Fate of long wavelength fluctuations & initial states of inflationary universe

Yuko Urakawa (U. of Barcelona)

Y.U. & T. Tanaka 1007.0468[hep-th], 1009.2947[hep-th],

Y.U. & T. Tanaka 1209.1914[hep-th],

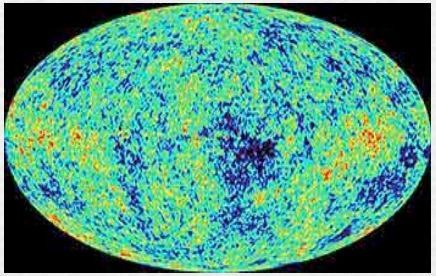
Y.U. & T. Tanaka / Y.U. & Y. Misonoh in preparation

@ AEI, Hannover

Why long wavelength fluctuations?

❏ Observational cosmology

Important information to access the early universe



Scale-invariance/Gaussian/Adiabatic, ...

⟶ Inflation

❏ Infrared divergence (IR) problem

Break down of predictability?

❏ Initial states of our universe

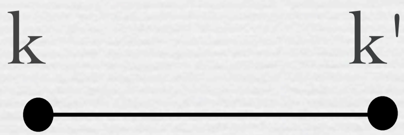
IR divergence from interactions

Massless field ϕ with $\lambda\phi^4$

(ex) Inflaton(~ Adiabatic perturbation), Spin-2 graviton

❑ Two-point function with loop corrections

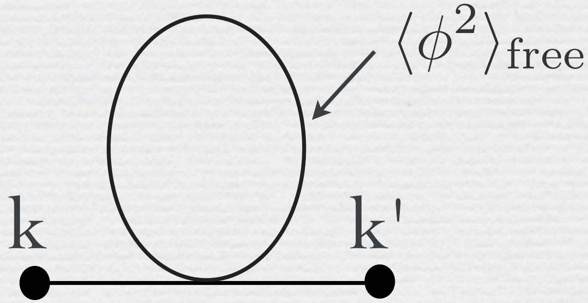
- Leading order



$$\langle \phi_k \phi_{k'} \rangle \propto k^{-3}$$

Scale-invariant

- Next to leading order (1loop)



