ANOMALY CANCELLATION IN SEVEN-DIMENSIONAL SUPERGRAVITY WITH A BOUNDARY.

A. Kehagias (NTU Athens)

Phys.Rev.D68:065019,2003

Phys.Rev.Lett.90:101601,2003

Kolymbari Crete, June 2003

2. THE MINIMAL $\mathcal{N}=2$ 7D SUGRA

There are 2 SUGRAs in 7D:

$$\boxed{\mathcal{N}=4 \qquad\qquad \text{R-symmetry} = SU(4)}\,,$$
$$\boxed{\mathcal{N}=2 \qquad\qquad \text{R-symmetry} = SU(2)}\,,$$

We will discuss the minimal $\mathcal{N}=2$ case the field content of which is ($M,N=0,...,6, \quad i,j=1,2,$)

$$(A_M, A_i{}^j, \psi^i)\,, \qquad\qquad \text{vector multiplet}$$
$$(g_{MN}, A_{MNK}, A_M{}^{ij}, \phi, \psi^i_M, \chi^i)\,, \qquad \text{gravity multiplet}$$

The vector multiplet contains:

- a vector A_M,
- an $SU(2)$ triplet of scalars $A_i{}^j$
- and an $SU(2)$ pseudo-Majorana spinor ψ^i

The gravity multiplet contains:

- the graviton g_{MN}
- an antisymmetric three-form A_{MNK}
- an $SU(2)$ triplet of vectors $A_M{}^{ij}$,
- a scalar ϕ
- the $SU(2)$ pseudo-Majorana gravitinos ψ^i_M
- and spinors χ^i.

GAUGING THE R-SYMMETRY

The $SU(2)$ R-symmetry of the minimal $\mathcal{N}=2$ 7D SUGRA can be gauged. The resulting $\mathcal{N}=2$ 7D gauged SUGRA has two version:

- with an antisymmetric two-form B_{MN} (*Salam & Sezgin (1983), Han, Koh & Lee (1985)*)

$$F_4 = dA_3 = *H_3 = *dB_2 \tag{0.1}$$

- with the three-form A_{MNP} *Townsend & Nieuwenhuizen (1983).*

The coupling of n vector multiplets to the 7D $\mathcal{N}=2$ supergravity leads to the irreducible multiplet

$$(g_{MN}, B_{MN}, A^I_M, \phi^\alpha, \psi, \chi^a, \psi^i_M)$$

$$I = 1, ..., n+3\,, \qquad a = 1, ..., n\,, \qquad \alpha = 1, ..., 3n$$

The scalars parametrize the coset

$$\frac{SO(n,3)}{SO(n)\times SO(3)}$$

We will consider the pure $\mathcal{N}=2$ 7D gauged supergravity with no vector multiplets.

THE LAGRANGIAN

The minimal $\mathcal{N} = 2$ 7D SUGRA is described by the Lagrangian

$$
\begin{aligned}
&\kappa^2 e^{-1}\mathcal{L}= \\
&\frac{1}{2}R - \frac{\sigma^{-4}}{48}F^2_{MNPQ} - \frac{\sigma^2}{4}F_{MNi}{}^j F^{MN}{}_j{}^i - \frac{1}{2}(\partial_M\phi)^2 - \frac{1}{2}\bar{\chi}^i\Gamma^M\mathcal{D}_M\chi_i \\
&-\frac{1}{2}\bar{\psi}^i_M\Gamma^{MNK}\mathcal{D}_N\psi_{Ki} - \frac{i\sigma}{2\sqrt{2}}\left(\frac{1}{2}\bar{\psi}^i_K\Gamma^{KMNR}\psi_{Rj} + \bar{\psi}^{Mi}\psi^N_j\right)F_{MNi}{}^j \\
&-\frac{\sigma^{-2}}{8\sqrt{2}}\left(\frac{1}{12}\bar{\psi}^i_K\Gamma^{KMNPQR}\psi_{Ri} + \bar{\psi}^{Mi}\Gamma^{NP}\psi^Q_i\right)F_{MNPQ} \\
&-\frac{\sigma}{2\sqrt{10}}\bar{\chi}^i\Gamma^M\Gamma^{NK}\psi_{Mj}F_{NKi}{}^j + \frac{\sigma^{-2}}{24\sqrt{10}}\bar{\chi}^i\Gamma^L\Gamma^{MNPQ}\psi_{Li}F_{MNPQ} \\
&+\frac{\sigma^{-2}}{160\sqrt{2}}\bar{\chi}^i\Gamma^{MNPQ}\chi_i F_{MNPQ} - \frac{3i\sigma}{20\sqrt{2}}\bar{\chi}^i\Gamma^{MN}\chi_j F_{MNi}{}^j \\
&+\frac{i}{48\sqrt{2}}F_{MNPQ}\left(F_{KLi}{}^j A_{Rj}{}^i - \frac{2ig}{3}\mathrm{tr}(A_K A_L A_R)\right)\epsilon^{MNPQKLR} \\
&+60(m - \frac{2}{5}h\sigma^4)^2 - 10(m + \frac{8}{5}h\sigma^4)^2 + (\frac{5}{2}m - h\sigma^4)\bar{\psi}^i_M\Gamma^{MN}\psi_{Ni} \\
&+\sqrt{5}(m+\frac{8}{5}h\sigma^4)\bar{\psi}^i_M\Gamma^M\chi_i + (\frac{3}{2}m+\frac{27}{5}h\sigma^4)\bar{\chi}^i\chi_i + \frac{1}{2}\bar{\chi}^i\Gamma^M\Gamma^N\partial_N\phi\psi_{Mi} \\
&+\frac{h}{36}\epsilon^{KLMNPQR}F_{KLMN}A_{PQR}\ ,
\end{aligned}
$$

where

$$F_{KLMN} = 4\partial_{[K}A_{LMN]}\ , \qquad \kappa^2 \text{ the 7D Newton's constant}$$

