Quantum Hall Physics in String Theory

Oren Bergman

Technion, Israel Institute of Technology

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Outline

1. Introduction

2. Branes and the 2d FQHF

3. Higher dimensional FQHFs

4. Future directions

Bergman,Brodie,Okawa hep-th/0107178

Bergman,Brodie to appear

1 Introduction

In certain backgrounds and in certain decoupling limits string theory exhibits fractional quantum Hall-like behavior.

This suggests a new direction for string theory, as an effective theory of the FQHE.

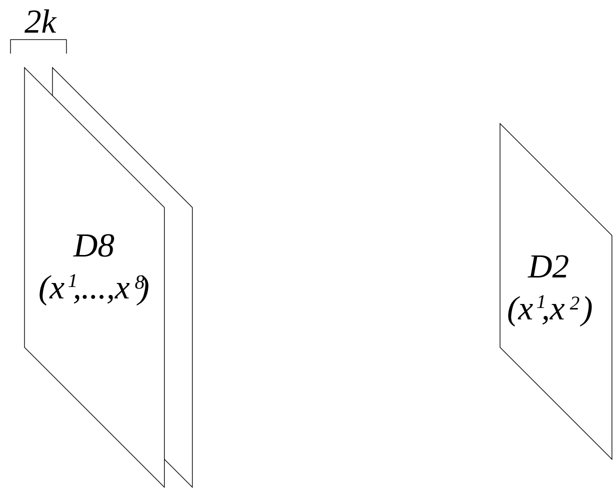
The basic principle can be thought of as replacing the coordinates of the electrons by large N matrices.

This "effective string theory" naturally incorporates:

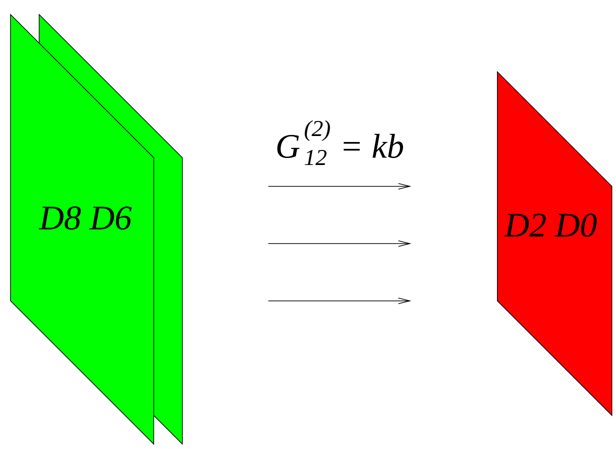
- Quantization of filling fraction $\nu = 1/k$.
- Fractional charges $\pm\nu e$ and statistics.
- Effective gauge theories of FQHE.

2 Branes and the 2d FQHF

D2-brane in massive IIA SUGRA, $G^{(0)} = k$:



Turn on B-field, $B_{12} = b \Rightarrow$



2d D0-brane QHF with filling fraction

$$\nu = \frac{\rho_0}{|G^{(2)}|} = \frac{b}{kb} = \frac{1}{k} \, .$$

D2-brane effective gauge theory

$$S = \frac{\mu_2}{g_s}\int d^3x\,\sqrt{\det(g+2\pi\alpha' F+B)}$$

$$+\mu_2\int C\wedge e^{2\pi\alpha' F+B}+\frac{k}{4\pi}\int A\wedge dA+\cdots$$

Decoupling limit $\alpha'\to 0$, with scalings

$$g_s\sim(\alpha')^{\frac{3}{2}+\epsilon}\,,\ g_{ij}\sim(\alpha')^{2+\delta}\,,\ b=\frac{B_{12}}{2\pi\alpha'}=\text{finite}$$

where $\delta>\epsilon>0$ ($\delta=\epsilon=0$ is SW limit).

