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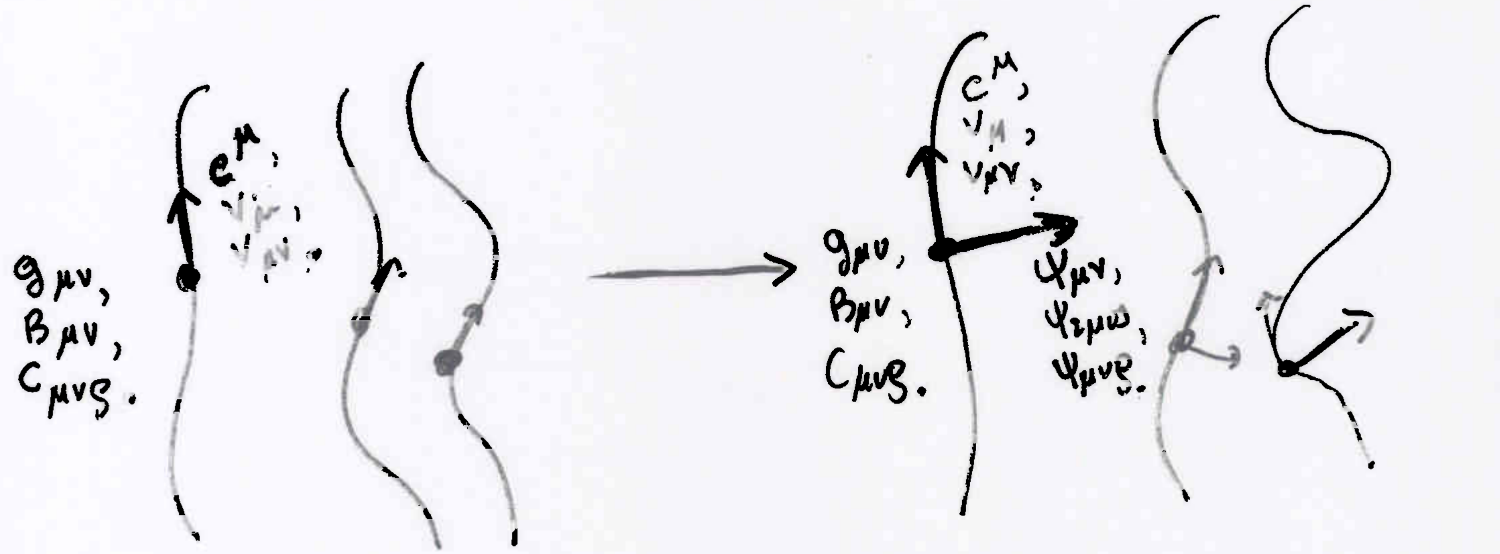
GRAVITATIONAL TQFTs → SUPERGRAVITY



$s B_{\mu\nu} = \partial_{[\mu} V_{\nu]}$ → $Q B_{\mu\nu} = \partial_{[\mu} V_{\nu]} + \Psi_{\mu\nu}$

etc... etc...

$\rightsquigarrow I = \int R + |\partial_\mu B_{\nu\rho}|^2 + |\partial_\mu C_{\nu\rho\sigma}|^2 + s(\dots)$ → $I_t = \int d(\quad) + \{Q, \dots\}$

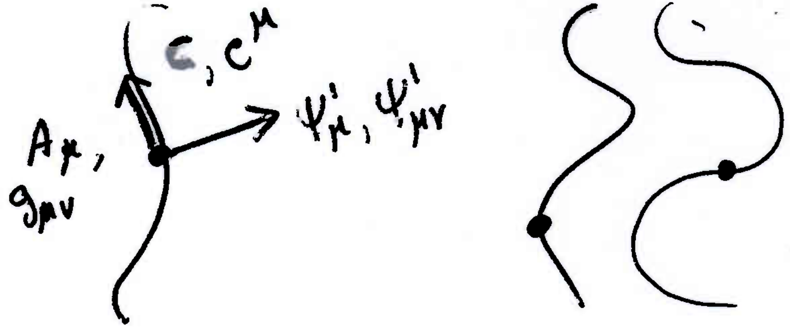
↓ "twist"

