Non-perturbative Effects in Stochastic Slow-roll Inflation

Alexei A. Starobinsky

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Perturbative results

Stochastic slow-roll inflation

Langevin and Fokker-Planck equations

Application: equilibrium state on the de Sitter background

Probabilities to go to different vacua after inflation

Local duration of inflation

Choice of an initial condition

Correlations and joint PDF

Conclusions

Perturbative anomalous growth of light scalar fields in the de Sitter space-time

Background - fixed - de Sitter or, more interestingly, quasi-de Sitter space-time (slow roll inflation).
Occurs for $0 \leq m^2 \ll H^2$ where $H \equiv \frac{\dot{a}}{a}$, $a(t)$ is a LFRW scale factor. The simplest and textbook example:
$m = 0,\ H = H_0 = const$ for $t \geq t_0$ and the initial quantum state of the scalar field at $t = t_0$ is the adiabatic vacuum for modes with $k/a(t_0) \gg H_0$ and some infrared finite state otherwise:

$$< \phi^2 >= \frac{H_0^2 N}{4\pi^2} + const$$

Here $N = \ln \frac{a}{a(t_0)} \gg 1$ is the number of e-folds from the beginning of inflation and the constant depends on the initial quantum state (Linde, 1982; AS, 1982; Vilenkin and Ford, 1982).
Straightforward generalization to the slow-roll case $|\dot{H}| \ll H^2$.

For $0 < m^2 \ll H^2$, the Bunch-Davies equilibrium value

$$< \phi^2 >= \frac{3H_0^4}{8\pi m^2} \gg H_0^2$$

is reached after a large number of e-folds $N \gg \frac{H_0^2}{m^2}$. Purely infrared effect - creation of real field fluctuations; renormalization is not important and does not affect it.

For the de Sitter inflation (gravitons only) (AS, 1979):

$$P_g(k) = \frac{16GH_0^2}{\pi}; \quad < h_{ik} h^{ik} >= \frac{16GH_0^2 N}{\pi} .$$

The assumption of small perturbations breaks down for $N \gtrsim 1/GH_0^2$. Still ongoing discussion on the final outcome of this effect. My opinion - no screening of the background cosmological constant, instead - stochastic drift through an infinite number of locally de Sitter, but globally non-equivalent vacua.

Reason: the de Sitter space-time is not the generic late-time asymptote of classical solutions of GR with a cosmological constant Λ both without and with hydrodynamic matter. The generic late-time (expanding) asymptote is (Starobinsky, 1983):

$$ds^2 = dt^2 - \gamma_{ik}\, dx^i dx^k$$

$$\gamma_{ik} = e^{2H_0 t} a_{ik} + b_{ik} + e^{-H_0 t} c_{ik} + \ldots$$

where $H_0^2 = \Lambda/3$ and the matrices a_{ik}, b_{ik}, c_{ik} are functions of spatial coordinates. a_{ik} contains two independent physical functions (after 3 spatial rotations and 1 shift in time + spatial dilatation) and can be made unimodular, in particular.

An always larger effect - due to scalar (adiabatic) fluctuations of an inflaton degree of freedom (Mukhanov and Chibisov, 1981; Hawking, 1982; AS, 1982; Guth and Pi, 1982):

$$P_\zeta(k) = \frac{GH_k^4}{\pi|\dot{H}|_k}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$.
The consistency relation for inflaton-driven inflation

$$r \equiv \frac{P_g}{P_\zeta} = \frac{16|\dot{H}_k|}{H_k^2} = 8|n_g| < 0.17$$

(the last inequality - from the most recent observations).
For sufficiently large $N_f = \ln\left(\frac{a_f}{a}\right)$, inhomogeneous fluctuations become larger than 1. Loop corrections proportional to higher powers of h may become important in this regime only.

Beyond small perturbations during slow-roll inflation

Locally – around our world-line – slow-roll inflation has both the beginning and the end.
Globally it has no beginning and no end in the most of interesting cases – in the sense that patches of an approximately de Sitter space-time (not necessarily expanding only) always exist somewhere in space and time (but outside our past and future light cones) - "eternal inflation".
Taking backreaction into account $\Longrightarrow$ quantum background.