The local supersymmetry transformation rules are

$$
\begin{aligned}
\delta e_M^A &= \frac{1}{2}\bar{\epsilon}^i \Gamma^A \psi_{Mi} \ , \\
\delta\psi_{Mi} &= \mathcal{D}_M \epsilon_i + \frac{\sigma^{-2}}{80\sqrt{2}} \left(\Gamma_M{}^{NKPQ} - \frac{8}{3}\delta_M^N \Gamma^{KPQ} \right) F_{NKPQ}\epsilon_i \\
&\quad + \frac{i\sigma}{10\sqrt{2}} \left(\Gamma_M{}^{NK} - 8\delta_M^N \Gamma^K \right) F_{NKi}{}^j \epsilon_j + (m - \frac{2}{5}h\sigma^4)\Gamma_M \epsilon_i \ , \\
\delta A_{MNK} &= \frac{3\sigma^2}{2\sqrt{2}} \bar{\psi}^i_{[M} \Gamma_{NK]}\epsilon_i + \frac{\sigma^2}{\sqrt{10}} \bar{\chi}^i \Gamma_{MNK}\epsilon_i \ , \\
\delta A_{Mi}{}^j &= \frac{i\sigma^{-1}}{\sqrt{2}} \left(\bar{\psi}^j_M \epsilon_i - \frac{1}{2}\delta_i^j \bar{\psi}^k_M \epsilon_k \right) - \frac{i\sigma^{-1}}{\sqrt{10}} \left(\bar{\chi}^j \Gamma_M \epsilon_i - \frac{1}{2}\delta_i^j \bar{\chi}^k \Gamma_M \epsilon_k \right) \ , \\
\delta\chi_i &= \frac{1}{2}\Gamma^M \partial_M \phi \epsilon_i - \frac{i\sigma}{2\sqrt{10}} \Gamma^{MN} F_{MNi}{}^j \epsilon_j + \frac{\sigma^{-2}}{24\sqrt{10}} \Gamma^{MNPQ} F_{MNPQ}\epsilon_i \\
&\quad - \sqrt{5}(m + \frac{8}{5}h\sigma^4)\epsilon_i \ , \\
\delta\phi &= \frac{1}{2}\bar{\epsilon}^i \chi_i \ . \qquad (0.2)
\end{aligned}
$$

The notation here is

$$
\mathcal{D}_M \chi_i = \partial_M \chi_i + \frac{1}{4}\omega_{MAB}\Gamma^{AB}\chi_i + igA_{Mi}{}^j \chi_j \, , \quad m = -\frac{g\sigma^{-1}}{5\sqrt{2}} \, ,
$$

$$
F_{MNi}{}^j = \partial_M A_{Ni}{}^j + igA_{Mi}{}^k A_{Nk}{}^j - M \leftrightarrow N \, , \quad \sigma = \exp\left(-\frac{\phi}{\sqrt{5}} \right) \ ,
$$

and

g is the SU(2) coupling

The potential of the scalar ϕ is

$$V(\phi) = 16h^2\sigma^8 + 80hm\sigma^4 - 50m^2 \ ,$$

For $h/g > 0$ there are two extrema,

- a non-supersymmetric local minimum
- and a supersymmetric local maximum.

At the supersymmetric vacuum

$$< \sigma >= \left(\frac{g}{8\sqrt{2}h}\right)^{1/5}$$

$$V(< \sigma >) = V_0 = -15 \left(\frac{hg^4}{16}\right)^{2/5}$$

$V_0 < 0$ is the cosmological constant and, thus,

$$\text{SUSY vacuum of } \mathcal{N} = 2 \text{ 7D SUGRA} \rightarrow \mathbf{AdS_7} \qquad (0.3)$$

S^1 AND $S^1/\mathbb{Z}_2$ COMPACTIFICATION

Although there exists no stable Minkowski vacuum, one can still perform a dimensional reduction of the theory (even in the presence of a cosmological constant). From the graviton multiplet we get:

$$\begin{aligned}
&g_{MN} \rightarrow (g_{\mu\nu},\ g_{77}=\xi,\ g_{\mu 7}=A_\mu) \\
&A_{MNP} \rightarrow (A_{\mu\nu\rho},\ A_{\mu\nu 7}=A_{\mu\nu}) \\
&A_{Mi}{}^{j} \rightarrow (A_{\mu i}{}^{j},\ A_{7i}{}^{j}=A_i{}^{j}) \\
&\psi^i_M \rightarrow (\psi^i_\mu, \psi^i_7=\psi^i) \\
&\chi^i \rightarrow \chi^i
\end{aligned} \tag{0.4}$$