SUPERGRAVITIES

SUGRA.		TQFT
$N=1$, $d=2$	←	$(g_{\mu\nu})$
$N=2$, $d=4$	←	$(g_{\mu\nu}, B_{\mu\nu})$
$N=1$, $d=8$	←	$(g_{\mu\nu}, B_{\mu\nu})$
$N=2$, $d=8$ $(\sim N=1, d=11)$	←	$(g_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu\rho})$

L.B., Bellon, Tanzini
Anselmi + Fré

↑ $(g_{\mu\nu}, C_3, 3B_2, 6A, 7S, 2\lambda^\alpha_\mu, 6\chi^\alpha)_{D=8} \sim (g_{\mu\nu}, C_3, \dots$



- $(s+d)(A^0_1 + c^1_0) = F^0_2 + \psi^1_1 + \phi^2_0$

i.e., $\boxed{s A_1 = -dc^1 + \psi^1_1}$ ... gluino!

- $(s+d)(B^0_2 + V^1_1 + m^2_0) = G^0_3 + \psi^1_2 + \phi^2_1 + R^3_0$

$\boxed{s B_2 = -d V^1_1 + \psi^1_2}$ ... gravitino & Higgsino

- $(s+d)(e^a + (\omega^{ab}_1 + \Omega^{ab\,1})\, e^b) = \exp i_c \, (T^a_2 + \psi^a_1 + \Phi^a)$

$\boxed{s e^a = \mathcal{L}_c e^a_\mu + \Omega^{ab} e^b + \psi^{1\,a}_1}$ ... gravitino

"TWIST": Mapping between forms and spinors, e.g., in $d=8$:

$$\begin{cases} \lambda^\alpha_\mu = \psi^a_\mu \, \gamma^a \, \eta^\alpha \\ \chi^\alpha = \phi^a \gamma^a \, \eta^\alpha \end{cases}$$

(needs special manifolds, e.g. spin(7) holonomy)

antighost can be introduced in this way – (explains why special manifolds are needed)

TOPOLOGICAL GRAVITY (d = 4, 8)

need something like $\int_{M_4} |F_{\mu\nu}|^2 = \int |F_{\mu\nu} + \tilde{F}_{\mu\nu}|^2$.

$$\int R = \int e^a_1 \wedge \cdots \wedge e^b \wedge (d\omega + \omega \wedge \omega)^{cd} \, \epsilon_{ab \ldots cd}$$

$$= \int d(\ldots) + \omega^{ab^-} \wedge \gamma^{bc} \wedge \omega^{ca^-}$$

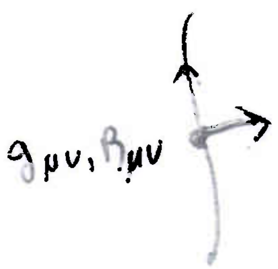
Here, $\omega^{ab} = \omega^{ab^-} + \omega^{ab^+}$ $\qquad \omega^{ab} = \omega^{ab}_\mu dx^\mu$

$$\begin{cases} d = 4 \to 6 = 3 \oplus 3 \\ d = 8 \to 28 \underset{spin(7)}{=} 7 \oplus 21 \end{cases}$$

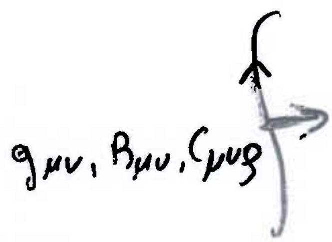
Using $Q e^a_\mu = \psi^a_\mu + \mathcal{L}_c e^a_\mu$

$$I = \int d(\ldots) + \left\{ Q, \bar{\Psi}^{ab^-}_\mu \left(\omega^{ab^-}_\mu(e) + G^{ab^-}_\mu(B_2) + H^{ab^-}_\mu + \ldots \right\} \right.$$

≡ supergravity action, up to a twist



or



$N = 1, d = 8$ $\qquad$ $N = 2 \quad d = 8$

N = 1, d = 8 supergravity.

