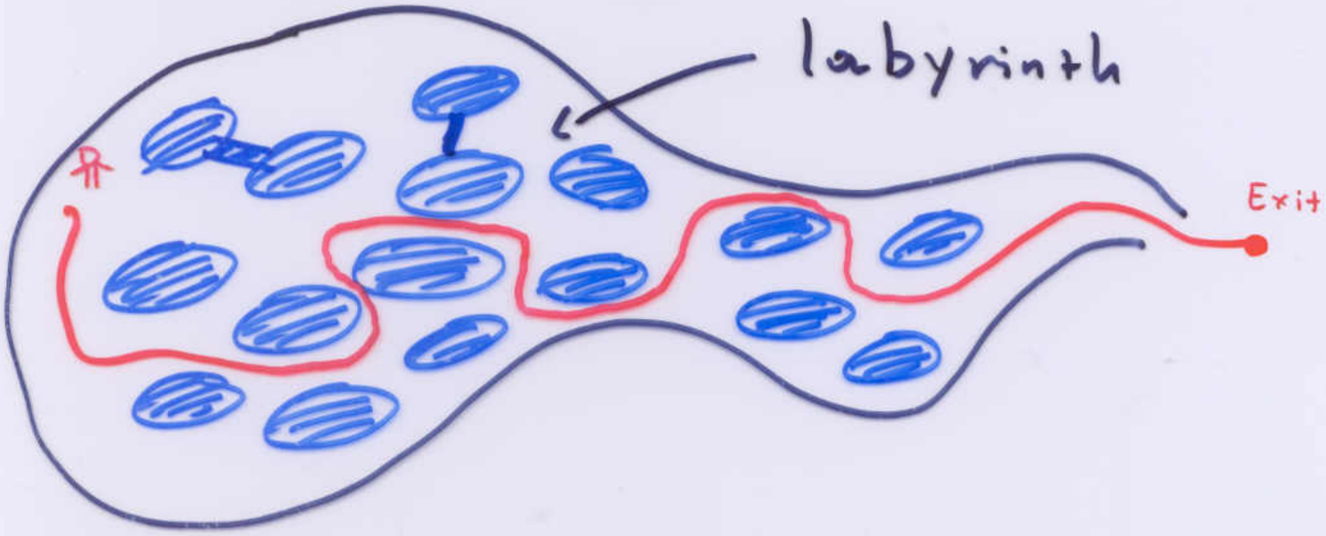
hep-th/0304198

Ariadne Strings

Herodotus
Thukydides
Evripides
Sophocles

Crete $\ll$ 1620 BC

- Theseas Problem



- Open Strings
- Dirichlet boundary conditions (D0-branes)
- Winding modes

$\therefore$ Solution still in use today!

Thera Waves

- **Marinatos Theory (1967)**

In ~ 1620 BC, there was a volcanic eruption at Thera. A 260 m wave washed at the north shores at Crete. The evidence are volcanic rock that has been found in Crete as well as writings from Egypt to China.

∴ Are these the first evidence for strings and waves ?

THE FIVE TESTS

Progress in String Theory the last 3,500 years since Ariadne Strings.

- Can we quantize strings in a string background?

- Can we quantize string's in a string soliton background?

- Can we quantize strings in a cosmological background?
- Superstring Unification?

- Can we prove the AdS/CFT correspondence

NO

NO

No despite the progress in String cosmology

There are models close to what we expect
but many others which are not

A lot of evidence in support but not
an actual proof.

Classical Strings

Action

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \; \sqrt{\gamma} \;\; \gamma^{\mu\nu} \; \partial_\mu X^M \; \partial_\nu X^N \;\; G_{MN}(X)$$
$$+ \dots$$

$\gamma_{\mu\nu}$ worldsheet metric

G_{MN} spacetime metric

$X: \;\; \Sigma \longrightarrow M$

Symmetries

$$\left.\begin{aligned} \delta \gamma_{\mu\nu} &= \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \\ \delta X^M &= \epsilon^\mu \partial_\mu X^M \end{aligned}\right\} \text{Diff.}$$

$$\left.\delta \gamma_{\mu\nu} = \phi \, \gamma_{\mu\nu} \right\} \text{Weyl}$$

Equations of motion

$$T_{\mu\nu} = \left(\partial_\mu X^M \partial_\nu X^N - \frac{1}{2} \gamma_{\mu\nu} \partial^\lambda X^M \partial_\lambda X^N \right) G_{MN}(X)$$
$$= 0$$

$$\gamma^{\mu\nu} \nabla_\mu \partial_\nu X^M = \gamma^{\mu\nu} \Big(\partial_\mu \partial_\nu X^M - \Gamma^\lambda_{\mu\nu}(\gamma) \, \partial_\lambda X^M$$
$$+ \Gamma^M_{NL}(G) \, \partial_\mu X^N \partial_\nu X^L \Big) = 0$$

∴ The equations of motion are non-linear and cannot be solved for general backgrounds

- α' corrections

TOPICS

- Eleven-dimensional supergravity
  - Maximal supersymmetry
- IIB Supergravity
  - Maximal supersymmetry
- Classification of maximally supersymmetric solutions in $D=11, 10$
- Penrose limits
  - Penrose limits and maximal supersymmetry
  - Penrose limits and string solitons
- Strings in plane waves
  - Homogeneous plane waves
  - Plane wave backgrounds
  - Homogeneous plane wave backgrounds
- Non-singular homogeneous plane wave backgrounds
  - Klein Gordon equation
  - Classical string mode equations

- Four-dimensional plane waves
- General solution
- Hamiltonian
- Examples of four-dimensional models

- Strings in homogeneous singular waves
  - Klein Gordon equation
  - Backreaction
  - Strings in singular waves
  - Hamiltonian
  - String mode creation
  - String transition through singularity

- Conclusions

ELEVEN-DIMENSIONAL SUPERGRAVITY

Cremmer
Julia
Scherk

Bosonic Fields $\quad G$ (metric), F (four-form)

$$dF = 0 \quad \Rightarrow \quad F = dA$$

A (three-form gauge potential)

Field Equations

$$R_{MN} - \frac{1}{2} G_{MN} R = \frac{1}{12} F_M{}^{PQR} F_{NPQR} - \frac{1}{4} G_{MN} F^2$$

$$d * F + \frac{1}{2} F \wedge F = 0$$

Killing Spinor Equation

$$\mathcal{D}_M \epsilon = 0$$

super-covariant derivative

$$\mathcal{D}_M = \nabla_M - \frac{1}{288} \left(\Gamma^{PQRS}{}_M + 8 \Gamma^{PQR} \delta^S_M \right) F_{PQRS}$$

- The Clifford algebra in $D=11$ admits a 32-dimensional spinor real rep. so ϵ is a 32-component real spinor
- The # of solutions of KSE is the number of susies preserved by the background.