Only the topological terms survive:

$$S=\frac{\mu_0}{2\pi}\int C^{(1)}\wedge dA+\frac{k}{4\pi}\int A\wedge dA\,.$$

This resembles Fradkin's effective gauge theory for the FQHE, where

$C^{(1)}$ = electromagnetic gauge field

A = "hydrodynamic" gauge field, describes fluctuations of the fluid

NCCS theory

Define a new gauge field $\widehat{A}$ by the SW map. In the SW limit, the worldvolume theory reduces to a non-commutative gauge theory:

- DBI action $\to$ NCYM (Seiberg+Witten)
- WZ action $\to$ NCWZ (Mukhi,et.al.)
- CS action $\to$ NCCS*

In our limit $\widehat{S} = \widehat{S}_{NCWZ} + \widehat{S}_{NCCS}$, where

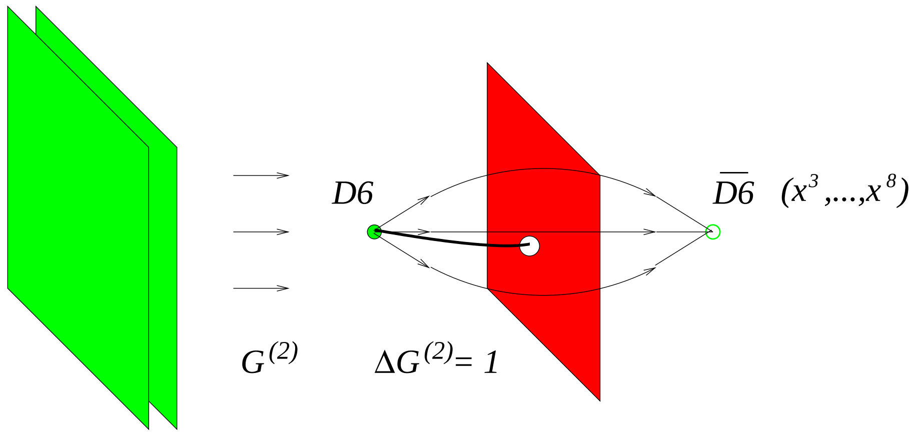
$$\widehat{S}_{NCCS} = \frac{k}{4\pi} \int \left(\widehat{A} \star d\widehat{A} + \frac{2i}{3} \widehat{A} \star \widehat{A} \star \widehat{A} \right),$$

in agreement with Susskind's conjecture.

*The simplicity of the SW transform of the CS action was demonstrated by Grandi+Silva and Okawa+Ooguri. It's related to the absense of B in the CS term.

Quasihole/quasiparticle excitations

Take a D6-brane across the D2-brane:



(a) $G^{(2)} \to G^{(2)} \pm 1/A$

(b) string created, end carries $\mp\frac{1}{k}$ D0 charge*

The D0-brane state has two components

$$N = \rho_0 A = (N \pm 1/k) \mp 1/k \ ,$$

a uniform fluid with

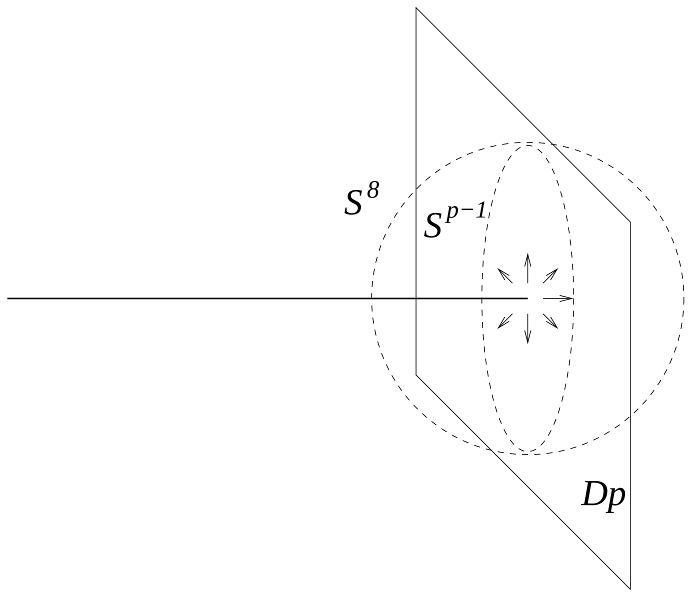
$$\nu = \frac{N \pm 1/k}{A|G^{(2)} \pm 1|} = \frac{1}{k} \ ,$$

and a localized excitation with $Q_0 = \mp 1/k$.