TWIST

$$
\left\{
\begin{aligned}
\Psi_\mu^\alpha &= \Psi_\mu^{(1)a}(\Gamma^a)^\alpha_\beta\phi^\beta \\
\Psi_{\mu\dot\alpha} &= \Psi_\mu^{(-1)ab^-}(\Gamma^{ab})_{\dot\alpha\beta}\phi^\beta + \Psi_\mu^{(-1)}\psi_{\dot\alpha} \\
\lambda^\alpha &= \chi_\mu^{(-1)}(\Gamma^a)^\alpha_\beta\phi^\beta \\
\lambda_{\dot\alpha} &= (\Psi^{(1)}_{\mu\nu-} - \Psi^{(1)a}_{[\mu} e^a_{\mu]-})(\Gamma^{\mu\nu})_{\dot\alpha\beta}\phi^\beta + \eta_{(\sigma)}\psi_{\dot\alpha}
\end{aligned}
\right.
\qquad (27)\ (28)
$$

The scalar field σ and vectors $A_\mu^{(\pm 2)}$ are respectively the dilaton and both graviphotons of supergravity.

GRA MULTIPLET : $e^a_\mu, \sigma, (\Psi_\mu^{(1)a}, \bar\Psi_\mu^{(-1)ab^-}, \bar\Psi_\mu^{(-1)}), B_{\mu\nu}, A_\mu^{(2)}, A_\mu^{(-2)}, (\bar\chi_\mu^{(-1)}, \Psi^{(1)}_{\mu\nu-}, \eta^{(1)}_{(\sigma)})$

...mmuting ghosts (spinors) / local SUSY: $(\Phi^{(2)a}, \Phi^{(0)ab-}, \Phi^{(0)})$, $(\bar\Phi^{(-2)a}, \bar\Phi^{(0)ab-}, \bar\Phi^{(0)})$

...rmionic lagrangy condition / ...a $\not\!\phi\, \gamma^\mu \psi_\mu = 0$: $(\bar\eta^{(-1)a}, \eta^{(1)ab^-}, \eta^{(1)})$

$$
\begin{array}{ccccccc}
 & & & e^a_\mu & & & \\
 & \Psi_\mu^{(1)a} & & & \bar\Psi_\mu^{(-1)ab^-}, \chi_\mu^{(-1)} & & \\
\Phi^{(2)a} & & & \sigma & & & \bar\Phi^{(-2)a} \\
 & \eta_\sigma^{(1)} & & & & &
\end{array}
\qquad (29)
$$

$$
\begin{array}{ccccccc}
 & & & B_{\mu\nu} & & & \\
 & & \Psi^{(1)}_{\mu\nu-} & & & & \\
 & A_\mu^{(?)} & & & & A_\mu^{(-2)} & \\
R^{(3)} & & S^{(1)} & & \bar\Psi_\mu^{(-1)} & & \bar R^{(-3)}
\end{array}
$$

9) express what is twisted supergravity.

→ simply comes from building the TQFT of

$A_{\mu\nu} = g_{\mu\nu} \oplus B_{[\mu\nu]} = e^a_\mu \oplus \Omega^{ab} \oplus B_{[\mu\nu]}$

$s^2 = 0$ or $\underline{s}^2 = 0$, modulo gauge transf...

$$\left\{\begin{array}{rcl} se^a_\mu &=& \Psi^{(1)a}_\mu - \Omega^{ab} e^b_\mu + \mathcal{L}_\xi e^a_\mu \\ s\omega^{ab}_\mu &=& \tilde{\Psi}^{(1)ab}_\mu + D_\mu \Omega^{ab} + \mathcal{L}_\xi \omega^{ab}_\mu \end{array}\right.$$

$$\left\{\begin{array}{rcl} s\Psi^{(1)a}_\mu &=& -\Omega^{ab}\Psi^{(1)b}_\mu - \mathcal{L}_\Phi e^a_\mu + \tilde{\Phi}^{(2)ab} e^b_\mu + \mathcal{L}_\xi \Psi^a_\mu \\ s\tilde{\Psi}^{(2)ab}_\mu &=& -\Omega^{ac}\tilde{\Psi}^{(2)cb} + D_\mu \tilde{\Phi}^{(2)ab} - \mathcal{L}_\Phi \omega^{ab}_\mu + \mathcal{L}_\xi \tilde{\Psi}^{(2)ab}_\mu \end{array}\right.$$