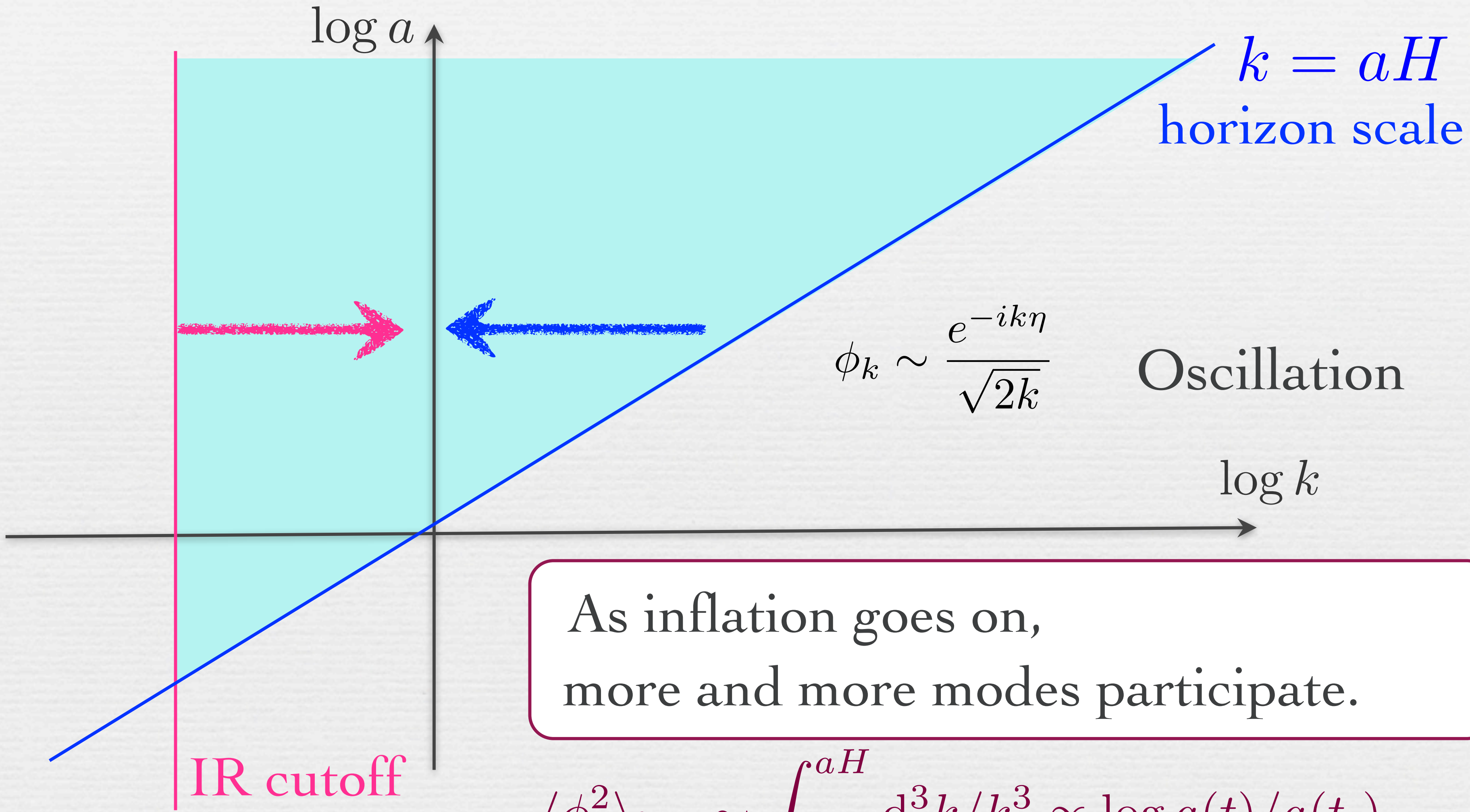
Momentum (Loop)integral

$$\langle \phi^2 \rangle_{\text{free}} \sim \int \frac{d^3\mathbf{k}}{k^3}$$

Logarithmic divergence

Origin of secular growth

Which modes participate in loop corrections?



As inflation goes on,
more and more modes participate.

$$\langle \phi^2 \rangle_{\rm free} \sim \int_{H_i}^{aH} d^3k/k^3 \propto \log a(t)/a(t_i)$$

Logarithmic corrections

Cosmological perturbations

Perturbation around FRW $\qquad g_{\mu\nu} = g_{\mu\nu}^{FRW} + \delta g_{\mu\nu}$

❑ "Massless" fields

GR

- gravitational field $\delta g_{\mu\nu}$
  - Spin-2 transverse traceless $\qquad \# = 2$
  - Spin-0 longitudinal

$$\zeta = \mathcal{R} - \frac{H}{\dot{\sigma}}\delta\sigma$$

- multi-scalar fields ϕ^I $\quad (I = 1, \cdots N)$
  - Adiabatic perturbation $\delta\sigma$ $\qquad \# = 1$
    - Coupled to longitudinal mode of gravitational field
  - Entropy perturbations $\mathcal{S}^I$ $\qquad \# = N - 1$
    $(I = 1, \cdots, N-1)$
    - behaves like a test field at a fixed background

Outline

✔ 1. Origin of Infrared(IR) divergence problem

2. Beginning with the free theory

3. Initial state with interaction

Gauge modes in local universe

Inflation driven by a scalar field ϕ

❑ Gauge conditions

- Time slicing $\delta\phi = 0$

  Uniform field slicing → Fix time slices

- Spatial coordinates

$$h_{ij} = a^2(t)e^{2\zeta}\left[e^{\delta\gamma}\right]_{ij} \qquad \partial_j \delta\gamma_{ij} = \delta\gamma_{ii} = 0$$

Residual gauge DOFs

Y. U. & T. Tanaka (2010)

$$x^\mu \to \tilde{x}^\mu = e^{\mathcal{L}_\xi} x^\mu \qquad \xi^\mu = (0,\, \xi^i)$$