Dualizing the three-form $A_{\mu\nu\rho}$ into a vector B_μ we have the $\mathcal{N}\ =\ (1,1)$ 6D massless spectrum

$$\begin{aligned}
&(g_{\mu\nu}, A_\mu, A_{\mu\nu}, A_{\mu i}{}^{j}, \phi, \psi^i_\mu, \chi^i)\,, &&\text{gravity multiplet} \\
&(B_\mu, A_i{}^{j}, \xi, \psi^i)\,, &&\text{vector multiplet}
\end{aligned}$$

The minimal $\mathcal{N}\ =\ 2$ 7D SUGRA is invariant under $x_7 \rightarrow -x_7$ provided we have

$$\begin{aligned}
&A_{MNP} \rightarrow -A_{MNP}\,, && A_{Mi}{}^{j} \rightarrow -A_{Mi}{}^{j} \\
&h \rightarrow -h\,, && m \rightarrow -m\,,
\end{aligned} \tag{0.5}$$

As the $\mathbb{Z}_2$ transformation $x_7 \rightarrow -x_7$ is a symmetry of the theory, we can mod it out, by considering the compactification on $S^1/\mathbb{Z}_2$.

$$\text{Surviving fields at the orbifold fixed points} \rightarrow \mathbb{Z}_2\text{singlets} \tag{0.6}$$

$\mathbb{Z}_2$-parity assignments

$$g_{\mu\nu},\ A_{\mu\nu},\ \phi,\ \xi,\ A_i{}^j,\ \psi^i_{\mu-},\ \chi^i_+,\ \psi^i_+ \qquad \text{even parity}$$
$$A_\mu,\ B_\mu,\ A_{\mu i}{}^j,\ \psi^i_{\mu+},\ \chi^i_-,\ \psi^i_- \qquad \text{odd parity}$$

For supersymmetry parameters we have ϵ_- is even whereas ϵ_+ is odd. The odd-parity fields are projected out while the even-parity fields survive the orbifold projection and are organized in 6D $\mathcal{N} = (0,1)$ representations. The surviving fields on the orbifold $S^1/\mathbb{Z}_2$ are arranged into the following 6D multiplets

$$(*) \qquad \begin{array}{ll} (g_{\mu\nu}, A^+_{\mu\nu}, \psi^i_\mu)\,, & \text{gravity} \\ (A^-_{\mu\nu}, \phi, \chi^i)\,, & \text{tensor} \\ (A_i{}^j, \xi, \psi^i)\,, & \text{hypermultiplet} \end{array}$$

where $\psi^i_\mu = \psi^i_{\mu-}$ are left-handed symplectic Majorana-Weyl fermions while $\chi^i = \chi^i_+$ and $\psi^i = \psi^i_+$ are right-handed.

The 6D spectrum (*) at the orbifold fixed points is anomalous!

The only way to make sense of such a theory is to introduce extra vector, hyper, and tensor multiplets at the fixed points to cancel the anomalies as in the HW case.

Massless representations of the $(0,1)$ supersymmetry in 6D, labeled by their $SU(2) \times SU(2)$ representations:

$$
\begin{array}{lll}
\text{(i)} & \text{gravity}: & (1,1) + 2(\tfrac{1}{2},1) + (0,1) = (g_{\mu\nu}, B^{+}_{\mu\nu}, \psi_{\mu})\,, \\
\text{(ii)} & \text{tensor}: & (1,0) + 2(\tfrac{1}{2},0) + (0,0) = (B^{-}_{\mu\nu}, \phi, \chi)\,, \\
\text{(iii)} & \text{vector}: & (\tfrac{1}{2},\tfrac{1}{2}) + 2(0,\tfrac{1}{2}) = (A\mu, \lambda)\,, \\
\text{(iv)} & \text{hyper}: & 2(\tfrac{1}{2},0) + 4(0,0) = (\phi^{\alpha}, \psi^{a}).
\end{array}
$$

THE VACUUM OF THE THEORY

The supersymmetric vacuum is the one in which the Killing equations

$$\delta\psi_{Mi} = \delta\chi_i = 0 \ , \tag{0.7}$$

are satisfied. Assuming that all bulk fields are zero except for the scalar ϕ we find from Eq.(0.7) that

$$\langle\sigma\rangle = \left(\frac{g}{8\sqrt{2}h}\right)^{\frac{1}{5}} \ . \tag{0.8}$$

The supersymmetric vacuum invariant under $y \to -y$ is then specified to be

$$ds^2 = e^{-2k|y|}dx_6^2 + dy^2 \ , \tag{0.9}$$

where $0 \leq y \leq \pi R$ and k is the AdS curvature scale which is given by

$$k = \left(\frac{hg^4}{16}\right)^{\frac{1}{5}} \ . \tag{0.10}$$

Namely it is a seven-dimensional Randall-Sundrum solution.

The cosmological constants are given by

$$\Lambda_7 = -15M^5k^2 \ \ ; \qquad \Lambda_{(0)} = -\Lambda_{(\pi R)} = 10M^5k \ , \tag{0.11}$$

where $M = \kappa^{-2/5}$ is the 7D Planck mass.

BULK-BOUNDARY LAGRANGIAN

The general boundary theory contains vector, hyper and tensor multiplets. The boundary action may collectively be written as

$$S_{boundary} = S_{YM} + S_H + S_T \ ,$$

$S_{YM},\ S_H,\ S_T$ are the actions for the vector, hyper, and tensor multiplets, respectively.