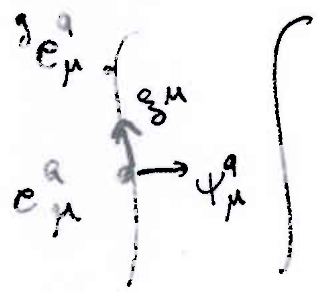
$$\left\{\begin{array}{rcl} s\Phi^{(2)a} &=& \mathcal{L}_\xi \Phi^{(2)a} - \Omega^{ac}\Phi^{(2)a} \\ s\tilde{\Phi}^{(2)ab} &=& -\Omega^{ac}\tilde{\Phi}^{(2)cb} + \mathcal{L}_\xi \tilde{\Phi}^{(2)ab} \end{array}\right.$$

$$\left\{\begin{array}{rcl} sB_{\mu\nu} &=& \Psi^{(1)}_{\mu\nu} + \mathcal{L}_\xi B_{\mu\nu} \\ s\Psi^{(1)}_{\mu\nu} &=& \mathcal{L}_\Phi B_{\mu\nu} + \partial_{[\mu} A^{(2)}_{\nu]} + \mathcal{L}_\xi \Psi^{(1)}_{\mu\nu} \\ sA^{(2)}_\mu &=& \partial_\mu R^{(3)} + \mathcal{L}_\xi A^{(2)}_\mu \\ sR^{(3)} &=& \mathcal{L}_\xi R^{(3)} \end{array}\right.$$

$$\begin{array}{rcl} s\xi^\mu &=& e^\mu_a \Phi^{(2)a} + \xi^\nu \partial_\nu \xi^\mu \\ s\Omega^{ab} &=& \tilde{\Phi}^{(2)ab} - \Omega^{ac}\Omega^{cb} + \mathcal{L}_\xi \Omega^{ab} \ . \end{array} \qquad (5)$$

$$\begin{array}{rclrcl} sA^{(-2)}_\mu &=& \bar{\Psi}^{(-1)}_\mu + \mathcal{L}_\xi A^{(-2)}_\mu & sA^{(0)}_\mu &=& \Psi^{(1)}_\mu + \mathcal{L}_\xi A^{(0)}_\mu \\ s\bar{\Psi}^{(-1)}_\mu &=& \partial_\mu \Phi^{(0)} + \mathcal{L}_\xi \bar{\Psi}^{-1}_\mu & s\Psi^{(1)}_\mu &=& \partial_\mu \Phi^{(2)} + \mathcal{L}_\xi \Psi^{(1)}_\mu \\ s\Phi^{(0)} &=& \mathcal{L}_\xi \Phi^{(0)} & s\Phi^{(2)} &=& \mathcal{L}_\xi \Phi^{(2)} \\ s\bar{\Phi}^{(0)} &=& \mathcal{L}_\xi \bar{\Phi}^{(0)} + \eta^{(1)} & s\bar{\Phi}^{(-2)} &=& \bar{\eta}^{(-1)} + \mathcal{L}_\xi \bar{\Phi}^{(-2)} \\ s\eta^{(1)} &=& \mathcal{L}_\xi \eta^{(1)} & s\bar{\eta}^{(-1)} &=& \mathcal{L}_\xi \bar{\eta}^{(-1)} \end{array} \qquad (6)$$

$$\left\{\begin{array}{l} \underline{s}\, e^a_\mu = \Psi^a_\mu \\ \underline{s}\, \Psi^a_\mu = \mathcal{L}_{\Phi^a}\, e^a_\mu \end{array}\right.$$



like: $(s+d)(A^0_1 + c^1_0) + (A+c)^2 = F^0_2 + \Psi^1_1 + \phi^2_0$

$(s+d)(F + \Psi + \Phi) = -[A + c, F + \Psi + \Phi]$

From geometry: $(g_{\mu\nu}, B_{\mu\nu}) \to (e^a_\mu, \Omega^{ab}, B_{\mu\nu})$

$e^a_\mu \to$

$$e^a_\mu \leftarrow \Psi^{(1)a}_\mu \leftarrow \Phi^{(2)a}, \quad \bar\Psi^{(-1)a}_\mu \leftarrow b^{(0)a}_\mu, \Phi^{(0)a} \leftarrow \eta^{(1)a}, \quad \bar\Phi^{(-2)a} \leftarrow \bar\eta^{(-1)a} \qquad (1)$$

$\omega^{ab}_\mu \to$

$$\omega^{ab}_\mu \leftarrow \tilde\Psi^{(1)ab}_\mu \leftarrow \tilde\Phi^{(2)ab}, \quad \bar{\tilde\Psi}^{(-1)ab}_\mu \leftarrow \tilde b^{(0)ab}_\mu, \tilde\Phi^{(0)ab} \leftarrow \tilde\eta^{(1)ab}, \quad \bar{\tilde\Phi}^{(-2)ab} \leftarrow \bar{\tilde\eta}^{(-1)ab} \qquad (2)$$

