Stochastic Generation of Quantum Fluctuations in Single Field Inflationary Models

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Outline

- Introduction
- Inflation and field-theoretical approach
- Growth of quantum fluctuation: stochastic approach
- Two field quadratic model: physical bounds from stochastic approach
- Test fields vs inflaton fluctuations
- Conclusions

Based on:
F. Finelli, G. M., A. A. Starobinsky, G. P. Vacca and G. Venturi, Phys. Rev. D 79, 044007 (2009);
F. Finelli, G. M., A. A. Starobinsky, G. P. Vacca and G. Venturi, Phys. Rev. D 82, 064020 (2010).

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Introduction

- The theory of quantum fields in expanding universe has evolved from its pioneering years into a necessary tool in order to describe the Universe on large scales.
  The de Sitter background has been the main arena where to compute quantum effects.
- In viable inflationary models, while $|\dot{H}| \ll H^2$, $\dot{H}$ may not be zero

  $$\Downarrow$$

  The study of quantum effects in a nearly de Sitter stage with $\dot{H} \neq 0$ is not of just pure theoretical interest.

Introduction

- In general inflationary model quantum scalar fields can be split into a long wave (coarse grained) component and a short-wave one.
  The former component effectively becomes quasi-classical and it experiences a random walk described by the stochastic inflation approach (Starobinsky (1986)).
- The non-perturbative nature of the stochastic approach to inflation is based on a number of heuristic approximations. Therefore, it is very important to check, whenever possible, results obtained by its application using the standard perturbative QFT in curved space-time.

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Inflation and field-theoretical approach

Let us consider, in a spatially flat FLRW background geometry, a two field model in which the dynamics is driven by a minimally coupled inflaton ϕ and a minimally coupled scalar field χ is present. We shall neglect the χ energy density and pressure in the background FLRW equations.

The action is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \bar{V}(\chi) \right]$$

and we can expand our background fields $\{\phi, \chi, g_{\mu\nu}\}$ up to second order in the non-homogeneous perturbations, without fixing any gauge, as follows:

$$\phi(t,\vec{x}) = \phi^{(0)}(t) + \phi^{(1)}(t,\vec{x}) + \frac{\phi^{(2)}(t,\vec{x})}{2},$$

$$\chi(t,\vec{x}) = \chi^{(0)}(t) + \chi^{(1)}(t,\vec{x}) + \frac{\chi^{(2)}(t,\vec{x})}{2}$$

Inflation and field-theoretical approach

$$g_{00} = -1 - 2\alpha^{(1)} - \alpha^{(2)}, \qquad g_{i0} = -\frac{a}{2}\left(\beta^{(1)}_{,i} + \frac{\beta^{(2)}_{,i}}{2}\right),$$

$$g_{ij} = a^2\left[\delta_{ij}\left(1 - 2\psi^{(1)} - \psi^{(2)}\right) + D_{ij}\left(E^{(1)} + \frac{E^{(2)}}{2}\right)\right],$$

where $D_{ij} = \partial_i\partial_j - \delta_{ij}(\nabla^2/3)$.
The scalar sector can be reduced to the study of the gauge invariant Sasaki-Mukhanov variable Q (Sasaki (1986), Mukhanov (1988)).
Q can be seen, order by order, as the scalar field fluctuations $\phi^{(n)}$ on uniform curvature hypersurface (see, for example, Malik (2005)).
At first order

$$Q^{(1)} = \phi^{(1)} + \frac{\dot{\phi}^{(0)}}{H}\left(\psi^{(1)} + \frac{1}{6}\nabla^2 E^{(1)}\right)$$

In the same way the physically meaningfull variable associated with χ is the gauge invariant variable Q_χ given, order by order, by its scalar field fluctuations $\chi^{(n)}$ on uniform curvature hypersurface.