Supersymmetric action:

$$S = S_{bulk} + S_{boundary}$$

How we can couples the 6D boundary theory of vectors, hypers and tensor multiplets to the 7D bulk? Or, how can we determine $S_{YM},\ S_H,\ S_T$?

Strategy:

- Start with the globally supersymmetric theory.
- Perform local supersymmetric variations to both the bulk and the boundary.
- Add the interaction $\psi_M \mathcal{S}_M$ to cancel part of the variation.
- Modify the Bianchi identity of F_4 to cancel the remaining variation.
- Add possible extra terms needed for making the total theory supersymmetric

Boundary vector multiplets

$$
\begin{aligned}
S_{YM} = -\frac{1}{\lambda^2}\int d^6x\sqrt{-g}\,\Big[&\frac{1}{4}\sigma^{-2}F^a_{\mu\nu}F^{a\mu\nu}+\frac{1}{2}\bar{\lambda}^a\Gamma^\mu D_\mu\lambda^a \\
&+\frac{1}{4}\sigma^{-1}\bar{\psi}_\mu\Gamma^{\nu\rho}\Gamma^\mu\lambda^a F^a_{\nu\rho}+\frac{1}{2\sqrt{5}}\sigma^{-1}\bar{\lambda}^a F^a_{\mu\nu}\Gamma^{\mu\nu}\chi \\
&-\frac{1}{24\sqrt{2}}\,\sigma^{-2}\bar{\lambda}^a\Gamma^{\mu\nu\rho}\lambda^a F_{\mu\nu\rho 7}+\frac{i}{2\sqrt{2}}\,\sigma\bar{\lambda}^{ai}\Gamma^\mu F_{\mu 7 i}{}^j\lambda^a_j\Big]\ ,
\end{aligned}
$$

Invariance under the supersymmetry transformations (up to fermionic bilinear terms):

$$
\begin{aligned}
\delta A^a_\mu &= \frac{1}{2}\,\sigma\,\bar{\epsilon}\Gamma_\mu\lambda^a\ , \\
\delta\lambda^a &= -\frac{1}{4}\,\sigma^{-1}\,\Gamma^{\mu\nu}F^a_{\mu\nu}\epsilon\ .
\end{aligned}
$$

Modification of the Bianchi identity:

$$
dF_{7\mu\nu\rho\sigma}=\frac{\kappa^2}{\sqrt{2}\lambda^2}F^a_{[\mu\nu}F^a_{\rho\sigma]}\delta(x^7) \tag{0.12}
$$

Under gauge transformations

$$
\delta A_{7\mu\nu}=\frac{\kappa^2}{\sqrt{2}\lambda^2}\mathrm{tr}(\epsilon F) \tag{0.13}
$$

Boundary hypermultiplets

The supersymmetric boundary action for hypermultiplets is:

$$\begin{aligned} S_H \;=\; & \int d^6x\sqrt{-g}\,\Big[-\frac{1}{2}g_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta - \frac{1}{2}\bar{\zeta}^Y\Gamma^\mu D_\mu\zeta_Y \\ & + \frac{1}{2}\sigma^{1/2}\bar{\psi}^i_\mu\,\Gamma^\nu\Gamma^\mu\partial_\nu\varphi^\alpha V_{\alpha i}{}^Y\zeta_Y + \frac{1}{2\sqrt{5}}\sigma^{1/2}V_{\alpha iY}\bar{\zeta}^Y\Gamma^\mu\partial_\mu\varphi^\alpha\chi^i \\ & + \frac{1}{24\sqrt{2}}\sqrt{-g}\;\sigma^{1/2}\bar{\zeta}^Y\Gamma^{\mu\nu\rho}\zeta_Y F_{7\mu\nu\rho}\Big]\ , \end{aligned}$$

It is invariant under the superymmetry transformations

$$\begin{aligned} \delta\varphi^\alpha &= \frac{1}{2}\sigma^{-1/2}V^\alpha{}_{iY}\bar{\epsilon}^i\zeta^Y\ , \\ \delta\zeta^Y &= \frac{1}{2}\sigma^{1/2}\,V_{\alpha i}{}^Y\Gamma^\mu\partial_\mu\varphi^\alpha\epsilon^i\ . \end{aligned}$$

Here, φ^α $(\alpha,... = 1,...,4n_H)$, ζ^Y $(Y,... = 1,...,2n_H)$ are, respectively, the scalar and fermion components of the n_H hypermultiplets on the boundary, and $g_{\alpha\beta}$ is the metric of the scalar manifold.The covariant derivative is defined as

$$D_\mu\zeta^Y = \partial_\mu\zeta^Y + \Gamma^Y_{\alpha X}\partial_\mu\varphi^\alpha\zeta^X\ ,$$

where $\Gamma^Y_{\alpha X}$ is the $Sp(n_H)$ connection.

We should make sure that the scalar manifold is quaternionic as required by $\mathcal{N} = (0,1)$ 6D local supersymmetry. This condition is obtained in a novel way by the boundary value of the field strength of the bulk $SU(2)$ gauged R-symmetry group:

$$F_{\mu\nu i}{}^j = \frac{i}{4\sqrt{2}}\kappa^2\partial_\mu\varphi^\alpha\partial_\nu\varphi^\beta J_{\alpha\beta i}{}^j\ .$$

$J_i{}^j$ is the triplet of complex structures.