$B_{\mu\nu} \to$

$$B_{\mu\nu} \leftarrow \Psi^{(1)}_{\mu\nu} \leftarrow A^{(2)}_\mu \leftarrow R^{(3)}, \quad \bar\Psi^{(-1)}_{\mu\nu} \leftarrow A^{(0)}_\mu, b^{(0)}_{\mu\nu} \leftarrow S^{(1)}, \Psi^{(1)}_\mu \leftarrow b^2_{S^{(1)}}, \Phi^{(2)}$$

$$A^{(-2)}_\mu \leftarrow \bar S^{(-1)}, \bar\Psi^{(-1)}_\mu \leftarrow b^{(0)}_{\bar S^{(-1)}}, \bar\Phi^{(0)}, \Phi^{(0)} \leftarrow \eta^{(1)}, \quad \bar R^{(-3)} \leftarrow b^{(-2)}_{\bar R^{(-3)}}, \bar\Phi^{(-2)} \leftarrow \bar\eta^{(-1)}$$

(3)

$$\xi^{(1)\mu} \quad b^{(0)\mu} \leftarrow \bar\xi^{(-1)\mu} \qquad \Omega^{(1)ab} \quad b^{(0)ab} \leftarrow \bar\Omega^{(-1)ab} \qquad (4)$$

reparametrization — Lorentz

$$(Q+d)\, e^a_\mu dx^\mu + (\Omega^{ab} + \omega^{ab})\, e^b_\mu dx^\mu = \exp i_\delta \left(T^a_{\mu\nu} dx^\mu dx^\nu + \Psi^{a(1)}_\mu dx^\mu + \Phi^{a(2)} \right)$$

$$(Q+d)\left(B_{\mu\nu} dx^\mu dx^\nu + V^{(1)}_\mu dx^\mu + m^{(2)} \right) = G_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho + \Psi^{(1)}_{\mu\nu} dx^\mu dx^\nu + A^{(2)}_\mu dx^\mu + R^{(3)}$$

Spin(7) ⊂ SO(8) ghost decomposition of e_μ, $B_{\mu\nu}$

$$e_\mu^a$$

$$\psi_\mu^{(1)a} \qquad \bar{\Psi}_\mu^{(-1)ab}, \ \bar{\chi}_\mu^{(-1)}$$

$$\phi^{(2)a} \qquad \sigma, \ \phi^{0ab} \qquad \bar{\phi}^{(-2)a}$$

$$\chi^{(1)}_{(\sigma)}, \ \eta^{(1)}_{ab} \qquad \bar{\eta}^{-1}(a)$$

$$B_{\mu\nu}$$

$$\Psi^{(1)}_{\mu\nu\pm} \qquad \bar{\Psi}^{(-1)}_{\mu\nu\pm}$$

$$A_\mu^2 \qquad A^{(0)}, \ A^{(0)}_{\mu\nu} \qquad A_\mu^{(-2)}$$

$$R^{3} \qquad S^{(1)}, \ \Psi^1, \ \Psi^1_{\mu\nu} \qquad S^{(-1)} \ \bar{\Psi}_\mu^{(-1)} \qquad R^{-3}$$

$$X^2 \qquad \phi_0 \bar{\phi}_0 \qquad \eta^{-2}$$

$$\eta_1$$

$$28 = 7 \oplus 21$$

↑

Spin(7)

$$\leadsto \begin{cases} \omega_\mu^{ab}(e) = G_\mu^{ab}(B_2) \\ \partial_\mu \sigma = \Omega_{\mu\nu\rho\sigma} \, G_{\nu\rho\sigma}(B_2) \end{cases}$$

$\Omega_{\mu\nu\rho\sigma}$ (from octonion C_{ijk}: $\Omega_{8ijk} = C_{ijk}$)

$\Rightarrow X_{\mu\nu} = X_{\mu\nu} + \Omega_{\mu\nu\rho\sigma} X_{\rho\sigma}$)