Inflation and field-theoretical approach

To fix a gauge in the scalar sector we can, in particular, set to zero two scalar variables among ϕ, α, β, ψ and E.

$$\psi = 0,\ E = 0 \qquad \text{Uniform Curvature Gauge}$$

$$\Downarrow$$

$$Q^{(n)} = \phi^{(n)} \qquad Q_\chi^{(n)} = \chi^{(n)}$$

In the UCG, at the leading order in the slow-roll approximation and in the long-wavelength limit, the equation of motions satisfied by the scalar fields ϕ and χ can be obtained order by order from the expansion around the classical solution of the equations:

$$\frac{d\phi}{dN} = -\frac{V_\phi}{3\,H(\phi)^2} \quad , \quad \frac{d\chi}{dN} = -\frac{\bar{V}_\chi}{3\,H(\phi)^2}$$

Inflation and field-theoretical approach

So for the inflaton field we have

$$\ddot{\phi}^{(0)} + 3H\dot{\phi}^{(0)} + V_\phi = 0 \quad , \quad 3H\dot{\phi}^{(1)} + \left[V_{\phi\phi} - \frac{V_\phi^2}{3H^2 M_{pl}^2} \right] \phi^{(1)} = 0$$

$$3H\dot{\phi}^{(2)} + \left[V_{\phi\phi} - \frac{V_\phi^2}{3H^2 M_{pl}^2} \right] \phi^{(2)} = - \left[V_{\phi\phi\phi} - \frac{V_{\phi\phi} V_\phi}{H^2 M_{\rm pl}^2} + \frac{2 V_\phi^3}{9H^4 M_{\rm pl}^4} \right] \phi^{(1)\,2}$$

while for the field χ one obtains

$$\ddot{\chi}^{(0)} + 3H\dot{\chi}^{(0)} + \bar{V}_\chi = 0 \quad , \quad 3H\dot{\chi}^{(1)} + \bar{V}_{\chi\chi}\chi^{(1)} = 2\frac{H_\phi}{H}\bar{V}_\chi \varphi^{(1)}$$

$$3H\dot{\chi}^{(2)} + \bar{V}_{\chi\chi}\chi^{(2)} = 2\frac{H_\phi}{H}\bar{V}_\chi\varphi^{(2)} + 2\left[\frac{H_{\phi\phi}}{H} - 3\left(\frac{H_\phi}{H}\right)^2\right]\bar{V}_\chi\varphi^{(1)2} + 4\bar{V}_{\chi\chi}\frac{H_\phi}{H}\varphi^{(1)}\chi^{(1)} - \bar{V}_{\chi\chi\chi}\chi^{(1)2}$$

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Growth of quantum fluctuation: stochastic approach

How does one re-derive these results in the stochastic approach?

The consideration above suggests that one has to choose the time variable $N = \int H(\phi)dt$ in the Langevin stochastic equation for the large-scale part of ϕ or χ

In the past both N (Gangui, Lucchin, Matarrese, Mollerach (1994), Enqvist, Nurmi, Podolsky, Rigopoulos (2008)) and the proper time t (Linde, Linde, Mezhlumian (1994), Martin, Musso (2005)) were considered as time variable in the Langevin stochastic equation.

The transformation from t to N is not a simple time reparametrization $t \to f(t)$, this is made using the stochastic function $H(\phi)$ and leads to a physically different stochastic process.

Statement: we *should* use the N variable when we consider any gauge invariant quantity containing metric fluctuations. Otherwise, incorrect results would be obtained using the stochastic approach which would then not coincide with those obtained using perturbative QFT methods.

Growth of quantum fluctuation: stochastic approach

Statement supported by exact non-perturbative results (valid to all orders of metric perturbations) from the general δN formalism which relates the value of the gauge invariant metric perturbation ζ after inflation to the difference in the number of e-folds N in different points of space (Starobinsky (1982,1985), Sasaki, Stewart (1996))

The choice of a proper time variable in the stochastic equation is not an absolute one.

This is dictated by the physical nature of 'clocks' relevant to observable effects.

Growth of gauge invariant quantum fluctuations $\Rightarrow$ N is the 'clock'.