Poisson eq. $\Delta\xi^i = \cdots$

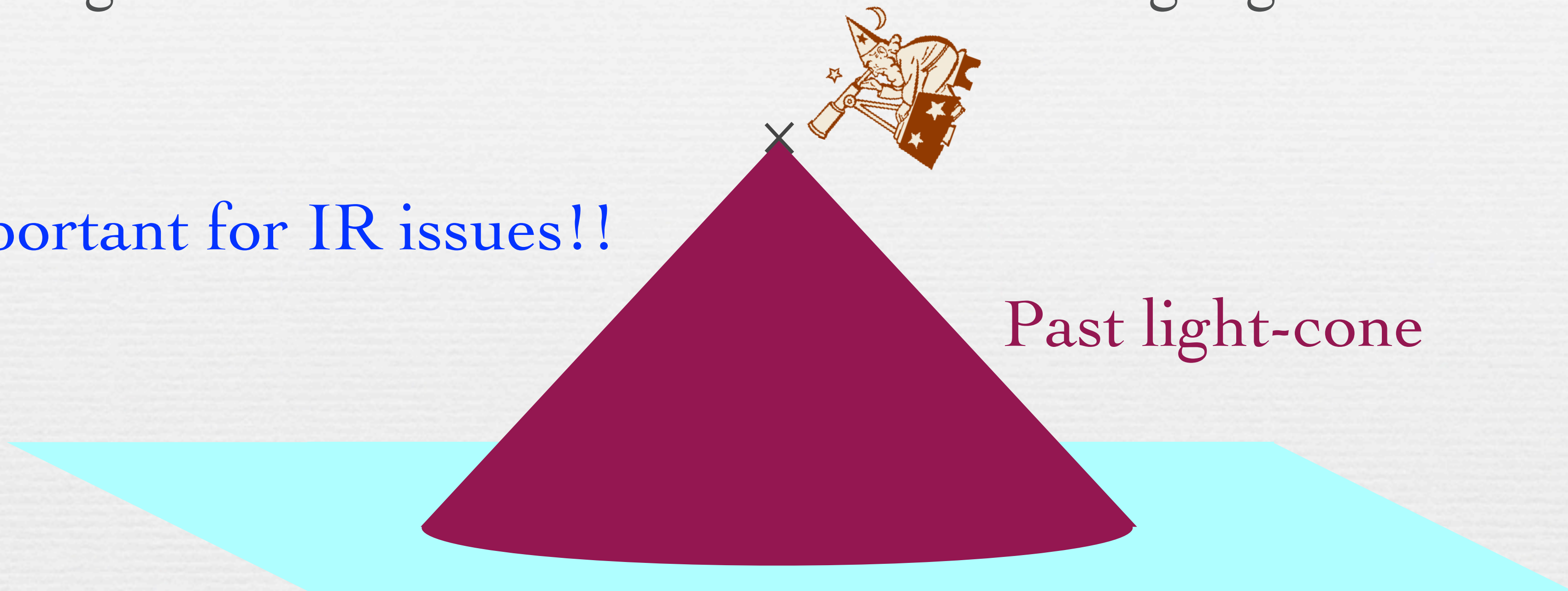
DOFs in boundary conditions

⟷ DOFs in changing invisible region

Gauge invariance in local universe

Cosmological fluctuations we observe should be gauge invariant.

Important for IR issues!!



<u>Definition of fluctuation</u>

$$\delta Q := Q - \bar{Q}$$

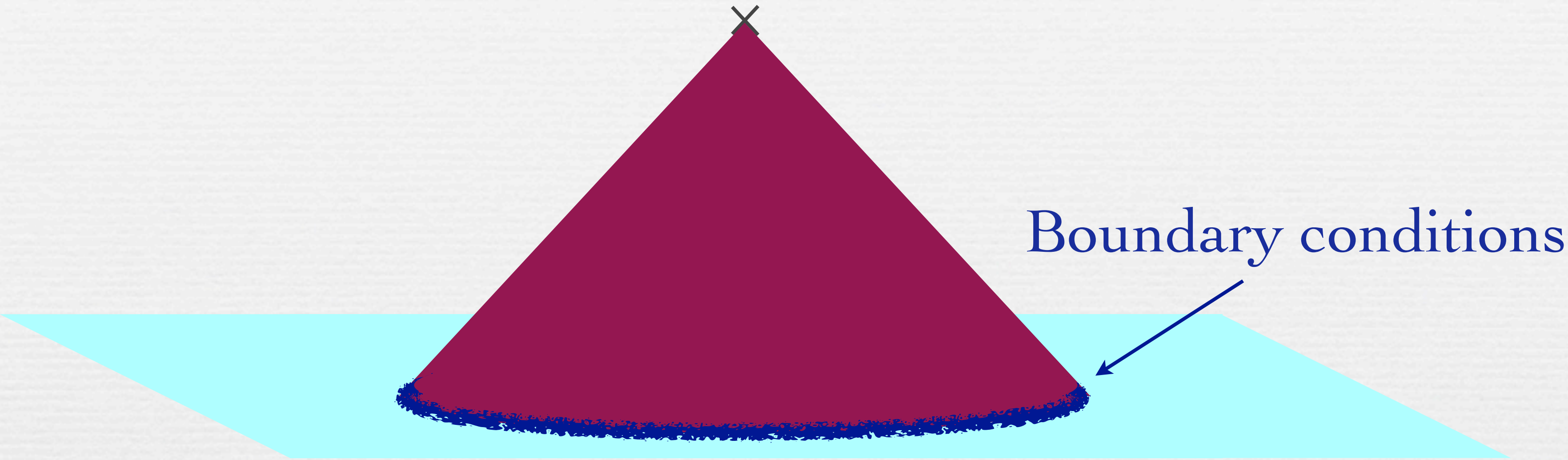
$$Q = \zeta,\ \delta\gamma_{ij}$$

Averaged value $\bar{Q} := \int_{\blacktriangle} \mathrm{d}^3\mathbf{x}\, Q(x) \Big/ \int_{\blacktriangle} \mathrm{d}^3\mathbf{x}$

Without changing δQ, we can shift

$$Q \to Q - f(t), \quad \bar{Q} \to \bar{Q} - f(t)$$

Gauge modes in local universe



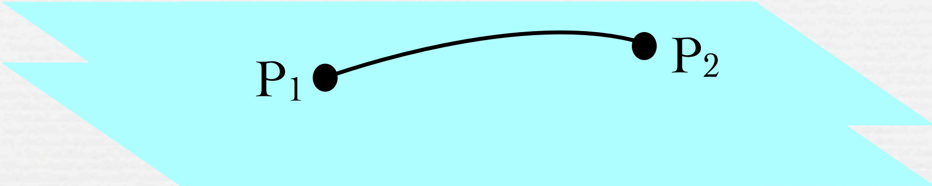
Fix the boundary conditions of the local universe

⟶ Quantization in the local universe

* Break the global translation symmetry
* Hilbert space??