MAXIMAL SUPERSYMMETRY

- $AdS_4 \times S^7$ and $AdS_7 \times S^4$.

Membrane

Duff Stelle

$$ds^2 = H^{-\frac{2}{3}} \underbrace{ds^2(\mathbb{R}^{1,2})}_{WV} + H^{\frac{1}{3}} \underbrace{dy \cdot dy}_{\text{transverse},\ \mathbb{R}^8}$$

$$F = d\mathrm{vol}(\mathbb{R}^{1,2}) \wedge dH^{-1} \qquad H = 1 + \frac{Q_2}{|y|^6}$$

Near Horizon:

$$ds^2 \sim |y|^4 ds^2(\mathbb{R}^{1,2}) + \frac{1}{|y|^2} dy \cdot dy =$$
$$= \frac{1}{4} \frac{1}{r^2}\left(dr^2 + ds^2(\mathbb{R}^{1,2})\right) + ds^2(S^7)$$
$$= \frac{1}{4} ds^2(AdS_4) + ds^2(S^7)$$

$$\boxed{\therefore\ AdS_4 \times S^7 \text{ has } 32 \text{ Killing spinors}}$$

M5-brane

Güven

$$ds^2 = H^{-\frac{1}{3}} \underbrace{ds^2(\mathbb{R}^{1,5})}_{WV} + H^{2/3} \underbrace{dy \cdot dy}_{\mathbb{R}^5}$$

$$F = \star dH \qquad H = 1 + \frac{Q_5}{|y|^3}$$

Near Horizon:

$$ds^2 \sim |y|\, ds^2(\mathbb{R}^{1,5}) + \frac{1}{|y|^2} dy \cdot dy$$
$$= \frac{4}{r^2}\left(dr^2 + ds^2(\mathbb{R}^{1,5})\right) + ds^2(S^4)$$
$$= 4\, ds^2(AdS_7) + ds^2(S^4)$$

$$\boxed{\therefore\ AdS_7 \times S^4 \text{ has } 32 \text{ Killing spinors}}$$

Kowalski-Glikman

Plane Wave

$$ds^2 = 2\,dx^+ dx^- - \frac{\mu^2}{36}\left(4\sum_{i=1}^{3}(x_i)^2 + \sum_{a=1}^{6}(y_i)^2\right)(dx^-)^2 + dx^2 + dy^2$$

$$F = \mu\, dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3$$

- This solution has 32 killing spinors

Minkowski Spacetime.

$$ds^2 = -dt^2 + \sum_{i=1}^{10}(dy_i)^2$$

$$F = 0$$

- 32 killing Spinors

Schwarz
West
Howe

IIB SUPERGRAVITY AND MAXIMAL SUPERSYMMETRY

Bosonic Fields $\left.\begin{matrix} G \text{ (metric)}, \; B \text{ (NS-NS 2-form pot)} \\ \phi \text{ (dilaton)} \end{matrix}\right\} NS$

$\left.\begin{matrix} C_4 \; (F_5^+ = dC_4 \text{ self-dual}), \; C_2 \\ \sigma \text{ (axion)} \end{matrix}\right\} RR$

D3-brane

Duff
Lu

$$ds^2 = H^{-\frac{1}{2}} ds^2(\mathbb{R}^{1,3}) + H^{\frac{1}{2}} \underbrace{dy \cdot dy}_{\mathbb{R}^6}$$

$$F_5^+ = \left(d\mathrm{vol}(\mathbb{R}^{1,3}) \wedge dH^{-1}\right) + (\qquad)^* \qquad H = 1 + \frac{Q_3}{|y|^4}$$

Near-Horizon:

$$ds^2 \sim y^2 ds^2(\mathbb{R}^{1,3}) + \frac{dy^2}{y^2} + ds^2(S^5) =$$

$$= \frac{1}{r^2}\left(ds^2(\mathbb{R}^{1,3}) + dr^2\right) + ds^2(S^5) = ds^2(AdS_5) + ds^2(S^5)$$

$\boxed{\bullet \; AdS_5 \times S^5 \text{ has 32 killing Spinors}}$

Plane Wave

M. Blau
J. Figueroa-O'Farrill
C. Hull
GP

$$ds^2 = 2dx^+dx^- - \mu^2 y^2 + \underbrace{dy \cdot dy}_{\mathbb{R}^8}$$

$$F_5 = \frac{\mu}{2} dx^- \wedge (dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 + dy^5 \wedge dy^6 \wedge dy^7 \wedge dy^8)$$

$\boxed{\bullet \; \text{32 killing Spinors}}$

Minkowski Spacetime $\qquad F = 0$

$\boxed{\bullet \; \text{32 killing Spinors}}$

CLASSIFICATION OF MAXIMALLY SUSY SOLUTIONS

J. Figueroa O'Farrill
GP

M-THEORY

Th: The maximally supersymmetric solutions of D=11 supergravity are $AdS_4 \times S^7$, $AdS_7 \times S^4$ plane wave, Minkowski space

Proof

- Compute supercoviariant curvature

$$\mathcal{R}_{MN} = [\mathcal{D}_M, \mathcal{D}_N] = \sum_{n=1}^{5} \Phi_{MN, A_1 \cdots A_n} \Gamma^{A_1 \cdots A_n} = 0$$

- $\mathcal{R}_{MN} = 0 \Rightarrow$ Spacetime symmetric space
  F parallel.

$$\boxed{\imath_X \imath_Y \imath_Z F \wedge F = 0}$$

Plücker relation

- Plücker relation $\Rightarrow$ F simple
  - $F^2 < 0 \Rightarrow AdS_4 \times S^7$
  - $F^2 > 0 \Rightarrow AdS_7 \times S^4$
  - F null $\Rightarrow$ plane wave
  - $F = 0 \Rightarrow$ Minkowski space

IIB

Th: The maximally supersymmetric solutions of IIB supergravity are $AdS_5 \times S^5$, plane wave and Minkowski spacetime.