THE TOPOLOGICAL ACTION

$$b^{ab^-}\wedge e^b\wedge e^c\wedge b^{ac^-}+b^{ab^-}\wedge e^b\wedge e^c\wedge(\omega^{ac^-}(e)+G^{ac^-}(B))$$
$$+|b^{(0)}_\mu|^2+b^{(0)}_\mu(\partial_\mu\sigma+\Omega_{\mu\nu\rho\sigma}G_{\nu\rho\sigma})$$
$$\sim|G_{abc}(B)|^2+R(e)+\omega_c^{ab^-}(e)G_{abc}+\partial_\mu\sigma\partial^\mu\sigma+\text{boundary term} \quad (16)$$

$$|G_b^{ac^-}(B)|^2+|\Omega_{abcd}G_{bcd}|^2=|G_{abc}(B)|^2$$

We thus consider the s-exact terms :

$$\mathcal{L}_{e,B} = s\Big[\bar\Psi^{(-1)ab^-}\wedge e^b\wedge e^c\wedge(b^{(0)ac^-}+\omega^{ac^-}(e)+G^{ac^-}(B)) +\bar\chi^{(0)}_\mu\Big(b^{(0)}_\mu+(\partial_\mu\sigma+\Omega_{\mu\nu\rho\sigma}G_{\nu\rho\sigma})\Big)\Big] \quad (17)$$

$$\mathcal{L}_{\bar\Psi_\mu}=s\Big[\partial_{[\mu}A^{(-2)}_{\nu]^-}\Psi^{(1)a}_{[\mu}e^a_{\nu]}\Big] \quad (18)$$

$$s\Big[\Psi^{(1)}_{\mu\nu}\partial_{[\mu}A^{(-2)}_{\nu]}\Big]=\partial_{[\mu}A^{(2)}_{\nu]}\partial_{[\mu}A^{(-2)}_{\nu]}+\Psi^{(1)}_{\mu\nu}\partial_{[\mu}\Psi^{(-1)}_{\nu]} \quad (19)$$

$$\begin{aligned}\mathcal{L}_{\Psi^{(1)}_{\mu\nu},\tilde\Phi^{(2)ab}} &= s\Big[\tilde e^\mu_c e^\nu_d\ \bar\Phi^{(-2)cd^+}(\Psi^{(1)}_{\mu\nu}-\Psi^{(1)a}_{[\mu}e^a_{\nu]})\Big]\\ &= e^\mu_c e^\nu_d\ \bar{\tilde\eta}^{(-1)cd^+}(\Psi^{(1)}_{\mu\nu}-\Psi^{(1)a}_{[\mu}e^a_{\nu]})+\bar{\tilde\Phi}^{ab^+}(\partial_{[a}A^{(2)}_{b]}-\tilde\Phi^{(2)}_{ab}+\dots)\end{aligned} \quad (20)$$

Define $\Psi^{(1)}_{\mu\nu^-}=\Psi'^{(1)}_{\mu\nu^-}+\Psi^{(1)a}_{[\mu}e^a_{\nu]^-}$ $\Psi'^{(1)}_{\mu\nu^-}$ and $\eta^{(1)}$ determine a chiral Majorana spinor that will one chiral component of the dilatino. $\bar\chi^{(-1)\mu}$ determines by twist the other chiral component.

$$\begin{aligned} se^a_\mu &= \Psi^{(1)a}_\mu+\Omega^{ab}e^b_\mu+\dots\\ sB_{\mu\nu} &= \Psi^{(1)a}_{[\mu}e^a_{\nu]}+\Psi'^{(-1)}_{\mu\nu^-}+\dots\end{aligned} \quad (21)$$

we see that $B_{\mu\nu}$ is truly the Kalb-Ramond Field

CONCLUSION

Yang-Mills :

$N=2,\ d=8$ (or 4) $\leftarrow$ TQFT (A)

Gravity

$N=1,\ d=2$	sugra	$\leftarrow$	TQFT $(g_{\mu\nu})$
$N=2,\ d=4$	sugra	$\leftarrow$	TQFT $(g_{\mu\nu}, B_{\mu\nu})$
$N=1,\ d=8$	sugra	$\leftarrow$	TQFT $(g_{\mu\nu}, B_{\mu\nu})$
$N=2,\ d=8$	sugra	$\leftarrow$	TQFT $(g_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu\rho})$

$N=1,\ d=11$

- The gravitino is obtained by "untwisting" a "topological ghost", that is, is a curvature.

- Local supersymmetry transformations are ghost of ghost transformations. The Faddeev-Popov ghost for local susy is obtained by untwisting the ghost of ghost of reparametrization.

Topological gravity has 2 phases → topological
→ supergravity