Using this boundary value one may prove

$$\Omega_i{}^j = -\frac{g\kappa^2}{4\sqrt{2}} J_i{}^j \ ,$$

where $\Omega_i{}^j$ is the $Sp(1)$ curvature two-form.

ANOMALY CANCELLATION

There are three sources that can contribute to 6D gravitational anomalies, namely fermions, gravitinos and antisymmetric tensor fields which may spoil 6D diffeomorphisms. In particular, the anomaly eight forms for a right-handed fermion, a right-handed gravitino and a self-dual two-form are

$$I_{1/2} = \frac{1}{(2\pi)^4}\frac{1}{5760}\Big[\mathrm{Tr}R^4 + \frac{5}{4}(\mathrm{Tr}R^2)^2\Big] = \frac{1}{5760}\left(7p_1^2 - 4p_2\right)$$
$$I_{3/2} = \frac{1}{(2\pi)^4}\frac{1}{5760}\Big[245\mathrm{Tr}R^4 - \frac{215}{4}(\mathrm{Tr}R^2)^2\Big] = \frac{1}{5760}\left(275p_1^2 - 980p_2\right)$$
$$I_A = \frac{1}{(2\pi)^4}\frac{1}{5760}\Big[28\mathrm{Tr}R^4 - 10(\mathrm{Tr}R^2)^2\Big] = \frac{1}{5760}\left(16p_1^2 - 112p_2\right)$$

The Pontryagin classes are defined as

$$(2\pi)^2 p_1 = -\frac{1}{2}\mathrm{Tr}R^2\,, \quad (2\pi)^4 p_2 = -\frac{1}{4}\mathrm{Tr}R^4 + \frac{1}{8}(\mathrm{Tr}R^2)^2$$

Vectors and hypers may have gravitational, gauge and mixed anomalies. The anomaly eight form is given in this case by the eight form in the expansion $ch(F)\hat{A}(M)$ where the Chern classes and the A-roof genus are given by

$$ch(F) = \mathrm{Tr}_\rho(e^{iF/2\pi})\,, \quad \hat{A}(M) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \ldots$$

Recalling that fermions in vectors and hypermultiplets have opposite chiralities, the anomaly for N_V vector and N_H hypermultiplets in 6D, is

$$I_{YM+H} = (N_V - N_H)I_{1/2} + \frac{1}{48}p_1\, X^{(2)} + \frac{1}{24}X^{(4)}$$

where $X^{(n)}$ are defined as

$$(2\pi)^n X^{(n)} = \mathrm{Tr}F^n - \sum_{\mathcal{R}} n_{\mathcal{R}} \mathrm{tr}_{\mathcal{R}} F^n$$

As usual, Tr denotes the trace in the adjoint representation, $\mathrm{tr}_{\mathcal{R}}$ the trace in the representation $\mathcal{R}$ of the group $\mathcal{G}$, and $n_{\mathcal{R}}$ is the number of hypermultiplets in the representation $\mathcal{R}$. Expressing $X^{(2)}$, $X^{(4)}$ as

$$(2\pi)^2 X^{(2)} = \beta\, \mathrm{tr}F^2 \,, \qquad (2\pi)^4 X^{(4)} = \alpha\, \mathrm{tr}F^4 + \gamma\, (\mathrm{tr}F^2)^2$$

where α, β, γ are constants, we may write

$$\begin{aligned} I_{YM+H} &= (N_V - N_H) I_{1/2} + \frac{1}{(2\pi)^2} \frac{\beta}{48} p_1\, \mathrm{tr}F^2 \\ &+ \frac{1}{(2\pi)^4} \frac{\gamma}{24} (\mathrm{tr}F^2)^2 + \frac{1}{(2\pi)^4} \frac{\alpha}{24} \mathrm{tr}F^4 \end{aligned}$$

In our case, there are three possible contributions to the 6D anomaly:

- Bulk anomalies due to $S^1/\mathbb{Z}_2$ compactification
- Anomalies in the boundary theory (due to vector, hyper and tensor multiplets)
- Anomalous variations of bulk Chern-Simons terms.

(a) Anomalies from the bulk fields

The bulk contribution is due to the localized graviton, hyper and tensor multiplet in the $S^1/\mathbb{Z}_2$ compactification. It is evenly distributed between the two fixed points and it is written as

$$
\begin{aligned}
I_{bulk} &= \frac{1}{2}\left(I_{3/2} - 2I_{1/2}\right)\,\delta(y) + \frac{1}{2}\left(I_{3/2} - 2I_{1/2}\right)\,\delta(y-\pi R) \\
&= \frac{1}{2\cdot 5760}\,\left(261p_1^2 - 972p_2\right)\,\delta(y) \\
&\quad + \frac{1}{2\cdot 5760}\,\left(261p_1^2 - 972p_2\right)\,\delta(y-\pi R)
\end{aligned}
$$

(b) Boundary-theory anomalies

The boundary theory contains vectors and hypers which contribute to gravitational, gauge and mixed anomalies whereas tensor multiplets contribute to gravitational anomalies. In particular, with $N_V(\tilde{N}_V)$, vectors, $N_H(\tilde{N}_H)$ hypers and $N_T(\tilde{N}_T)$ tensors at $y = 0(\pi R)$, the boundary anomaly eight-form is

$$
\begin{aligned}
I_{bd} &= \left(I_{YM+H} - N_T I_A - N_T I_{1/2}\right)\,\delta(y) \\
&\quad + \left(\tilde{I}_{YM+H} - \tilde{N}_T I_A - \tilde{N}_T I_{1/2}\right)\,\delta(y-\pi R)
\end{aligned}
$$