Proof

- The Killing spinor equations imply that the active fields are G, F_5^+ the rest vanish or are constant.

- $\mathcal{R}_{MN} = 0 \Rightarrow$ spacetime symmetric space
  F_5^+ parallel

$$\boxed{i_X i_Y i_Z (F_5)^L \wedge (F_5)_L = 0}$$ New Plücker relation

- Plücker relation $\Rightarrow$ $F_5 = G + {}^*G$ simple
  - $G^2 \neq 0 \Rightarrow AdS_5 \times S^5$
  - G null $\Rightarrow$ plane wave
  - $G = 0 \Rightarrow$ Minkowski spacetime.

Penrose
Güven

PENROSE LIMITS

- A Penrose limit is a first order approximation to a Lorentzian spacetime
- The space at the limit is a plane wave

Limit

The metric of any spacetime at the nbh of a null geodesic can be written as

$$ds^2 = dV\left(dU + \alpha\, dV + \sum_i \beta_i\, dY^i\right) + C_{ij}\, dY^i dY^j$$

U affine parameter null geodesic

α, β_i, C_{ij} depend on all coordinates

Set $U = u$ $\quad V = \Omega^2 v \quad Y^i = \Omega y^i$

$\Omega > 0$ parameter

$$\bar{g} = \lim_{\Omega \to 0} \Omega^{-2} g(\Omega) \qquad \text{metric}$$

$$\bar{\phi} = \lim_{\Omega \to 0} \phi(\Omega) \qquad \text{scalar}$$

$$\bar{A}_p = \lim_{\Omega \to 0} \Omega^{-p} A_p(\Omega) \qquad \text{p-form gauge potentials}$$

The metric at the limit

$$d\bar{s}^2 = du\,dv + \bar{C}_{ij}(u)\, dy^i dy^j$$

Setting

$$u = 2x^- \qquad v = x^+ - \frac{1}{2} M_{ij}(x^-)\, x^i x^j$$

$$y^i = Q^i{}_j(u)\, x^j \qquad \bar{C}_{ij}\, Q^i{}_k\, Q^j{}_\ell = \delta_{k\ell}$$

$$M_{ij} = C_{k\ell}\, Q'^k{}_i\, Q^\ell{}_j$$

$$\boxed{d\bar{s}^2 = 2\,dx^+dx^- + A_{ij}(x^-)\, x^i x^j\, (dx^-)^2 + \delta_{ij}\, dx^i dx^j}$$

- $d\bar{s}^2$ is a plane wave metric.

Properties

- The Penrose limits of supergravity solutions are supergravity solutions because supergravity actions are homogeneous under Ω- scalling.

- All isometries of a spacetime are inherited in the limit.

- All supersymmetries of a supersymmetric solution are inherited in the limit

- The symmetry superalgebra of a Penrose limit contains a contraction of the symmetry superalgebra of spacetime

- The limit depends on the choice of null geodesic

  ie. a spacetime can have many different Penrose limits.

PENROSE LIMITS AND SUPERSYMMETRY

M. Blau
J. Figueroa-O'Farrill
C. Hull
GP

Metric on $AdS_{p+2} \times S^n$

$$ds^2 = R^2_{AdS}\left[-d\tau^2 + \sin^2\tau\left(\frac{dr^2}{1+r^2} + r^2 ds^2(S^p)\right)\right]$$
$$+ R_S^2\left[d\psi^2 + \sin^2\psi\, ds^2(S^{n-1})\right]$$

Set $U = \psi + \rho\tau \qquad V = \psi - \rho\tau \qquad \rho = \frac{R_{AdS}}{R_S}$

$$R_S^{-2} ds^2 = dU\,dV + \rho^2 \sin^2\left(\frac{U-V}{2\rho}\right)\left(\frac{dr^2}{1+r^2} + r^2 ds^2(S^p)\right)$$
$$+ \sin^2\left(\frac{U+V}{2}\right) ds^2(S^{n-1})$$

Take the limit.

$$R_S^{-2} d\bar{s}^2 = du\,dv + \sum_{i=1}^{p+n} \frac{\sin^2\lambda_i u}{(2\lambda_i)^2}\, dy^i dy^i$$

$$\lambda_i = \begin{cases} \frac{1}{2\rho} & i = 1, \ldots, p+1 \\ \frac{1}{2} & i = p+2, \ldots, p+n \end{cases}$$

Set $x^- = \frac{u}{2} \qquad x^+ = v - \frac{1}{4}\sum_i (y^i)^2 \frac{\sin 2\lambda_i u}{2\lambda_i}$

$$x^i = y^i \frac{\sin\lambda_i u}{2\lambda_i}$$

$$\boxed{R_S^{-2} d\bar{s}^2 = 2\,dx^+ dx^- - 4\sum_i \lambda_i^2 (x^i)^2 (dx^-)^2 + (dx^i)^2}$$

Another limit

$$ds^2 = \frac{R_{AdS}^2}{(r+c)^2}\left(du\,dv + ds^2(\mathbb{R}^{p-1}) + dr^2\right) + R_S^2\, ds^2(S^n)$$

v Affine parameter

$$d\bar{s}^2 = ds^2(\mathbb{R}^{n+p+1,1})$$ Minkowski space

- There are <u>no</u> other PLs for $AdS_{p+2} \times S^n$.