Stochastic approach to inflation ("stochastic inflation"):

$$\hat{R}_\mu^\nu - \frac{1}{2}\delta_\mu^\nu \hat{R} = 8\pi G\, \hat{T}_\mu^\nu(\hat{g}_{\alpha\beta})$$

- not as a function of $< \hat{g}_{\alpha\beta} >$!

Leads to QFT in a stochastic background.

Stochastic inflation:

1) can deal with an arbitrary large (though sufficiently smooth) global inhomogeneity;

2) takes backreaction of created fluctuations into account;

3) goes beyond any finite order of loop corrections.

Fully developed in Starobinsky (1984,1986) though the first simplified application (but beyond the one-loop approximation) was already in Starobinsky (1982).

Langevin equation for the large-scale field

The first main idea: splitting of the inflaton field ϕ into a large-scale and a small-scale parts with respect to H. More exactly, the border is assumed to lie at $k = \epsilon a H$ with $\exp\left(-\frac{H^2}{|\dot{H}|}\right) \ll \epsilon \ll 1$.

The second main idea: a non-commutative part of the large-scale field is very small (it is composed from decaying modes), so we may neglect it. Then the remaining part is equivalent (not equal!) to a stochastic c-number (classical) field with some distribution function.

$$\frac{d\phi}{d\tau^{(n)}} = -\frac{1}{3H^{n+1}}\frac{dV}{d\phi} + f ,$$

$$< f(\tau_1^{(n)})\, f(\tau_2^{(n)}) >= \frac{H^{3-n}}{4\pi^2}\, \delta(\tau_1^{(n)} - \tau_2^{(n)})\,.$$

The time-like variables $\tau^{(n)} = \int H^n(t, \mathbf{r})\, dt$, where $H^2 = 8\pi G V(\phi)/3$.

This is not a time reparametrization $t \to f(t)$ in GR. Different $\tau^{(n)}$ describe different stochastic processes and even have different dimensionality. Different "clocks" are needed to measure them:

1) $n = 0$: phase of a wave function of a massive particle ($m \gg H$);
2) $n = 1$: scalar metric perturbations (δN formalism);
3) $n = 3$: dispersion of a light scalar field generated during inflation

$$< \chi^2 >= \frac{1}{4\pi^2} < \int H^3\, dt >= \frac{< \tau^{(3)} >}{4\pi^2} \, .$$

See F. Finelli *et al.*, Phys. Rev. D **79**, 044007 (2009) for more details.

f describes the flow of small-scale linear field modes through the border $k = \epsilon aH$ to the large-scale region in the course of the universe expansion. In the leading approximation, it is constructed from solutions of the massless scalar field equation in the de Sitter space-time with the adiabatic vacuum initial condition for a given spatial Fourier mode:

$$\phi_k = H_0 \,(2k)^{-1/2} \left(\eta - \frac{i}{k}\right) e^{-ik\eta} \ , \quad \eta = -(a(t)H_0)^{-1}$$

$$f(t, \mathbf{r}) = \frac{\epsilon a H_0^2}{(2\pi)^{3/2}} \int d^3k \, \delta(k - \epsilon a H_0) \, \frac{(-i)H_0}{\sqrt{2}k^{3/2}} \left[a_k e^{-i\mathbf{k}\mathbf{r}} - a_k^\dagger e^{i\mathbf{k}\mathbf{r}}\right]$$

Though formally an operator, $f(t, \mathbf{r})$ is equivalent to a classical Gaussian white noise.

Applicability conditions – the standard slow-roll ones:

$$V'^2 \ll 48\pi GV^2, \quad |V''| \ll 8\pi GV/3$$

Einstein-Smoluhovsky (Fokker-Planck) equation

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial \phi}\left(\frac{V'}{3H^{n+1}}\rho\right) + \frac{1}{8\pi^2}\frac{\partial^2}{\partial \phi^2}\left(H^{3-n}\rho\right) .$$

Probability conservation: $\int \rho \, d\phi = 1$.

Remarks

- More generally, the last term can be written the form

$$\frac{1}{8\pi^2}\frac{\partial}{\partial \phi}\left(H^{(3-n)\alpha}\frac{\partial}{\partial \phi}\left(H^{(3-n)(1-\alpha)}\rho\right)\right)$$

  with $0 \leq \alpha \leq 1$.
  $\alpha = 0$ – Ito calculus.
  $\alpha = 1/2$ – Stratonovich calculus.
  However, keeping terms explicitly depending on α exceeds the accuracy of the stochastic approach. Thus, α may put 0.