*Explain next.

Strings ending on branes in massive SUGRA

Ordinarily, the string end is a source of world-volume electric flux (Strominger):



B equation of motion from SUGRA + DBI:

$$d(*H) = \pm\delta^{(8)} + *_p F \wedge \delta^{(9-p)}$$

$$\int_{S^8} d(*H) = 0 = \pm 1 + \int_{S^{p-1}} *_p F$$

Corollary: Strings cannot end on D0-branes.

In massive IIA SUGRA there are additional terms in the B equation of motion:

$$\begin{aligned} d(*H) &= \pm\delta^{(8)} + *_p F \wedge \delta^{(9-p)} \\ &+ k\left(G^{(8)} + B\wedge G^{(6)} + B^2\wedge G^{(4)}\right) \\ &+ \cdots \end{aligned}$$

Integrate over S^8 surrounding the string end:

$$0 = \pm 1 + \int_{S^{p-1}} *_p F + k\int_{S^8} G^{(8)}$$

But electric flux vanishes asymptotically due to CS term, so the end carries D0 charge:

$$Q_0 = \int_{S^8} G^{(8)} = \pm\frac{1}{k}\,.$$

(Bergman+Brodie)

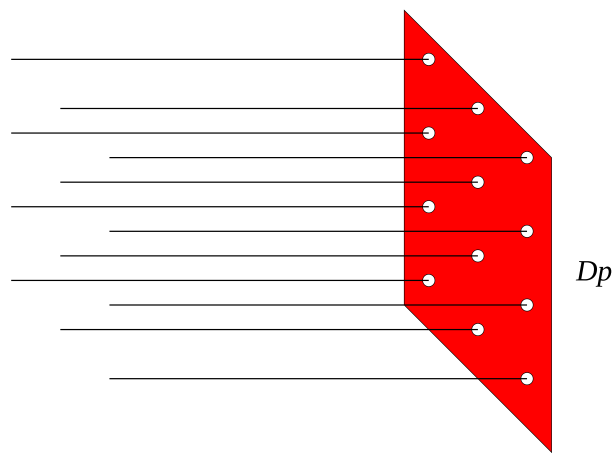
Corollary: D0-branes carry k string ends.

(Polchinski+Strominger)

More strings ending on branes

2d array of strings ending on a Dp with B:

$$\rho_s\,\delta^{(6)}\wedge dx^1\wedge dx^2\,,\quad B_{12}=b$$



Relevant terms in B equation

$$d(*H)=\rho_s\,\delta^{(6)}\wedge dx^1\wedge dx^2+k\left(G^{(8)}+B\wedge G^{(6)}\right)$$

Integrate over S^8 $\Rightarrow$ D0 density $\rho_0=\rho_s/k$ (secondary quasiparticle fluid).

Integrate over an S^6 surrounding the end-plane, find that it also carries D2-brane charge:

$$\Rightarrow Q_2=\int_{S^6}G^{(6)}=\frac{\rho_s}{kb}\,.$$

D0-brane matrix model

Describe the QHF in terms of the large N D0-brane matrix model:

$$S = S_{BFSS} + S_{RR} + S_{CS} + S_{\psi}$$

$$\begin{aligned} S_{RR} &= \frac{\mu_0}{2}\int dt\, \mathrm{Tr}\left[G^{(2)}_{ij} X^i D_0 X^j\right] \quad \text{(Lorentz interaction)} \\ S_{CS} &= -k\int dt\, \mathrm{Tr}\left[A_0 - X^9\right] \quad \text{(classical 0-8 strings)} \\ S_{\psi} &= \int dt\, \mathrm{Tr}\left[\psi^{\dagger}_a (iD_0 - X^9)\psi_a\right] \quad \text{(quantum 0-8 strings)} \end{aligned}$$

In our scaling limit this reduces to

$$S = \int dt\, \mathrm{Tr}\left[G^{(2)}_{12}\, X^1 D_0 X^2 - kA_0 + i\psi^{\dagger}_a D_0 \psi_a\right] ,$$

where $G^{(2)}_{12} = kb$ in the basic configuration.