M-THEORY

$$
\begin{tikzcd}
AdS_4 \times S^7 \arrow[d, "\rho = 2"] \arrow[dr] & AdS_7 \times S^4 \arrow[d] \arrow[dl, "\rho = \frac{1}{2}"'] \\
\text{Plane wave} \arrow[r] & \text{Minkowski}
\end{tikzcd}
$$

IIB STRINGS

$$
\begin{tikzcd}
& AdS_5 \times S^5 \arrow[dl, "\rho = 1"'] \arrow[dr] & \\
\text{Plane wave} \arrow[rr] & & \text{Minkowski}
\end{tikzcd}
$$

- There are no other maximally supersymmetric solutions in other $D=10$ supergravities apart from Minkowski space.

PENROSE LIMITS OF STRING SOLITONS

M. Blau
J. Figueroa-O'Farrill
GP

Dp-branes (IIA + IIB)

$$ds^2 = H^{-\frac{1}{2}} ds^2(\mathbb{R}^{1,p}) + H^{\frac{1}{2}} \underbrace{dy \cdot dy}_{\mathbb{R}^{9-p}}$$

$$e^{2\phi} = H^{\frac{p-3}{2}} \qquad H = 1 + \frac{Q_p}{|y|^{7-p}}$$

$$F_{p+2} = d\mathrm{vol}(\mathbb{R}^{1,p}) \wedge dH^{-1}$$

Set $|y| = r$, $V = t + r$ $U = t - r$

Near-Horizon $H \sim \frac{Q_p}{r^{7-p}}$

Penrose limit

$$\boxed{\begin{aligned} d\bar{s}^2 &= 2\,du\,dv - k\frac{x^2}{u^2}du^2 + (dx)^2 \\ e^{2\bar{\phi}} &= u^{8k} \qquad k = \frac{(7-p)(p-3)}{16} \end{aligned}}$$

Fundamental String

$$ds^2 = H^{-1} ds^2(\mathbb{R}^{1,1}) + \underbrace{dy \cdot dy}_{\mathbb{R}^8}$$

$$e^{2\phi} = H^{-1}$$

$$H = 1 + \frac{Q_1}{|y|^6}$$

$$F_3 = d\mathrm{vol}(\mathbb{R}^{1,1}) \wedge dH^{-1}$$

Near-Horizon, Penrose limit $k = \frac{3}{16}$

A cosmological metric

$$ds^2 = -dt^2 + t^{\alpha}\left(dr^2 + r^2\, ds^2(S^n)\right)$$

Penrose limit $\qquad k = \frac{1}{1+\alpha} - \frac{1}{(1+\alpha)^2}$

$$\frac{1}{3} \leq \alpha \leq \frac{2}{3}$$

- Brane solitons have other Penrose limits associated with other choices of null geodesics

- Similar limits can be taken for intersecting branes, black holes, wrapped branes and others.

STRINGS IN PLANE WAVES

Amati
Klimcik
Horowitz
Steif
Nappi
Witten
Kiritsis
Kounnas

We cannot quantize strings in general backgrounds

Can strings be quantized in plane waves?

General plane wave metric

$$ds^2 = 2\,du\,dv + A_{ij}(u)\, x^i x^j\, du^2 + (dx^i)^2$$

Light cone gauge fix

$\gamma_{\mu\nu} = \eta_{\mu\nu}$ (remaining symmetry 2-d conf. transf. $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \phi \eta_{\mu\nu}$)

$U = p^u \tau$ (remaining symmetry $\sigma \to \sigma + a$ closed string level matching condition.)

(τ, σ) world sheet coordinates

$V(\tau,\sigma)$ is determined from the transverse coordinates X^i

Equation for the transverse coordinates

$$\partial_\tau^2 X^i - \partial_\sigma^2 X^i - A_{ij}(p^u \tau)\,(p^u)^2 X^j = 0$$

Closed string $\quad X^i = \sum_n X_n^i(\tau)\, e^{2in\sigma}$

$$0 < \sigma \leq \pi$$

$$\partial_\tau^2 X_n^i + 4n^2 X_n^i - A_{ij}(P^u \tau)(P^u)^2 X_n^j = 0$$

$$P^u = P_v$$

- At every level n, the string (transverse) coordinates X^i do not exhibit self-interactions

- The string equations reduce to a set of harmonic oscillators with possibly time dependent frequencies

$$\boxed{\omega_{n,i}^2 = 4n^2 - \lambda_i(\tau)}$$

for

$$(A_{ij}) = \mathrm{diag}\left(\frac{\lambda_i(\tau)}{P_v^2}\right)$$

- Can string theory be solved in any plane wave ?

M. Blau
M. O'Loughlin

HOMOGENEOUS PLANE WAVES

There are two classes of homogeneous plane waves (spacetime G/H)

- Non-singular I

$$ds^2 = 2\,du\,dv + \left(e^{-fu}\,\tilde{k}\,e^{fu}\right)_{ij} y^i y^j du^2 + (dy^i)^2$$

$$f_{ij} = -f_{ji} \qquad \tilde{k}_{ij} = \tilde{k}_{ji} \qquad \text{constants}$$

$$[\tilde{k}, f] \neq 0.$$

- Singular II

$$ds^2 = 2\,du\,dv + u^{-2}\left(e^{-f\ln u}\,\tilde{k}\,e^{f\ln u}\right)_{ij} y^i y^j + (dy^i)^2$$

For $f = 0$, $\tilde{k}_{ij} = -\delta_{ij}$

I $\Rightarrow$ Maximal supersymmetric IIB plane wave
(The spacetime is symmetric space)

II $\Rightarrow$ PL limit of D-branes, Fundamental String, Cosmological backgrounds.