Tilded quantities will be used to denote that they are referred to the theory at $y = \pi R$. Then the boundary anomaly is

$$
\begin{aligned}
I_{bd} \;=\; & \left\{ \frac{1}{5760}\Big[4(N_H+29N_T-N_V)p_2+(7N_V-7N_H-23N_T)p_1^2\Big] \right. \\
& \left. +\frac{1}{(2\pi)^2}\frac{\beta}{48}p_1\,\mathrm{tr}F^2+\frac{1}{(2\pi)^4}\frac{\gamma}{24}(\mathrm{tr}F^2)^2+\frac{1}{(2\pi)^4}\frac{\alpha}{24}\mathrm{tr}F^4\right\}\delta(y) \\
& +\left\{ \frac{1}{5760}\Big[4(\tilde N_H+29\tilde N_T-\tilde N_V)p_2+(7\tilde N_V-7\tilde N_H-23\tilde N_T)p_1^2\Big] \right. \\
& \left. +\frac{1}{(2\pi)^2}\frac{\tilde\beta}{48}p_1\,\mathrm{tr}\tilde F^2+\frac{1}{(2\pi)^4}\frac{\tilde\gamma}{24}(\mathrm{tr}\tilde F^2)^2+\frac{1}{(2\pi)^4}\frac{\tilde\alpha}{24}\mathrm{tr}\tilde F^4\right\}\delta(y-\pi R)
\end{aligned}
$$

(c) Anomalies from bulk terms with anomalous variations

As in the HW case, there exists an anomalous variation of the bulk three-form A_{MNP} which will contribute to the anomaly. Up to two derivative level, the only such contribution comes from the Chern-Simons term in the bulk Lagrangian (**??**), which in the "downstairs" approach is

$$
S_{CS} = \frac{2}{\kappa^2}\int_M h\; F_4 \wedge A_3 \;,
$$

with A_3 the three-form gauge field of 7D supergravity and $F_4 = dA_3$. We also expect, as in the 11D supergravity higher derivative terms which are expected to contribute to the anomaly. There are two possible such terms, namely

$$
S_R = \xi_R \int A_3 \wedge p_1 \;,
$$

and the gravitational Chern-Simons term (in "downstairs" approach)

$$S_{GCS} = \xi_G \int X_7 \ ,$$

where ξ_R, $\xi_G = n\pi$ are constants and X_7 is a 7D Chern-Simons form satisfying

$$X_8 = dX_7 = \frac{1}{48}\left(p_2 - \frac{p_1^2}{4}\right)$$

The above S_R and S_{GCS} terms should exist in the gauged 7D $\mathcal{N} = 2$ supergravity (resulting from the $K3$ compactification of the 11D five-brane anomaly term $F_4 \wedge X_7$), and they are expected to survive after gauging. Under diffeomorphism, X_7 transforms as $X_7 \rightarrow X_7 + dX_6^{(0)}$. Thus, on a manifold with boundary, the anomalous variation of S_{GCS} is

$$\delta S_{GCS} = \xi_G \int X_6^{(0)}$$

and the corresponding anomaly eight form is then

$$I_{GCS} = \frac{\xi_G}{2\pi} X_8$$

The anomalous variations of S_{CS} and S_R can be found by employing the boundary value and variation of A_3. The latter can be determined by recalling that on each component of the boundary, we will have

$$F_4|_{\partial M} = \frac{\kappa^2}{\sqrt{2}\lambda^2} Q_4 \ ,$$

where the four-form Q_4 is defined as

$$Q_4 = \xi_{CS}\,\mathrm{tr}R^2 - \mathrm{tr}F^2$$

and ξ_{CS} is a numerical constant. We may now define $Q_3 = \xi_{CS}\,\omega_{3L} - \omega_{3Y}$ where as usual $\omega_{3Y,L}$ are the Yang-Mills and Lorentz Chern-Simons terms

$$\begin{aligned}\omega_{3Y} &= \mathrm{tr}\left(AF - \frac{1}{3}A^3\right)\\ \omega_{3L} &= \mathrm{tr}\left(\omega R - \frac{1}{3}\omega^3\right)\end{aligned}$$

Then, we have the descent equations

$$Q_4 = dQ_3\,, \quad \delta Q_3 = dQ_2^1$$

for δ gauge and Lorentz transformations which follows from

$$\begin{aligned}d\omega_{3L} = \mathrm{tr}R^2\,, &\quad \delta\omega_{3L} = d\omega_{2L}^1\,,\\ d\omega_{3Y} = \mathrm{tr}F^2\,, &\quad \delta\omega_{3Y} = d\omega_{2Y}^1\end{aligned}$$

In the following we will not need the explicit forms of ω_3, ω_2^1. Then, we have

$$A_3|_{\partial M} = \frac{\kappa^2}{\sqrt{2}\lambda^2}Q_3$$

so that

$$\delta A_3|_{\partial M} = \frac{\kappa^2}{\sqrt{2}\lambda^2}dQ_2^1$$

This variation is extendable to the bulk by writing

$$\delta A_3|_{\partial M} = d\Lambda\,, \quad \Lambda|_{\partial M} = \frac{\kappa^2}{\sqrt{2}\lambda^2}Q_2^1$$