This is similar to matrix models proposed by Susskind and Polychronakos. (Here ψ_a are fermions rather than bosons.)

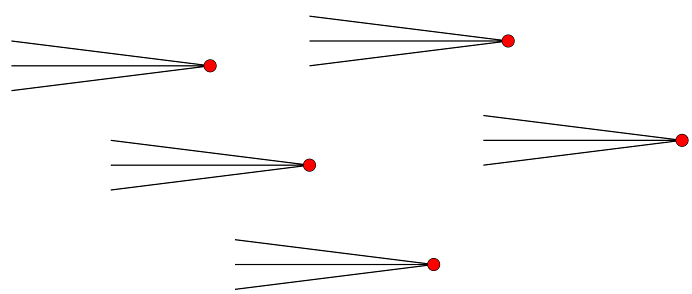
Matrix model configurations

Gauss' law constraint:

$$G_{12}^{(2)}[X^1, X^2] + i\psi^\dagger\psi = ik\mathbf{1}$$

N finite $\Rightarrow$ Tr$[X^1, X^2] = 0 \Rightarrow$ Tr$\psi^\dagger\psi = Nk$

Gas of D0-branes, each carrying k strings:



$N \to \infty$ allows D2-brane (BFSS):

$$\text{Tr}[X^1, X^2] = iA = \frac{iN}{b}$$

The form of the solution depends on the value of $G^{(2)}$. In particular:

$$n_s = \text{Tr}\psi^\dagger\psi = N\left(k - \frac{1}{\nu}\right) \qquad \left(\nu \equiv \frac{N}{AG^{(2)}}\right)$$

(a) $\underline{G^{(2)} = kb}$:

$$[X^1, X^2] = \frac{i}{b}\mathbf{1}$$
$$\psi^\dagger \psi = 0$$

D2-brane with uniform D0-brane density $\rho_0 = b$.

$\Rightarrow$ Ground state of the QHF at $\nu = 1/k$.

(b) $\underline{G^{(2)} = kb + 1/A}$:

$$[X^1, X^2] = \frac{i}{b + \frac{1}{kA}}\left(\mathbf{1} + \frac{1}{k}|0\rangle\langle 0|\right)$$
$$\psi^\dagger \psi = -|0\rangle\langle 0|$$

D2-brane with uniform D0-brane density $\rho_0 = b + 1/(kA)$ and one string.

$\Rightarrow$ Quasihole excitation of the QHF at $\nu = 1/k$.

D0-brane statistics (Susskind)

In general D0-branes obey non-Abelian statistics, but they simplify in the QHF state.

Exchanging two D0-branes in the QHF ground state gives a phase $\phi = \exp(i\pi k)$.

So D0-branes behave like 2d fermions for odd k, and bosons for even k.

From D0-branes to NCCS theory

NC gauge theory can be formulated in terms of a NC configuration in a lower dimensional gauge theory. (Seiberg)

Expand around the QHF ground state,

$$X^i = x^i - \frac{\epsilon_{ij}}{b} \hat{A}_j$$

and take the continuum limit $N \to \infty$. This gives the 3d NCCS theory. (Polychronakos)

3 Higher dimensional FQHFs

Compact 4d generalization of the FQHE:

Particles in a large representation of $SU(2)$ on S^4 in the presence of an $SU(2)$ instanton gauge field. (Zhang,Hu)

Equivalent to 6d system of charged particles on CP^3 in the presence of a large $U(1)$ magnetic field. (Topologically S^7 in both cases.)

The $U(1)$ QHF can be generalized to CP^n.

(Karabali,Nair)

In the large volume limit it becomes a planar $2n$-dimensional QHF, with a magnetic field in each of the n independent planes.