Setting $y_i = (e^{-fu})_{ij} x^j$

(1) $\Rightarrow$ $ds^2 = 2\,du\,dv + k_{ij} x^i x^j du^2 + 2 f_{ij} x^i dx^j du + (dx^i)^2$

$$\tilde{k}_{ij} = k_{ij} - f_{ik} f_{jk}$$

PLANE WAVE BACKGROUNDS

Common sector fields: G metric

H NS 3-form

ϕ dilaton

Field equations

$$R_{MN} - \frac{1}{4} H^R{}_{ML} H^L{}_{RN} + 2 \nabla_M \partial_N \phi = 0$$

$$\nabla_L \left(e^{-2\phi} H^L{}_{MN} \right) = 0$$

$$M, N = 0, \dots, 9$$

Background

$$ds^2 = 2\,du\,dv + K(u,x)\,du^2 + 2 A_i(u,x)\,dx^i du + (dx^i)^2$$

$$B = B_i(u,x)\ dx^i \wedge du \qquad H = dB$$

$$\phi = \phi(u)$$

Field equations

$$-\frac{1}{2} \partial_i^2 K + \partial_u \partial^i A_i + \frac{1}{4} F_{ij} F^{ij} - \frac{1}{4} H_{ij} H^{ij} + 2 \partial_u^2 \phi = 0$$

$$\partial^i F_{ij} = 0 \qquad \partial^i H_{ij} = 0$$

$$F_{ij} = \partial_i A_j - \partial_j A_i \qquad H_{ij} = \partial_i B_j - \partial_j B_i$$

M. Blau
M. O'Loughlin
GP
A. Tseytlin

HOMOGENEOUS PLANE WAVE BACKGROUNDS
(NON-SINGULAR)

I $\quad \mathbb{K}(u,x) = k_{ij}\, x^i x^j \qquad k_{ij}$ constant

$F_{ij} = 2 f_{ij} \qquad f_{ij}$ constant

$H_{ij} = 2 h_{ij} \qquad h_{ij}$ constant

Field equations

$\phi(u) = \phi_0 + c\, u - \frac{1}{2} \mu u^2 \qquad c$ constant

$$\mu = -\frac{1}{2}\left(k^i{}_i - f_{ij} f^{ij} + h_{ij} h^{ij} \right)$$

- $\mu > 0$, $\mu = 0 \;\; c = 0$,

  String coupling small everywhere

- Four dimensional models

$k_{ij} = \mathrm{diag}(k_1, k_2)$, $f_{ij} = f\, \varepsilon_{ij}$, $h_{ij} = h\, \varepsilon_{ij}$

$k_1 \neq k_2$

If $k_1 = k_2$ the "rotation" f_{ij} can be eliminated with a coordinate transformation.

KLEIN-GORDON EQUATION

Metric $\quad ds^2 = 2\,du\,dv + k_{ij}\,x^i x^j\,du^2 +$

$$+ 2 f_{ij}\, x^i dx^j\, du + (dx^i)^2$$

KG: $\left[2\,\partial_u \partial_v - K\,\partial_v^2 + (\partial_i - A_i \partial_v)^2 - M^2 \right] \Psi = 0$

Fourier transf.

$$\Psi(u,v,x) = \int dp_u\, dp_v\; e^{i p_u u + i p_v v}\, \psi(p_u, p_v; x)$$

Substitute

$$H\psi = \mathcal{E}\psi \qquad \mathcal{E} = -\tfrac{1}{2} M^2 - p_u p_v$$

$$H = -\frac{1}{2}\left[(\partial_i - i p_v A_i)^2 + p_v^2\, K(x) \right]$$

For $\quad A_i = -f\,\varepsilon_{ij}\, x^j \qquad K = k_1 x_1^2 + k_2 x_2^2$

Set $\quad x = x_1 \qquad y = x_2$

The Hamiltonian operator

$$H = -\frac{1}{2}\Big[(\partial_x + i p_v f y)^2 + (\partial_y - i p_v f x)^2 + p_v^2 (k_1 x^2 + k_2 y^2) \Big]$$

- This is the Hamiltonian of a charged harmonic oscillator in magnetic field.

Yonei

The Hamiltonian can be diagonalized with a unitary transformation

$$H = U(\theta_1, \theta_2)\,(H_1 + H_2)\,U^\dagger(\theta_1, \theta_2)$$

$$U(\theta_1, \theta_2) = e^{-i\theta_1 xy}\, e^{-i\theta_2 p_x p_y}$$

$$H_i = \frac{1}{2m_i} p_i^2 + \frac{m_i}{2} \omega_i^2 x_i^2$$

such that

$$\theta_1 = -P_v \frac{(k_1 - k_2) + D}{4f} \qquad \theta_2 = -\frac{2f}{P_v D}$$

$$D^2 = (k_1 - k_2)^2 - 8f^2(k_1 + k_2) + 16 f^4$$

the frequencies and masses

$$\omega_{1,2}^2 = \frac{1}{2}\left(4f^2 - k_1 - k_2 \mp D\right)$$

$$m_{1,2} = \frac{2D}{D + k_1 - k_2 \mp 4f^2}$$

- The Hilbert space of the theory is that of two harmonic oscillators.

CLASSICAL STRING MODE EQNS

Background

$$ds^2 = 2\,du\,dv + k_{ij}\,x^i x^j\,du^2 + 2\,f_{ij}\,x^i dx^j\,du + (dx^i)^2$$

$$B = -h_{ij}\,x^j\,dx^i \wedge du \qquad i,j = 1,\dots,d$$

Action

$$L = \frac{1}{4\pi\alpha'}\,(G+B)_{MN}\,\partial_+ X^M \partial_- X^N$$

Equations of motion

light-cone gauge $\qquad U = p^u \tau$

mode expansion $\qquad X^i = \sum_n X_n^i\, e^{2in\sigma}$

$$-\ddot{X}_n^i + 2\,p_v\,f_{ij}\,\dot{X}_n^j + \left(p_v^2\,k_{ij} - 4n^2\delta_{ij}\right)X_n^j + 4in\,p_v\,h_{ij}\,\dot{X}_n^j = 0$$

- Without loss of generality

$$k_{ij} = \mathrm{diag}(k_1, k_2, \dots) \qquad p_v = 1.$$

- This is a differential equation with constant coefficients.