⟹ 1. Quantization at the whole universe
2. Construction of the gauge-invariant quantity

Gauge-invariant operator



fix slicing $\delta\phi = 0$ η:const

3D scalar curvature ${}^sR$

$$x^\mu \to \tilde{x}^\mu = e^{\mathcal{L}_\xi} x^\mu \qquad \xi^\mu = (0, \xi^i)$$

$${}^s\tilde{R}(x) = {}^sR(e^{-\mathcal{L}_\xi} x)$$

❑ Correlation fns. of ${}^sR$ in geodesic normal coordinates

Y.U. & T. Tanaka (2010)

$\langle {}^sR {}^sR \rangle (l)$ $\quad l$: Geodesic distance between P_1 and P_2

spatial line element $\quad \mathrm{d}l^2 = e^{2\zeta} \left[e^{\delta\gamma} \right]_{ij} \mathrm{d}x^i \mathrm{d}x^j$

x^i : Global coordinates
X^i : Geodesic normal coordinates

$$x^i(X) \simeq e^{-\zeta} \left[e^{-\delta\gamma} \right]^i{}_j X^j$$

Gauge-invariant operator $\quad {}^gR(X) = {}^sR(x^i(X))$

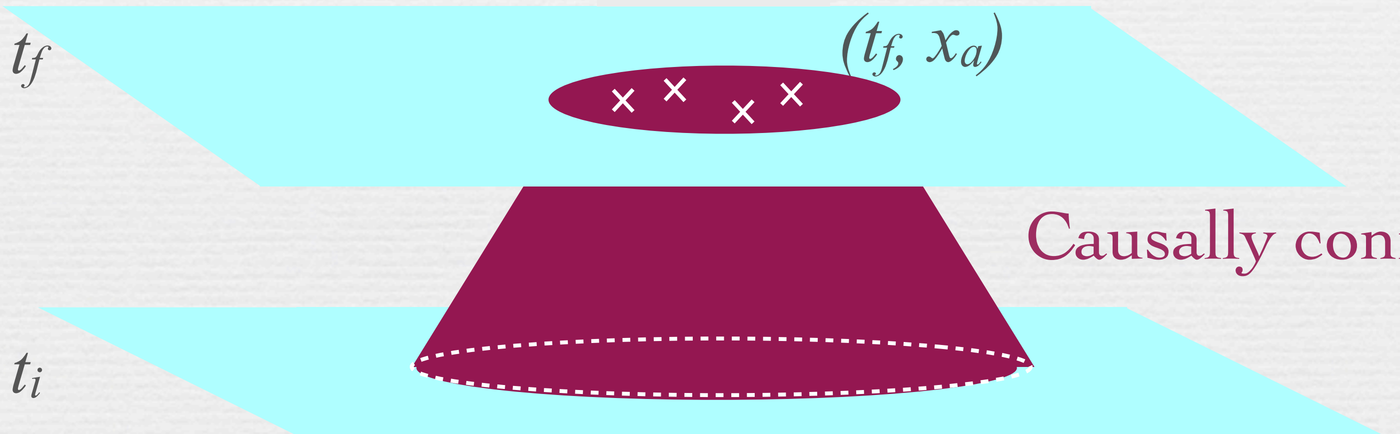
Quantum state?

❑ Gauge invariant n-point functions

at the end of inflation $t = t_f$

$\langle {}^g R(t_f, x_1^i) {}^g R(t_f, x_2^i) \cdots {}^g R(t_f, x_n^i) \rangle \ni G^+(x_1, x_2)$

Non-local



Conditions on initial states

- We request the locality of the n-point fns.

  = IR regularity of the n-point fns.

⟷ Connection to gauge-invariance

Outline

1. Origin of Infrared(IR) divergence problem

✔ 2. Beginning with the free theory

3. Initial state with interaction

Beginning with free theory

Interaction picture

Expands Heisenberg field ζ by free field ψ

$$\begin{cases} \zeta(t_i, \mathbf{x}) = \psi(t_i, \mathbf{x}) \\ \pi(t_i, \mathbf{x}) = \pi_\psi(t_i, \mathbf{x}) \end{cases}$$

- Unitary boundary conditions

$$\zeta(t, \mathbf{x}) = U^\dagger(t, t_i)\psi(t, \mathbf{x})U(t, t_i)$$

U(t, t') :unitary operator

$$\left[\zeta(t, \mathbf{x}),\, \pi_\zeta(t, \mathbf{y})\right] = \left[\psi(t, \mathbf{x}),\, \pi_\psi(t, \mathbf{y})\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

Retarded Green function

Another expression of $\zeta(t,\mathbf{x}) = U^\dagger(t,t_i)\psi(t,\mathbf{x})U(t,t_i)$

Heisenberg equation

$$\mathcal{L}\zeta = \mathcal{S}_{NL}[\zeta] \qquad\qquad \mathcal{L} := \partial_t^2 + (3 - \cdots)\dot{\rho}\partial_t - e^{-2\rho}\partial_{\mathbf{x}}^2$$

- Solution with the retarded Green fn.

$$\zeta(x) = \psi(x) + \int_{t_i}^{t} \mathrm{d}t' \int \mathrm{d}^3\mathbf{x}' G_R(x,x')\mathcal{S}_{\rm NL}[\zeta(x')]$$

$$\mathcal{L}G_R(x,\, x') = \delta^{(4)}(x - x')$$

n-point fun. from the solution with the retarded Green fun. agree with n-point fun. in the in-in formalism.