Main properties:

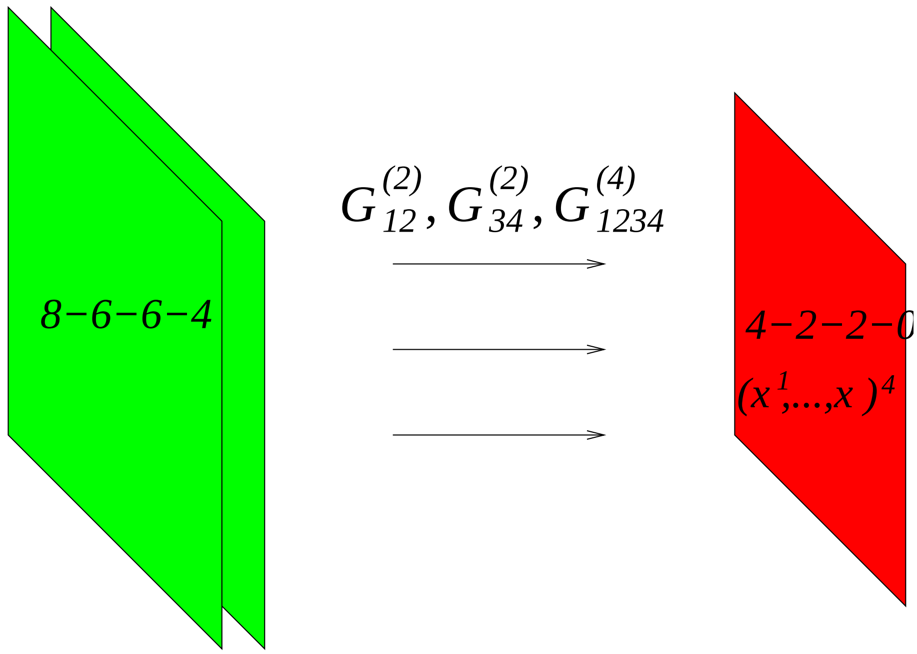
(a) Filling fraction $\nu = 1/k^n$, k odd.

(b) QH/QP excitations with charge $\pm\nu$.

(c) Brane excitations, fractional statistics.

Brane configuration for 4d FQHF

Replace D2-brane with D4-brane, turn on two B-field components: $B_{12} = b$, $B_{34} = b'$.



RR charges: $\rho_0 = bb'$, $\rho_2 = b$, $\rho'_2 = b'$

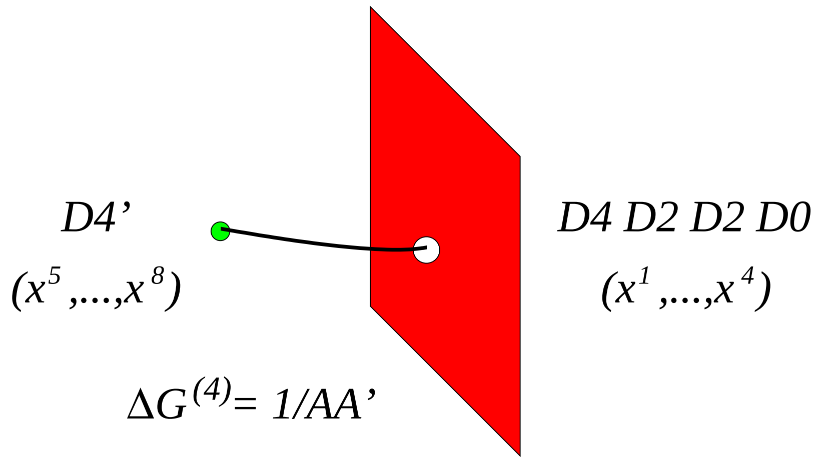
RR fields: $G^{(2)}_{12} = kb$, $G^{(2)}_{34} = kb'$, $G^{(4)}_{1234} = kbb'$