Then, the anomalous variation of the bulk action is

$$\delta S_{CS} = -\frac{\kappa^2 h}{\lambda^4} \int_{M^6} Q_2^1 \wedge Q_4$$

where M^6 is the boundary at $y = 0$, and it should be compensated by the anomaly of the boundary theory. Thus, the bulk anomaly eight form from the Chern-Simons term S_{CS} is

$$I_{CS} = -\frac{\kappa^2 h}{2\pi\lambda^4}\, Q_4 \wedge Q_4$$

However, these are not the only sources which contribute to the anomaly. In particular, we expect a term (see Appendix B)

$$S_R = -\xi_R \int_{M^7} A_3 \wedge \mathrm{tr} R^2 \ ,$$

in the 7D action where ξ_R is a dimensionful constant. Such a term exists in the gauged 7D $\mathcal{N} = 2$ supergravity theory resulting from the $K3$ compactification of the 11D five-brane anomaly term, and is expected to survive after gauging. The anomalous variation of S_R is

$$\delta S_R = -\frac{\kappa^2 \xi_R}{\sqrt{2}\lambda^2} \int_{M^6} Q_2^1 \wedge \mathrm{tr} R^2 \ ,$$

so that the corresponding anomaly eight-form becomes

$$I_R = -\frac{\kappa^2 \xi_R}{2\pi\sqrt{2}\lambda^2} Q_4 \wedge \mathrm{tr} R^2 \ .$$

In addition, the anomaly eight-form for the boundary theory is

$$I_{bdy} = \frac{1}{(2\pi)^4}\left[\frac{1}{4608}\Big(N_V - N_H + 7N_T\Big)(\mathrm{tr}R^2)^2 - \frac{\beta}{96}\mathrm{tr}R^2\mathrm{tr}F^2 + \frac{\gamma}{24}(\mathrm{tr}F^2)^2\right]$$

The reducible part of the anomaly eight-form from the bulk fields (**??**), is evenly distributed between the two fixed points and contributes a term

$$I_{bulk} = \frac{1}{(2\pi)^4} \frac{-1}{2 \cdot 5760} \frac{225}{4} (\mathrm{tr} R^2)^2 \ ,$$

at each fixed point. Finally, there exists a contribution to the anomaly arising from the gravitational Chern-Simons term (0.18). The irreducible $\mathrm{tr} R^4$ part in Eq.(0.21) has been cancelled against the bulk and boundary irreducible parts of the anomaly. The remaining contribution of the gravitational Chern-Simons term to the anomaly is then

$$I_{GCS} = \frac{1}{(2\pi)^4} \frac{n}{8 \cdot 192} (\mathrm{tr} R^2)^2 \ .$$

The total anomaly eight-form coming from the bulk, the boundary theory and the Chern-Simons terms is

$$I_{\text{total}} = I_{bulk} + I_{bdy} + I_{CS} + I_R + I_{GCS} \ .$$

It is a polynomial in $\mathrm{tr} R^4, (\mathrm{tr} R^2)^2,\ \mathrm{tr} R^2 \mathrm{tr} F^2$ and $(\mathrm{tr} F^2)^2$ and the vanishing of the total anomaly is equivalent to the vanishing of the coefficients of these terms. We get:

$$\begin{aligned}
\mathrm{tr} R^4 : \quad & N_H + 29 N_T - N_V = \frac{243}{2} - 15n, \qquad y = 0 \\
& \tilde{N}_H + 29 \tilde{N}_T - \tilde{N}_V = \frac{243}{2} + 15n \ , \qquad y = \pi R \\
(\mathrm{tr} R^2)^2 : \quad & 320 \gamma \xi_{CS}^2 - 80 \beta \xi_{CS} + 2(N_V - N_H + N_T) + 3 = 0 \, , \\
\mathrm{tr} R^2 \mathrm{tr} F^2 : \quad & 384 \sqrt{2} \pi^3 \xi_R \frac{\kappa^2}{\lambda^2} = \beta - 8 \gamma \xi_{CS} \ ,
\end{aligned}$$

respectively, whereas the vanishing of the $(\mathrm{tr}F^2)^2$ specifies the dimensionless ratio, η, as

$$\eta \equiv \frac{h\kappa^2}{\lambda^4} = \frac{\gamma}{3(4\pi)^3} \ .$$

The last relation fixes the gauge coupling, λ, in terms of the gravitational coupling, κ, and the topological mass parameter, h, of the Chern-Simons term. This is similar to the relation obtained in the HW theory except for the presence of the extra parameter, h. The difference is due to the fact that in the 11D HW theory the Chern-Simons term is fixed by supersymmetry, whereas in seven dimensions the theory is supersymmetric up to an arbitrary topological mass parameter, h.