FREQUENCY BASE ANSATZ

Set $$X_n^i = \sum_{J=1}^{2d} \xi_J^{(n)} a_{iJ}^{(n)} e^{i\omega_J^{(n)}\tau}$$

$\omega_J^{(n)}$ frequencies

$a_{iJ}^{(n)}$ eigen-direction for frequency $\omega_J^{(n)}$

ξ_J parameters to be determined.
(they can be absorbed in $a_{iJ}^{(n)}$)

Equations of motion

$$M_{ik}(\omega_J^{(n)}, n)\, a_{kJ}^{(n)} = 0 \qquad J = 1, 2, \dots, 2d$$

where

$$M_{ik}(\omega, n) = (\omega^2 + k_i - 4n^2)\,\delta_{ik} + 2i\omega f_{ik} + 4in\, h_{ik}$$

Equation for classical frequencies

$$\det M = 0.$$

- $n = 0, \qquad M(-\omega, 0) = \tilde{M}(\omega, 0)$ ~ transpose

$\Rightarrow \det M(\omega, 0) = \det M(-\omega, 0) \Rightarrow$

frequencies $\boxed{\pm\,\omega \text{ pairs}}$

- $n \neq 0, \qquad M(-\omega, -n) = \tilde{M}(\omega, n)$

$\Rightarrow \det M(\omega, n) = \det M(-\omega, -n) \Rightarrow$

$$\boxed{\omega_J^{(-n)} = -\,\omega_J^{n}}$$

FOUR DIMENSIONAL PLANE WAVES

$$M = \begin{pmatrix} w^2 + k_1 - 4n^2 & 2ifw + 4inh \\ -2ifw - 4inh & w^2 + k_2 - 4n^2 \end{pmatrix}$$

Eqn for frequencies

$\det M = 0 \Leftrightarrow$

$$\omega^4 + (k_1 + k_2 - 4f^2 - 8n^2)\,\omega^2 - 16nfh\,\omega + (k_1 - 4n^2)(k_2 - 4n^2) - 16n^2h^2 = 0$$

- It can be solved explicitly.
- $n = 0$, $\quad \omega^4 + (k_1 + k_2 - 4f^2)\,\omega^2 + k_1 k_2 = 0 \Rightarrow$

$$w_{1,2}^2 = \frac{1}{2}(4f^2 - k_1 - k_2) \mp \frac{1}{2} D$$

i.e the frequencies of the QM model.

Eigen-directions a_{iJ}

$$(a_{1J}, a_{2J}) = (-M_{22}(\omega_J), M_{21}(\omega_J))$$

Then

$$M_{1K}(\omega_J)\, a_{KJ} = -\det M(\omega_J) = 0$$

$$M_{2K}(\omega_J)\, a_{KJ} = -M_{21}M_{22} + M_{22}M_{21} = 0$$

- a_{iJ} are related to minors of M.

GENERAL SOLUTION

$n=0$

$$\boxed{X_0^i = (-1)^i \sum_{j=1}^{d} \left[\xi_j^+ \, m_{1i}(\omega_j)\, e^{i\omega_j \tau} + \xi_j^- \, m_{i1}(\omega_j)\, e^{-i\omega_j \tau} \right]}$$

$n\neq 0$

$$\boxed{X_n^i = (-1)^i \sum_{J=1}^{2d} \xi_J^{(n)} \, m_{1i}(\omega_J^{(n)})\, e^{i\omega_J^{(n)} \tau}}$$

m_{ij} is the i,j minor of M.

Canonical Commutation relations

$$[\xi_j^-, \xi_j^+] = c_j$$

$$[\xi_J^{(-n)}, \xi_I^{(n)}] = \delta_{IJ}\, C_J^{(n)}$$

$$c_j = \frac{1}{2 m_{11}(\omega_j)\, \omega_j \prod_{k\neq j} (\omega_j^2 - \omega_k^2)}$$

$$C_J^{(n)} = \frac{}{m_{11}(\omega_J^{(n)}) \prod_{K\neq J} (\omega_J^{(n)} - \omega_K^{(n)})}$$

$$\alpha_j^{\pm} = \xi_j^{\pm} / c_j^{1/2} \qquad c_j > 0$$

$$\alpha_j^{\pm} = \xi_j^{\mp} / |c_j|^{1/2} \qquad c_j < 0$$

$$\alpha_J^{(n)} = \xi_J^{(n)} / C_J^{1/2} \qquad C_J > 0$$

$$\alpha_J^{(n)} = \xi_J^{(-n)} / |C_J|^{1/2} \qquad C_J < 0.$$

Then

$$[\alpha_j^- , \alpha_k^+] = \delta_{jk}$$

$$[\alpha_J^{(n)}, \alpha_K^{(m)}] = -\operatorname{sign}(n)\, \delta_{n+m}\, \delta_{JK}$$

- The (classical) frequencies are the roots of M
- The eigen-directions are constructed from the minors of M
- The commutators are determined by the frequencies

HAMILTONIAN

The light cone hamiltonian

$$H = \frac{1}{2\pi}\int_0^{\pi} d\sigma \left[\delta_{ij} \left(\dot{X}^i \dot{X}^j + X'^i X'^j - k_i X^i X^j \right) - 2 h_{ij} X^i X'^j \right]$$

Then $H = H^{(0)} + \sum_{n>0} H^{(n)}$

$$H^0 = \sum_{j=1}^{d} \mathrm{sign}(C_j)\, \Omega_j \left(\alpha_j^{+} \alpha_j + \tfrac{1}{2} \right)$$

$$H^{(n)} = \sum_{J=1}^{2d} \mathrm{sign}\left(C_J^{(n)} \right) \Omega_J^{(n)} \left(\alpha_J^{(n)} \alpha_J^{(-n)} + \tfrac{1}{2} \right)$$

and

$$\Omega_j = \frac{\sum_i (\omega_j^2 - k_i)\, m_{ii}(\omega_j)}{2\omega_j \prod_{k\neq j} (\omega_j^2 - \omega_k^2)} \quad (= \omega_j)$$

$$\Omega_J^{(n)} = 2\omega_J^{(n)} C_J^{(n)} m_{11}(\omega_J^{(n)}) \sum_{i,j} \left[\omega_J^{(n)} \delta_{ij} + i (-1)^{i+i} f_{ij} \right] m_{ij}(\omega_J^{(n)})$$

$$(= \omega_J^{(n)})$$

- In many cases the classical frequencies are equal to those of the hamiltonian.