- All results are independent of the form of a cutoff in the momentum space as far as it occurs for $k \ll aH$ $(\epsilon \ll 1)$.
- Backreaction is taken into account: $\delta T_\mu^\nu = (V - V_{clas})\, \delta_\mu^\nu$.
- No necessity in any infrared cutoff. Problems with the so called "volume weighting" arise because quantities like $a^3\rho$ are considered which are not normalizable, thus, they may not be considered as probabilities of anything from the mathematical point of view ("unitarity breaking"). Their physical justification is also flawed since it based on the wrong assumption that all Hubble physical volumes ("observers") emerging from expansion of a previous inflationary patch are clones of each other while it is not so.

- Another possible source of apparent infrared divergences: use of "gauge invariant" (with respect to a background space-time metric) variables like $\zeta(\mathbf{r}, t)$ which are not generally covariant with respect to the full metric and, therefore, not directly observable. In contrast, quantities like $\zeta(\mathbf{r}, t) - \zeta(0, t_0)$ are generally covariant and observable though non-local.
- The accuracy of the stochastic approach is not sufficient for calculating quantities $\sim H^2$ in $< \phi^2 >$ and $\sim H^4$ in EMT average values because of the omission of a contribution from the small-scale part (including the conformal anomaly). However, all larger quantities (if exist) can be calculated quantitatively correctly. Also, the small-scale part is mainly the one-loop correction from a massless minimally coupled scalar field, so it can be added.

Transition to predictions for the post-inflationary evolution

From $\rho(\phi, \tau)$ during inflation to the distribution $w(\tau)$ over the total local duration of inflation:

$$w(\tau) = \lim_{\phi \to \phi_{end}} j = \lim_{\phi \to \phi_{end}} \frac{|V'|}{3H^{n+1}} \rho(\phi, \tau) \, .$$

For the graceful exit to a post-inflationary epoch, the stochastic force should be much less than the classical one during last e-folds of inflation.

The same way to obtain the joint distribution $w(0, \tau_1; \ |\mathbf{r}|, \tau_2)$ from the 2-point joint probability distribution $\rho(\phi_1, 0, \tau_1; \phi_2, |\mathbf{r}|, \tau_2)$ during inflation.

From δN- to N-formalism

Let $n = 1$. When

$$ds^2 = dt^2 - a^2(t) e^{2\zeta(\mathbf{r})} d\mathbf{r}^2 + \text{small terms}$$

after inflation and complete thermalization where

$$\zeta(\mathbf{r}) = N(\mathbf{r}) \equiv \tau^{(1)}(\mathbf{r}) \ .$$

QFT of a self-interacting scalar field in the de Sitter background

A.A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994).

The equilibrium (static) solution for the 1-point distribution:

$$\rho_{eq}(\phi) = \mathrm{const}\, e^{-2v}\,, \quad v = \frac{4\pi^2 V(\phi)}{3H_0^4}\ .$$

Arbitrary Green functions and n-point distributions can be constructed, too, using solutions of the same Fokker-Planck equation.

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 , \quad 0 < \lambda \ll 1, \; H_0^2 = \frac{8\pi G V_0}{3} .$$

Three regimes:

1. Perturbative regime $\sqrt{\lambda}H_0^2 \ll m^2 \ll H_0^2$.

$$< \phi^2 >= \frac{3H_0^4}{8\pi m^2}\left(1 - 3\beta + \frac{429}{16}\beta^2 + ...\right) , \quad \beta = \frac{3\lambda H_0^4}{8\pi^2 m^4}.$$

Compare to the same result in the one-loop (Gaussian) approximation:

$$< \phi_G^2 >= \frac{3H_0^4}{8\pi m^2}\left(1 - 3\beta + 18\beta^2 + ...\right) .$$

2. Massless self-interacting regime $|m^2| \ll \sqrt{\lambda} H_0^2$.

$$< \phi^2 >= \sqrt{\frac{3}{2\pi^2}} \frac{\Gamma(0.75)}{\Gamma(0.25)} \frac{H_0^2}{\sqrt{\lambda}} \approx 0.132 \frac{H_0^2}{\sqrt{\lambda}}$$

$$< \phi_G^2 >= \frac{1}{\pi\sqrt{8}} \frac{H_0^2}{\sqrt{\lambda}} \approx 0.113 \frac{H_0^2}{\sqrt{\lambda}}$$

The scale $m^2 \sim \lambda H_0^2$ proposed in arXiv:1005.3551 is not critical at all!