The DR of the $F_4 \wedge X_7$ term of 11D SUGRA

The dimensional reduction of the interaction term $F_4 \wedge X_7$ of 11D supergravity gives rise to a gravitational Chern-Simons term, S_{GCS} and the term S_R in (**??**). In 11D supergravity the correct normalization of the $F_4 \wedge X_7$ term is

$$S_{GCS} = \frac{1}{2}\left(\frac{4\pi^2}{3\kappa_{11}^2}\right)^{1/3} \int F_4 \wedge X_7 \ , \tag{0.14}$$

where κ_{11} is the 11D Newton's constant. Upon compactification on $K3$, the above interaction gives rise to two terms in the 7D effective supergravity action

$$S_{GCS} = \frac{1}{2}\left(\frac{4\pi^2}{3\kappa_{11}^2}\right)^{1/3} \left(\int_{K3} F_4\right) \int X_7 \equiv \xi_G \int X_7 \ , \tag{0.15}$$

and

$$S_R = \frac{1}{2}\left(\frac{4\pi^2}{3\kappa_{11}^2}\right)^{1/3} \int F_4 \wedge \int_{K3} X_7 \ . \tag{0.16}$$

According to [**?**],[**?**], the F_4-fluxes are quantized according to

$$\int_{\mathcal{C}_4} F_4 = (6\pi)^{1/3} \kappa_{11}^{2/3}\, n \ , \tag{0.17}$$

where n is an integer or half-integer. If $\mathcal{C}_4$ is the $K3$ surface, we obtain for S_{GCS} the result

$$S_{GCS} = n\pi \int X_7 \ , \tag{0.18}$$

so that $\xi_G = n\pi$ and X_7 is a 7D Chern-Simons term satisfying

$$X_8 = dX_7 = \frac{1}{(2\pi)^4}\frac{1}{192}\left[\frac{1}{4}(\mathrm{tr}R^2)^2 - \mathrm{tr}R^4\right] \ . \tag{0.19}$$

Under diffeomorphisms, X_7 transforms as $X_7 \to X_7 + dX_6^{(0)}$. Thus, on a manifold with a boundary, the anomalous variation of S_{GCS} is

$$\delta S_{GCS} = \xi_G \int X_6^{(0)} , \tag{0.20}$$

and the corresponding anomaly eight form is then

$$I_{GCS} = \frac{\xi_G}{2\pi} X_8 \left[\delta(y) - \delta(y - \pi R)\right] , \tag{0.21}$$

and similarly at $y = \pi R$.

Performing the integral in (0.16), it is not difficult to see that

$$X_3 = \int_{K3} X_7 = \frac{1}{4 \cdot (2\pi)^2} \omega_{3L} , \tag{0.22}$$

so that the term S_R can be written as

$$S_R = -\frac{1}{8 \cdot (2\pi)^2} \left(\frac{4\pi^2}{3\kappa_{11}^2}\right)^{1/3} \int A_3 \wedge \mathrm{tr} R^2 . \tag{0.23}$$

Thus the coefficient ξ_R is determined to be

$$\xi_R = \left(\frac{\pi^2}{48\kappa_{11}^2}\right)^{1/3} = \left(\frac{\pi^2}{48\kappa^2}\right)^{1/3} {V_{K3}}^{-1/3} , \tag{0.24}$$

where V_{K3} is the volume of $K3$ and the relation $\kappa_{11}^2 = V_{K3}\kappa^2$ between the 11D and 7D Newton constants κ_{11} and κ, respectively, has been used.

Solutions

We tabulate here all the solutions which satisfy the anomaly constraint conditions for $n_T = 1$.

$\mathcal{G}_1 \times \mathcal{G}_2$	(n_1, n_2, n_S)
$E_8 \times E_7$	(0,10,64), (1,5,96)
$E_8 \times E_6$	(0,18,83) , (1,8,105)
$E_8 \times F_4$	(0,17,101), (1,7,113)
$E_8 \times G_2$	(0,11,428) , (0,46,183), (1,16,145), (2,1,2)
$E_7 \times E_7$	$(n_1, 8 - n_1, 61)$
$E_7 \times E_6$	$(n_1, 14 - 2n_1, 76 - 2n_1)$, (2,7,153)
$E_7 \times F_4$	$(n_1, 13 - 2n_1, 90 - 4n_1)$, (2,6,160)
$E_7 \times G_2$	$(n_1, 34 - 6n_1, 152 - 14n_1)$, (1,12,250),(2,13,187), (6,7,5)
$E_6 \times E_6$	$(n_1, 12 - n_1, 75)$, (2,7,156)
$E_6 \times F_4$	$(n_1, 11 - n_1, 87 - n_1)$, (2,6,163),(5,9,4), (7,1,158)
$E_6 \times G_2$	$(n_1, 28 - 3n_1, 139 - 6n_1)$, (0,12,251), (2,13,190), (3,14,156), (5,22,46), (9,6,50), (10,7,16)
$F_4 \times F_4$	$(n_1, 10 - n_1, 87)$, (1,6,165),(4,9,9)
$F_4 \times G_2$	$(n_1, 25 - 3n_1, 134 - 5n_1)$, (1,13,192), (2,14,159), (4,22,51), (8,6,59), (9,7,26)
$G_2 \times G_2$	$(n_1, 20 - n_1, 131)$, (1,14,166), (6,19,96), (7,22,68), (8,28,19)

CONCLUSIONS

- A Horava-Witten type $S^1/\mathbb{Z}_2$ compactification of gauged $\mathcal{N} = 2$ 7D SUGRA leads to an anomalous 6D theory.
- The vacuum of this theory is a 7D Randal-Sundrum bac-ground.
- Matter should exists at the orbifold fixed points.
- Anomalies can be cancelled by inflow mechanism leading to sensible 6D theory.
- Further compactification to 4D.
- String or M-theory embedding.
- Various limits of the background. Taking the UV brane to infinity decouples gravity. Tensionless strings.
- AdS/CFT correspondence. Dual CFT theory.