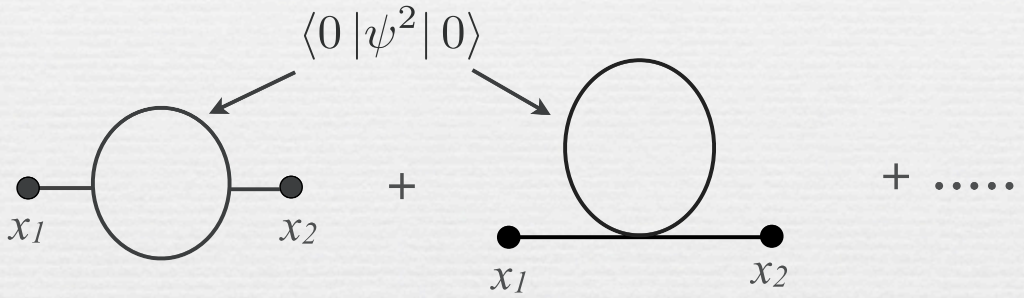
IR regularity condition

(ex) 2-point fn. up to one loop $\langle 0 |{}^g R(x_1) {}^g R(x_2) | 0 \rangle$

Y.U. & T. Tanaka (2012)

$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} (v_k a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \text{h.c.})$$

vacuum state $\quad a_{\mathbf{k}} | 0 \rangle = 0$



...could be divergent

IR regularity condition

$$\langle 0 |{}^g R(x_1) {}^g R(x_2) | 0 \rangle_{1\,loop} \sim \langle 0 | \psi^2 | 0 \rangle \times \mathcal{F}$$

$$\mathcal{F} \to 0 \quad \text{if} \quad \left[\mathcal{L}_{R,k}^{-1} 2 \left(\frac{k}{aH} \right)^2 + k^{-3/2} \mathbf{k} \cdot \partial_{\mathbf{k}} k^{3/2} \right] v_k(t) = 0$$

Retarded integral $\quad \mathcal{L}_{R,k}^{-1} f_{\mathbf{k}}(t) \sim \int_{t_i}^{t} \mathrm{d}t' [v_k(t) v_k^*(t') - \text{c.c.}] f_{\mathbf{k}}(t')$

Connection to gauge invariance

Residual gauge DOFs ∋ Scale transformation

$$x^i \to e^{-s} x^i, \quad \zeta \to \zeta - s$$

IR regularity conditions
= Invariance under the scale transformation

Dilatation symmetry

Symmetry of action for ζ $$x^i \to e^{-s} x^i, \quad \zeta \to \zeta - s$$

s: constant parameter (in x^i)

$$S = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}\mathcal{L}[x; \zeta(t, x^i)] = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}\mathcal{L}[x; \zeta(t, e^{-s} x^i) - s]$$

$$x^i \to e^{-s} x^i$$

- ..., because ζ appears in *S* only in the form of $e^\zeta \mathrm{d}x^i$

$$\mathrm{d}l^2 = e^{2\zeta} \mathrm{d}\mathbf{x}^2$$

- Dilatation symmetry should be thought of as a gauge DOF in the local universe.
- Non-perturbative argument without use of slow-roll approx.

Hamiltonian & dilatation symmetry

Canonical transformation

$$\zeta(x), \pi(x) \rightarrow \tilde{\zeta}(x) := \zeta(t,\, e^{-s}x^i),\, \tilde{\pi}(x)$$

To satisfy... $\left[\zeta(t,\mathbf{x}),\, \pi(t,\mathbf{y})\right] = \left[\tilde{\zeta}(t,\mathbf{x}),\, \tilde{\pi}(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$

$$\longrightarrow \quad \tilde{\pi}(x) := e^{-3s}\pi(t,\, e^{-s}x^i)$$

❑ Equivalence of two systems

- Lagrangian density

  from dilatation sym. $\quad \mathcal{L}[x;\zeta(x)] = \mathcal{L}[x;\tilde{\zeta}(x)-s]$

- Hamiltonian density

Key to IR issues!!

$$\mathcal{H}[x;\tilde{\zeta}(x)-s] = \mathcal{H}[x;\zeta(x)],\;\; \underline{\mathcal{H}_{int}[x;\tilde{\zeta}(x)-s] = \mathcal{H}_{int}[x;\zeta(x)]}$$