FOUR-DIMENSIONAL WAVES

(i) Ricci flat, Constant dilaton, $f=h=0$

$k_1+k_2=0 \qquad D=k_1-k_2 \qquad \omega_1^2=-k_1$

$\omega_2^2=-k_2 \qquad m_{1,2}=1$

- One of the frequencies is imaginary.

(ii) Ricci flat, Constant dilaton, $h=0$, $f\neq 0$.

$k_1+k_2=2f^2 \qquad D=k_1-k_2 \qquad \omega_1^2=k_1$

$\omega_2^2=k_2 \qquad m_1=-\frac{k_1-k_2}{2k_2} \qquad m_2=\frac{k_1-k_2}{2k_1}$

Energy $\qquad \mathcal{E}=-\omega_1\left(n_1+\frac{1}{2}\right)+\omega_2\left(n_2+\frac{1}{2}\right)$

- Discrete spectum but unbounded.

(iii) Ricci flat, Const. dilaton (Anti-Mach), $h=0$, $f\neq 0$, $k_2=0$

$k_1=2f^2, \quad \omega_1^2=0, \quad \omega_2^2=2f^2$

$m_1=-\infty \qquad m_2=\frac{1}{2}$

- Spectrum continuous along ω_1 and discrete along ω_2.

$$\mathcal{E}=\omega_2\left(n_2+\frac{1}{2}\right)-\frac{1}{2}p^2$$

A. Tseytlin
J. Russo
GP

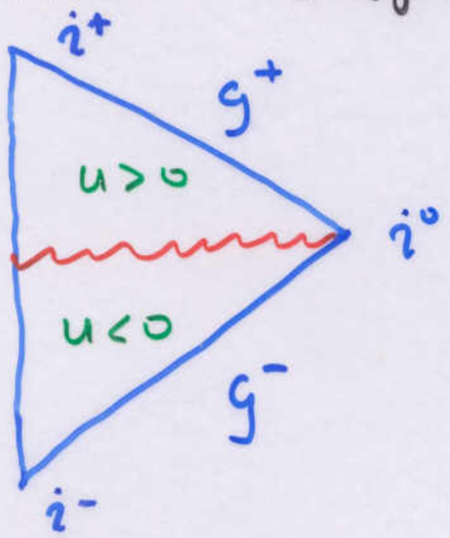
STRINGS IN HOMOGENEOUS SINGULAR WAVES

Background

$$ds^2 = 2\,du\,dv - \frac{K}{u^2}\, x^2\, du^2 + (dx^i)^2 \qquad i=1,\dots,d$$

$$e^{2\Phi} = u^{Kd}\, e^{-2u} \qquad u>0 \quad K>0$$

Penrose Diagram



singularity

- Dilaton small at $u \to +\infty$ $\quad u \to 0$
- Metric can be extended to $u<0$ but dilaton cannot unless $u \to |u|$
- Metric has a scalling symmetry
  $u \to \ell u \qquad v \to \ell^{-1} v$
  The metric is homogeneous
- Metric is geodesically incomplete at $u=0$. Some geodesics cannot be extended from $u<0$ to $u>0$.

KLEIN GORDON EQN.

$$\left(2\partial_u\partial_v + \frac{K}{u^2}x^2\partial_v^2 + \delta^{ij}\partial_i\partial_j\right)\phi = 0$$

Fourier transform in v

$$\phi = \int dP_v \; e^{iP_v v} \; \psi(u, x; P_v)$$

Gives

$$\left(2i\,P_v\partial_u - \frac{K}{u^2}P_v^2 + \delta^{ij}\partial_i\partial_j\right)\psi = 0$$

Set $\quad u = P_v \tau \qquad (P_v = p^u)$

$$i\partial_\tau \psi = \frac{1}{2}\left(-\delta^{ij}\partial_i\partial_j + \frac{K}{\tau^2}x^2\right)\psi$$

- This is the Schrödinger equation of a harmonic oscillator with time-dependent frequency.

TIME-DEPENDENT SCHRÖDINGER EQUATION

$$i\,\partial_\tau - \hat{H}(\tau)$$

Lewis
Riesenfeld

Define

$$\hat{A}(\tau) = i\left(\chi^* \hat{P} - \partial_\tau \chi^* \hat{X}\right)$$

$$\hat{A}^+(\tau) = -i\left(\chi \hat{P} - \partial_\tau \chi\, \hat{X}\right)$$

$\hat{P}$ momentum, $\hat{X}$ position

χ complex classical solution

$$\chi\, \partial_\tau \chi^* - \chi^* \partial_\tau \chi = i$$

Then

$$[\hat{A}, \hat{A}^+] = 1, \quad i\,\partial_\tau \hat{A} = [\hat{H}, \hat{A}]$$

$$i\,\partial_\tau \hat{A}^+ = [\hat{H}, \hat{A}^+]$$

Basis $\quad |\ell,\tau\rangle = \frac{1}{\sqrt{\ell!}}\left(\hat{A}^+(\tau)\right)^\ell |0,\tau\rangle$

$$\hat{A}\,|0,\tau\rangle = 0$$

Then $\quad (i\,\partial_\tau - \hat{H})\,|\ell,\tau\rangle = \Lambda_\ell(\tau)\,|\ell,\tau\rangle$

Solution

$$\boxed{\begin{aligned} |\psi\rangle &= \sum_\ell c_\ell |\psi_\ell\rangle = \sum_\ell c_\ell\, e^{i\gamma_\ell(\tau)}\, |\ell,\tau\rangle \\ \gamma_\ell(\tau) &= \int^\tau ds\ \Lambda_\ell(\tau) \end{aligned}}$$

Back to Klein Gordon

$$\hat{A}^i(\tau) = i\left(\chi^* \hat{P}^i - \partial_\tau \chi \hat{X}^i\right)$$

$$\chi(\tau) = \frac{i}{\sqrt{2\sqrt{k}}\,(2\nu-1)}\Big[-\left(\sqrt{k} - i(1-\nu)\right)\tau^{\nu} + \left(\sqrt{k} - i\nu\right)\tau^{1-\nu}\Big]$$

$$\nu = \frac{1}{2}\left(1 + \sqrt{1-4k}\right)$$

Hamiltonian operator

$$\hat{H} = c(\tau)\,\hat{A}^2 + c^*(\tau)\left(\hat{A}^+\right)^2 + b(\tau)\left(\hat{A}^+ A + \frac{d}{2}\right)$$