3. Symmetry breaking regime $m^2 < 0, \quad \sqrt{\lambda} H_0^2 \ll |m^2| \ll H_0^2$.

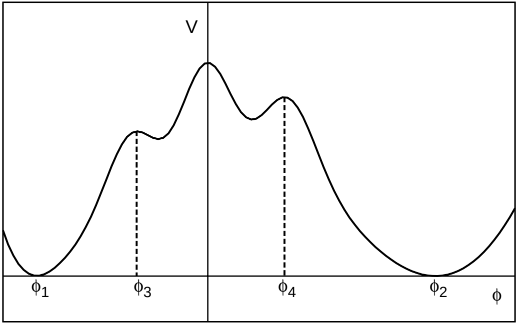
$$< \phi^2 >= \frac{|m^2|}{\lambda} + \frac{3H_0^4}{16\pi^2|m^2|} + \mathcal{O}\left(e^{-1/(4\beta)}\right)$$

The (modulus of) exponent is the action for the Hawking-Moss instanton. See also F. Finelli *et al.*, Phys. Rev. D **82**, 064020 (2010).

Account of scalar loops from the small-scale part → running of λ with energy → $\lambda \to \lambda(H_0)$.

Probabilities to go to different vacua after inflation

Let inflation may end in two vacua: $\phi = \phi_1$ and $\phi = \phi_2$ with $V(\phi_1) = V(\phi_2) = 0$ (to consider a larger number of post-inflationary vacua, ϕ should have more than one-dimensional internal space).



Boundary conditions at the end of inflation:
$\rho(\phi_1, \tau) = \rho(\phi_2, \tau) = 0$.

Method of calculation (Starobinsky (1984,1986)): consider the quantities

$$Q_m(\phi) = \int_0^\infty \tau^m \rho(\phi, \tau)\, d\tau$$

where $\tau = 0$ corresponds to the local beginning of inflation.
$Q_m(\phi_1) = Q_m(\phi_2) = 0$.

By integrating the Fokker-Planck equation over τ, we get for $m = 0$:

$$Q_0(\phi) = \frac{8\pi^2}{H^{3-n}} \exp\left(\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} d\psi \exp\left(-\frac{\pi}{GH^2(\psi)}\right) \times$$

$$\left(C_0 - \int_{\phi_1}^{\psi} \rho_0(\psi_1)\, d\psi_1\right) ,$$

$$C_0 = \frac{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} \rho_0(\psi)\, d\psi}{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right)} .$$

$P_1 = C_0$ – the absolute probability to go to the vacuum $\phi = \phi_1$;
$P_2 = 1 - C_0$ – the absolute probability to go to the vacuum $\phi = \phi_2$.
No n dependence in C !

Local duration of inflation

$$Q_1(\phi) = \frac{8\pi^2}{H^{3-n}} \exp\left(\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} d\psi \exp\left(-\frac{\pi}{GH^2(\psi)}\right) \times$$

$$\left(C_1 - \int_{\phi_1}^{\psi} Q_0(\psi_1)\, d\psi_1\right) ,$$

$$C_1 = \frac{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right) \int_{\phi_1}^{\phi} Q_0(\psi)\, d\psi}{\int_{\phi_1}^{\phi_2} d\phi \exp\left(-\frac{\pi}{GH^2(\phi)}\right)} .$$

$$< \tau_1 >= \frac{C_1}{C_0}, \quad < \tau_2 >= \frac{\tilde{C}_1}{1 - C_0},$$

$$< \tau >_{tot}= C_0 < \tau_1 > +(1 - C_0) < \tau_2 >= \int_{\phi_1}^{\phi_2} Q_0(\phi)\, d\phi .$$

$\tilde{C}_1$ is C_1 with ϕ_1 and ϕ_2 interchanged.