Invariance of vacuum

Y.U. & T. Tanaka (2012)

n-point functions calculated before/after the canonical transformation agree with each other

$$\langle\,\Omega\,|\zeta(t,e^{-s}x_1^i)\cdots\zeta(t,e^{-s}x_n^i)|\,\Omega\,\rangle_{\zeta/\pi}=\langle\,\tilde{\Omega}\,|\tilde{\zeta}(x_1)\cdots\tilde{\zeta}(x_n)|\,\tilde{\Omega}\,\rangle_{\tilde{\zeta}/\tilde{\pi}}$$

Beginning with free theory

$$\psi(x)=\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}(v_k a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}+\text{h.c.})\qquad\tilde{\psi}(x)=\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}(v_k\tilde{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}+\text{h.c.})$$

$$a_{\mathbf{k}}|\,\Omega\,\rangle=\tilde{a}_{\mathbf{k}}|\,\tilde{\Omega}\,\rangle=0$$

$$\Rightarrow\quad\left[\mathcal{L}_{R,k}^{-1}2\left(\frac{k}{aH}\right)^2+k^{-3/2}\mathbf{k}\cdot\partial_{\mathbf{k}}k^{3/2}\right]v_k(t)=0$$

IR regularity condition!!

IR regular initial condition → failed

No function v_k can satisfy both of *Y.U. & T. Tanaka (2012)*

1. IR regularity condition/Scaling invariance

$$\left[\mathcal{L}^{-1}_{R,k}2\left(\frac{k}{aH}\right)^2 + k^{-3/2}\mathbf{k}\cdot\partial_{\mathbf{k}}k^{3/2}\right]v_k(t) = 0$$

2. Wronskian condition

$$v_k(t)\dot{v}^*_k(t) - \text{c.c.} = \frac{i}{2a^3\varepsilon}$$

From condition 1, $\mathcal{L}^{-1}_{R,k}2\left(\frac{k}{aH}\right)^2 v_k(t_i) = \partial_{t_i}\mathcal{L}^{-1}_{R,k}2\left(\frac{k}{aH}\right)^2 v_k(t_i) = 0$

$$\longrightarrow \quad k^{-3/2}\mathbf{k}\cdot\partial_{\mathbf{k}}k^{3/2}v_k(t_i) = k^{-3/2}\mathbf{k}\cdot\partial_{\mathbf{k}}k^{3/2}\dot{v}_k(t_i) = 0$$

Incompatible with $\mathbf{k}\cdot\partial_{\mathbf{k}}$ (Wronskian condition)

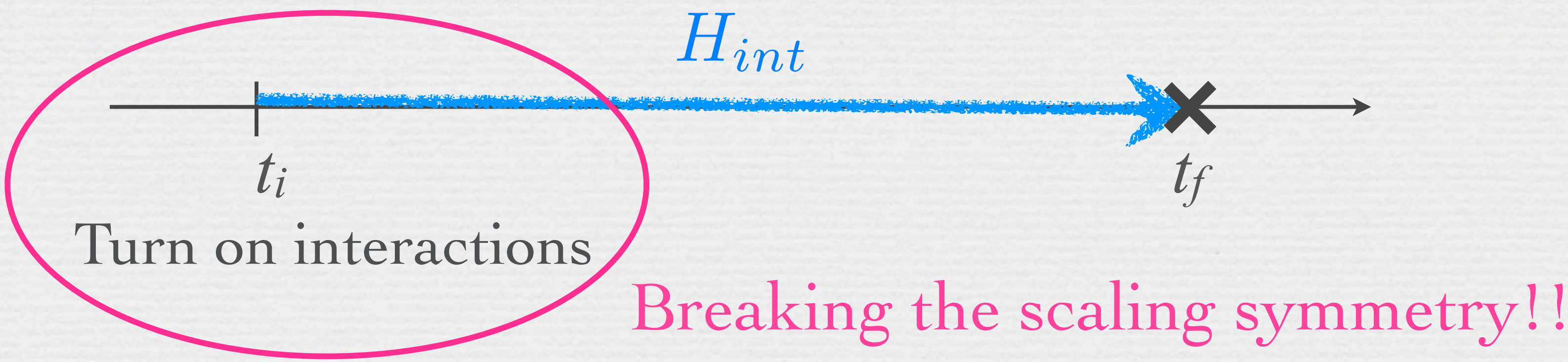
Connection to gauge invariance

Residual gauge DOFs ∋ Scale transformation

$$x^i \to e^{-s} x^i, \quad \zeta \to \zeta - s$$

IR regularity conditions
= Invariance under the scale transformation

Begging with free theory



Outline

1. Origin of Infrared(IR) divergence problem

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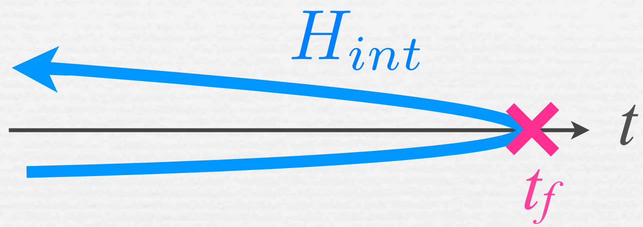
✔ 3. Initial state with interaction

$i\varepsilon$ prescription

Sending $t_i \rightarrow -\infty$ in CTP,



$i\varepsilon$ prescription provides a convergent expression by shutting off contributions of vertexes at distant past.