$\Rightarrow$ 4d FQHF with filling fraction:

$$\nu = \frac{\rho_0}{G^{(2)}_{12} G^{(2)}_{34}} = \frac{1}{k^2}\,.$$

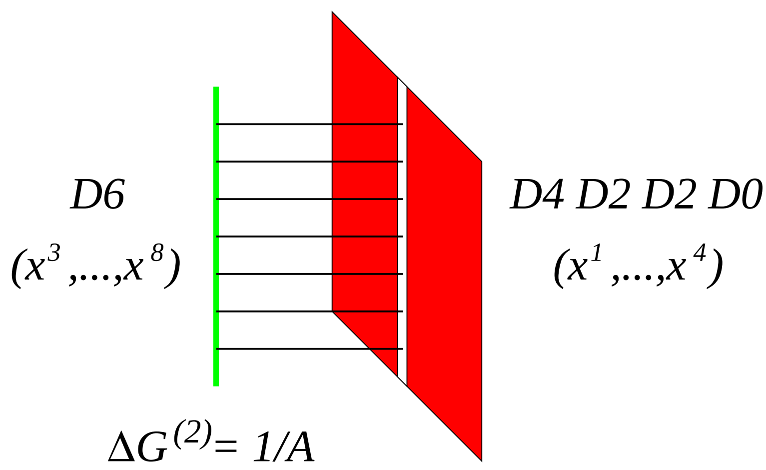
Excitations of the 4d FQHF

Quasihole/quasiparticle:



$\Rightarrow$ $Q_0 = 1/k$.

Quasimembrane:



2d array of strings with $\rho_s = \rho_2 = b$

$\Rightarrow$ $\rho_0 = b/k$, $Q_2 = 1/k$.

D4-brane effective gauge theory

In the scaling limit:

$$g_s \sim (\alpha')^{\frac{p+1}{2}+\epsilon}\,,\; g_{ij} \sim (\alpha')^{2+\delta}\,,\; b, b' = \text{finite}\,,$$

we are left effectively with*

$$S = \frac{\mu_0}{4\pi^2}\int C^{(1)}\wedge(dA)^2 + \frac{\mu_2}{2\pi}\int C^{(3)}\wedge dA + \frac{k}{24\pi^2}\int A\wedge(dA)^2.$$

We expect the CS part to become 5d NCCS theory under the SW map $A \to \widehat{A}$:

$$\int\left(d\widehat{A}\star d\widehat{A}\star\widehat{A} + \frac{3i}{2}d\widehat{A}\star\widehat{A}\star\widehat{A}\star\widehat{A} - \frac{3}{5}\widehat{A}\star\widehat{A}\star\widehat{A}\star\widehat{A}\star\widehat{A}\right).$$

*Some proposals for effective gauge theories of higher-dimensional FQHFs were made by Bernevig, Chern, Hu, Toumbas, and Zhang. It would be interesting to compare.

Matrix model for 4d FQHF

$G^{(2)}, G^{(4)}$ background, scaling limit $\Rightarrow$

$$S = \int dt \, \mathrm{Tr} \left[-\frac{i}{4} G^{(4)}_{ijkl} D_0 X^i X^j X^k X^l + \frac{1}{2} G^{(2)}_{ij} X^i D_0 X^j - k A_0 + i\psi^\dagger D_0 \psi \right].$$

Gauss' law:

$$\frac{i}{2} G^{(4)}_{ijkl} X^i X^j X^k X^l + \frac{1}{2} G^{(2)}_{ij} [X^i, X^j] + i\psi^\dagger \psi = ik\mathbf{1}.$$

4d QHF ground state:

$$\begin{aligned} \epsilon_{ijkl} X^i X^j X^k X^l &= -\frac{2}{bb'}\mathbf{1} \\ [X^i, X^j] &= -i\theta^{ij}\mathbf{1} \quad (\theta = i \,\mathrm{diag}[\sigma_2/b, \sigma_2/b']) \\ \psi^\dagger \psi &= 0 \end{aligned}$$

Excited states are obtained by changing the RR fields. This gives strings:

$$n_s = \mathrm{Tr}\, \psi^\dagger \psi = \frac{N}{bb'} \left(\Delta G^{(4)}_{1234} - b' \Delta G^{(2)}_{12} - b \Delta G^{(2)}_{34} \right).$$

4 Future directions

1. Other geometries, boundaries (branes ending on branes), edge states.

2. Dynamical issues: hierarchy of filling fractions, phase transitions.

3. Gauge/gravity correspondence, 2-brane solution in massive IIA SUGRA.

4. Other string theory $\leftrightarrow$ condensed matter physics connections?