Choice of an initial condition

- Static solutions – not normalizable in the inflationary (i.e. unstable) case.
- $\rho_0(\phi) = \delta(\phi - \phi_0)$ – why?
- "Eternal inflation as an initial condition": $\rho_0(\phi) \propto \rho_{E_1}(\phi)$ – the wave function of the lowest energy level of the Schrodinger equation arising through the separation of variables in the Fokker-Planck equation ($E_0 = 0$ due to hidden supersymmetry of the former).
  1) Not possible in the continuum spectrum case.
  2) In the discrete spectrum case, generically $E_2 - E_1 \sim E_1$ – not enough time for relaxation.
  As a whole, "eternal" inflation seems not be eternal enough to fix the initial condition uniquely.

However, if inflation had occurred at all, the dependence of predictions on $\rho_0(\phi)$ is comparatively weak: for almost all $\rho_0(\phi)$ except from the HH-like one $\rho_0(\phi) \propto \exp\left(\frac{\pi}{GH^2(\phi)}\right)$, the main contribution comes from the highest maximum of $V(\phi)$ without any necessity of a "tunneling" initial condition.

On the other hand, if $\rho_0(\phi) \propto \exp\left(\frac{\pi}{GH^2(\phi)}\right)$, there is practically no inflation at all, and final probabilities P_1 and P_2 are equal to the initial ones.

Correlations and joint PDF

Following A.A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994).

In the leading approximations, all Green functions and joint n-point probability distributions of the inflaton field can be expressed through solutions of the same Fokker-Planck equation with different initial conditions only. In particular, in the case $H \approx H_0$ during inflation (for simplicity only), the general two-point PDF for 4D-points lying outside each other future light cones in the stochastic approach is:

$$\rho_2[\phi_1(\mathbf{r_1}, t_1), \phi_2(\mathbf{r_2}, t_2)] =$$

$$\int \Pi[\phi_1(\mathbf{r_1}, t_1)|\phi_r(\mathbf{r_1}, t_r)]\Pi[\phi_2(\mathbf{r_2}, t_2)|\phi_r(\mathbf{r_2}, t_r)]\rho_1(\phi_r, t_r)\, d\phi_r$$

where t_r is the time in the past when both corresponding 3D-points were inside one Hubble volume and $\Pi[\phi_1(\mathbf{r}, t_1)|\phi_2(\mathbf{r}, t_2)]$ satisfies the Fokker-Planck equation with respect to both its time and field variables with the initial condition

$$\Pi[\phi_1(\mathbf{r}, t_1)|\phi_2(\mathbf{r}, t_1)] = \delta(\phi_1 - \phi_2)$$

Through the N-formalism - joint probability distributions of a space-time metric after inflation.

Otherwise, if the 4D-points are inside the future light cone of one of them, the spatial points $\mathbf{r_1}$ and $\mathbf{r_2}$ are inside one elementary averaging volume, so they coincide in terms of the stochastic approach. Then, for $t_1 < t_2$,

$$\rho_2[\phi_1(\mathbf{r}, t_1), \phi_2(\mathbf{r}, t_2)] = \Pi[\phi_2(\mathbf{r}, t_2)|\phi_1(\mathbf{r}, t_1)]\rho_1(\phi_1, t_1))$$

Conclusions

- During slow-roll inflation, backreaction of created inflaton fluctuations has to be taken into account for sufficiently long inflation and leads to QFT in a stochastic background.
- No problems of principle in predicting all joint probability distributions for light scalar fields, including the inflaton itself, during and after inflation (N-formalism) in the original (probability conserving) stochastic approach, once an initial condition $\rho_0(\phi)$ is given. No necessity to refer to other universes outside our light cone.
- New "clocks" apart from metric perturbations are probably needed to measure large infrared effects like the total local duration of an inflationary stage in our Universe.

- No satisfactory principle to fix $\rho_0(\phi)$ uniquely.
- Some dependence on $\rho_0(\phi)$ remains in final answers, so a possibility to get some knowledge on it from observational data does not seem hopeless. However, if inflation had occurred at all, the dependence of predictions on $\rho_0(\phi)$ is weak and mainly produced by the region around the highest maximum of $V(\phi)$. For this, no specific "tunneling" initial condition is needed.