$i\varepsilon$ prescription selects the ground state

$$U(t_0,\, t_i)|0\rangle = \sum_\alpha U(t_0,\, t_i)|\alpha\rangle\langle\alpha|0\rangle \qquad \sum_\alpha |\alpha\rangle\langle\alpha| = 1$$

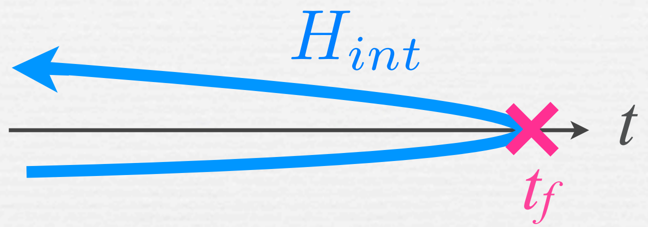
For constant Hamiltonian $\quad U(t_0,\, t_i)|\alpha\rangle \sim e^{-iE_\alpha(-t_i)}|\alpha\rangle$

$$\lim_{t_i\rightarrow-\infty(1-i\epsilon)} U(t_0,\, t_i)|0\rangle = \lim_{t_i\rightarrow-\infty(1-i\epsilon)} \sum_\alpha U(t_0,\, t_i)|\alpha\rangle\langle\alpha|0\rangle \sim |\Omega\rangle$$

where $|\Omega>$ is the ground state of this system

$i\varepsilon$ prescription 2

Sending $t_i \longrightarrow -\infty$ in CTP,



$i\varepsilon$ prescription provides a convergent expression by shutting off contributions of vertexes at distant past.

❑ n-point functions from the in-in formalism

$$\langle \Omega \,| \zeta(x_1) \cdots \zeta(x_n) |\, \Omega \rangle$$

η:conformal time

$$\ni \int \mathrm{d}\eta \, v_k(\eta) v_p(\eta) v_q(\eta) \cdots \,, \quad \int \mathrm{d}\eta \, v_k^*(\eta) v_p^*(\eta) v_q^*(\eta) \cdots \,, \cdots$$

$$\sim e^{-i(k+p+q+\cdots)\eta} \longrightarrow 0 \qquad \sim e^{i(k+p+q+\cdots)\eta} \longrightarrow 0$$

$$\eta \to -i\infty(1+i\varepsilon) \qquad \eta \to -i\infty(1-i\varepsilon)$$

By rotating time paths, all vertex integrals can be convergent.

Def. of Euclidean vacuum

Euclidean vacuum

= n-point functions $\langle\zeta(x_1)\cdots\zeta(x_n)\rangle$ are convergent

in the limit $t_i \longrightarrow -\infty(1\pm i\varepsilon)$

Remark 1

In the de Sitter limit, this definition gives the original Euclidean vacuum, which agrees with the adiabatic vacuum.

Remark 2

Euclidean vacuum is unique and irrelevant to choices of canonical variables. (ζ, π), $(\tilde{\zeta}, \tilde{\pi})$

$\longrightarrow$ Invariance under the scale transformation

$\longrightarrow$ IR regularity

Technical details

n-point functions for Euclidean vacuum *Y.U. & T. Tanaka (2012)*

$$\langle\,\Omega|\,{}^gR(t_f,x^i_1)\,{}^gR(t_f,x^i_2)\cdots{}^gR(t_f,x^i_n)|\Omega\,\rangle$$

is IR regular up to all orders in perturbation.

❋ To cure UV behavior, *Tsamis & Woodard (1992)*

$${}^gR(t_f,x^i)\to -4e^{-2\rho}\partial^2\zeta(t_f,e^{-\bar\zeta}x^i)$$

$\bar\zeta$:local spatial average

Key ingredient

$$(\zeta,\,\pi),\quad \mathcal{H}_{int}[\zeta(x)]$$

$$\longrightarrow\quad (\tilde\zeta,\,\tilde\pi),\quad \tilde{\mathcal{H}}_{int}[\tilde\zeta(x)]=\tilde{\mathcal{H}}_{int}[\tilde\zeta(x)-s(t)]+\cdots$$

IR regular

After quantization,

$$\tilde{\mathcal{H}}_{int}[\tilde\zeta(x)]\ni\tilde\zeta(x)-\bar\zeta(t),\,\partial_i\tilde\zeta(x),\,\partial_t\tilde\zeta(x)$$

Technical details 2

n-point functions for Euclidean vacuum *Y.U. & T. Tanaka (2012)*

$$\langle\, \Omega |{}^g R(t_f, x_1^i)\, {}^g R(t_f, x_2^i) \cdots {}^g R(t_f, x_n^i) | \Omega \,\rangle$$