$c(\tau),\ b(\tau) \sim \tau^{-2\nu} \qquad \tau \to 0$

- The hamiltonian operator is not diagonal in $\hat{A}, \hat{A}^+$

- $\langle \psi | \hat{H} | \psi \rangle \sim \tau^{-2\nu} \qquad \tau \to 0$

  $\langle \psi | \hat{H} | \psi \rangle \sim \tau^{2\nu-2} \qquad \tau \to +\infty$

BACKREACTION

Backreaction is measured by

$$e^{2\phi} \langle \psi | \hat{H} | \psi \rangle$$

and

$$\boxed{e^{2\phi} \langle \psi | \hat{H} | \psi \rangle \sim \tau^{kd-2\nu} \qquad \tau \to 0}$$

Backreaction small for

$$\boxed{\frac{3}{16} \leq k \leq \frac{1}{4} \qquad d = 8.}$$

STRINGS IN SINGULAR WAVES

Equations of motion (light-cone)

$U = p^u \tau$

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^i + \frac{K}{\tau^2} X^i = 0$$

Solution

$$X^i(\sigma,\tau) = x_0^i(\tau) + \frac{i}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{n} \Big[Z(2n\tau) \left(\alpha_n^i e^{2in\sigma} + \tilde{\alpha}_n^i e^{-2in\sigma}\right) - cc \Big]$$

where

$$x_0^i = \frac{1}{\sqrt{2\nu-1}} \left(x^i \tau^{1-\nu} + 2 p^i \tau^\nu \right)$$

$$Z(2n\tau) = e^{-\frac{i\pi\nu}{2}} \sqrt{\pi n \tau} \Big\{ \underbrace{J_{\nu-\frac{1}{2}}(2n\tau) - i\, Y_{\nu-\frac{1}{2}}(2n\tau)}_{\text{Bessel functions}} \Big\}$$

- $Z(2n\tau) \sim e^{-2in\tau} \left[1 + O\left(\frac{1}{\tau}\right) \right]$

  Oscillator modes have a flat limit at $\tau \to \infty$.

  Zero modes <u>do not</u>.

HAMILTONIAN

CCRS
$$[\alpha_n^i, \alpha_m^j] = n\,\delta^{ij}\,\delta_{n+m}$$
$$[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\,\delta^{ij}\,\delta_{n+m}$$
$$[x^i, p^j] = i\,\delta^{ij}$$

Lightcone hamiltonian

$$H = H_0 + \sum_{n>0} H^{(n)}$$

$$H^{(n)} = \frac{1}{2}\Big[\Omega_n(\tau)\left(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i\right) - B_n(\tau)\,\alpha_n^i\tilde{\alpha}_n^i - B_n^*(\tau)\,\alpha_{-n}^i\tilde{\alpha}_{-n}^i\Big]$$

H_0 zero mode hamiltonian

$\Omega_n(\tau) \sim 2 \qquad B_n(\tau) \sim 0 \qquad \tau \to \infty.$

- Hamiltonion non-diagonal.

Set

$$A_n^i = \alpha_n^i f_n(\tau) + \tilde{\alpha}_{-n}^i g_n^*(\tau)$$
$$A_n^{+i} = \alpha_{-n}^i f_n^*(\tau) + \tilde{\alpha}_n^i g_n(\tau)$$

Similarly for $\tilde{A}_n^i$ $\tilde{A}_n^{+i}$

Then

$$H(\tau) = H_0(\tau) + \sum_{n=1}^{\infty} w_n(\tau) \left[A^{+i}_n(\tau) A^i_n(\tau) + \tilde{A}^{+i}_n(\tau) \tilde{A}^i_n(\tau) \right] + h(\tau)$$

$$w_n = \sqrt{n^2 + \frac{k}{4\tau^2}}$$

STRING MODE CREATION

Define the vacuum $|0, p_v\rangle$

$$\alpha_n^i |0, p_v\rangle = \tilde{\alpha}_n^i |0, p_v\rangle = 0 \qquad \text{at } u \to -\infty$$

The expectation value at number operator is

$$\bar{N}_n(\tau) = \langle 0, p_v | \left(A^{+i}_n A^i_n + \tilde{A}^{+i}_n \tilde{A}^i_m \right) | 0, p_v\rangle$$

$$= 2 d n \, g_n^*(\tau) \, g_n(\tau)$$

which

$$\bar{N}(\tau) = \sum_n \bar{N}_n(\tau) \sim \frac{1}{\tau^2} \qquad \tau \to 0$$

- There is mode creation which is observer dependent

- There is <u>no</u> mode creation if the in-vacuum is with respect to A_n^i, A_n^{+i} operators.

STRING TRANSITION THROUGH SINGULARITY

There is an analytic continuation of Bessel functions

$$H^{(2)}_\mu (e^{-i\pi} z) = -e^{i\pi\mu} H^{(1)}_\mu (z)$$

$$H^{1,2}_\mu (z) = J_\mu(z) \pm i \, Y_\mu(z)$$

giving

$$Z(e^{-i\pi} 2n\tau) = Z^*(2n\tau)$$

$$Z^*(e^{-i\pi} 2n\tau) = Z(2n\tau)$$

- For $n>0$ the string can be analytically continued from $\tau<0$ to $\tau>0$

- For $n=0$, it cannot

CONCLUSIONS

- Supergravity has given an insight into string theory. In particular the maximal supersymmetric backgrounds Minkowski space, AdS and plane waves
- The Penrose limit together with the solvability of string theory on plane waves may allow us to understand string theory in curved backgrounds, e.g. brane and cosmological
- Many aspects remain to be understood

(i) Systematic investigation of PL of "interesting" backgrounds

(ii) The relation of PL limits and original backgrounds

(iii) The nature of string singularities.