- If we take the canonical variable of $\tilde{\zeta}$, vertex integrals include $\tilde{\zeta}$ with differentiation or in the form of $\tilde{\zeta} - \bar{\zeta}$

  - Spatial derivative $\partial_i \tilde{\zeta}(x) \to k_i\, \tilde{\zeta}_{\mathbf{k}}$

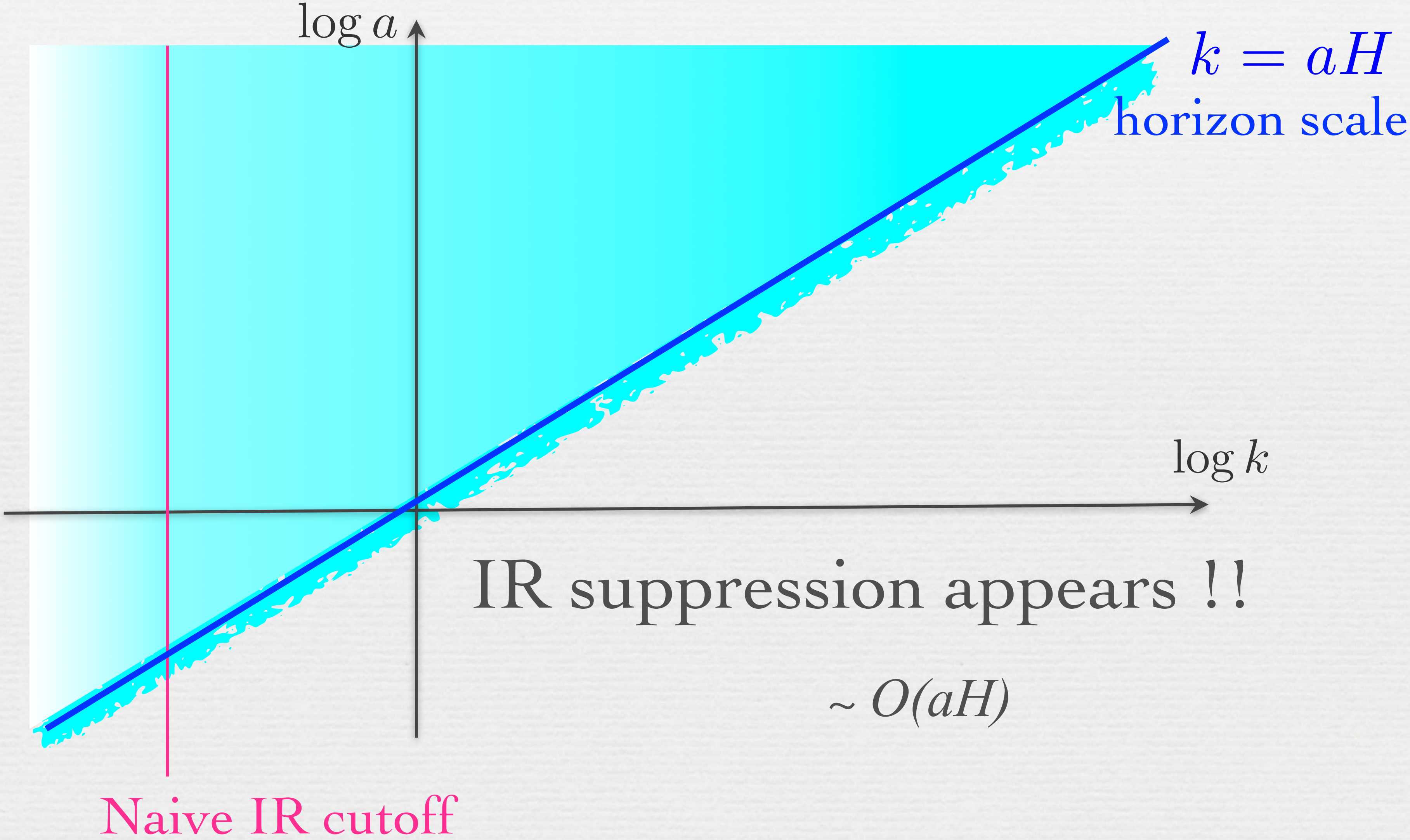
  - Time derivative $\partial_t \tilde{\zeta}(x) \to (k/aH)^2 \tilde{\zeta}_{\mathbf{k}}$

  - Subtraction of $\bar{\zeta}$ $\to |\mathbf{k}| L \tilde{\zeta}_{\mathbf{k}}$ L: Scale of local region

  These suppression makes n-point fns. IR regular and also makes irrelevant to change of outside the local region.

Revisit of secular growth

Which modes participate in loop corrections?



Concluding remarks

- Observable fluctuation should be gauge-invariant in the local universe.

❑ Strong restriction on initial states

- As long as we consider a finite initial time, the manifestly unitary time evolution is not compatible with IR regularity and the consistency of the canonical system.

- Taking Euclidean vacuum with $i\varepsilon$ prescription can provide an IR regular prescription which is independent of outside the observable region.