Growth of quantum fluctuation: stochastic approach

The Langevin stochastic equations can then be written as

$$\frac{d\phi}{dN} = -\frac{V_\phi}{3H^2} + \frac{f_\phi}{H}\,, \qquad \frac{d\chi}{dN} = -\frac{\bar{V}_\chi}{3H^2} + \frac{f_\chi}{H}$$

$$\langle f_\phi(N_1) f_\phi(N_2)\rangle = \frac{H^4}{4\pi^2}\,\delta(N_1 - N_2)\,, \qquad \langle f_\chi(N_1) f_\chi(N_2)\rangle = \frac{H^4}{4\pi^2}\,\delta(N_1 - N_2)$$

where $H^2 = V(\phi)/3M_{pl}^2$ is a function of ϕ, and the stochastic noise terms are given, to the leading order in the slow-roll approximation, by

$$\begin{aligned} f_\phi(t,\mathbf{x}) &= \epsilon a H^2 \int \frac{d^3k}{(2\pi)^{3/2}} \delta(k - \epsilon a H) \left[\hat{a}_k \phi_k(t) e^{-i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_k^\dagger \phi_k^*(t) e^{+i\mathbf{k}\cdot\mathbf{x}}\right] \\ f_\chi(t,\mathbf{x}) &= \epsilon a H^2 \int \frac{d^3k}{(2\pi)^{3/2}} \delta(k - \epsilon a H) \left[\hat{b}_k \chi_k(t) e^{-i\mathbf{k}\cdot\mathbf{x}} + \hat{b}_k^\dagger \chi_k^*(t) e^{+i\mathbf{k}\cdot\mathbf{x}}\right]\,. \end{aligned}$$

On expanding to first order one obtains the following stochastic equation for the inflaton fluctuation

$$\frac{d}{dN}\phi^{(1)} + 2M_{pl}^2 \left(\frac{H_{\phi\phi}}{H} - \frac{H_\phi^2}{H^2}\right)\phi^{(1)} = \frac{f_\phi}{H}$$

Growth of quantum fluctuation: stochastic approach

The general stochastic solution is given by

$$\phi^{(1)} = \frac{V_\phi}{V} \int_{t_i}^{t} d\tau \left(\frac{V}{V_\phi} f_\phi \right) ,$$

with the following result for the growth of quantum fluctuations

$$\langle (\phi^{(1)})^2 \rangle = \frac{1}{4\pi^2} \left(\frac{V_\phi}{V} \right)^2 \int_{t_i}^{t} dt' H^3 \left(\frac{V}{V_\phi} \right)^2$$

For example, for a chaotic model $V = \frac{m^2 \phi^2}{2}$ one obtains

$$\langle (\phi^{(1)})^2 \rangle = \frac{H_0^6 - H^6}{8\pi^2 m^2 H^2} ,$$

which corresponds to the QFT result (Finelli, GM, Vacca, Venturi (2004)).

Growth of quantum fluctuation: stochastic approach

In the same way, to second order, we have the following evolution equation

$$\frac{d}{dt}\langle\delta\phi^{(2)}\rangle = \frac{H^3}{8\pi^2}\left(\frac{V_\phi}{V}\right) - \left(\frac{1}{3H}V_{\phi\phi} + 2\frac{\dot{H}}{H}\right)\langle\delta\phi^{(2)}\rangle + \left[-\frac{1}{3H}V_{\phi\phi\phi} + \left(\frac{1}{H}V_{\phi\phi} + 4\frac{\dot{H}}{H}\right)\frac{V_\phi}{V}\right]\langle(\phi^{(1)})^2\rangle\,.$$

with general solution

$$\langle\phi^{(2)}\rangle=\left(\frac{V_\phi}{V}\right)\int_{t_i}^{t}dt'\left(\frac{V}{V_\phi}\right)\left\{\frac{H^3}{8\pi^2}\left(\frac{V_\phi}{V}\right)+\left[-\frac{1}{3H}V_{\phi\phi\phi}+\left(\frac{1}{H}V_{\phi\phi}+4\frac{\dot{H}}{H}\right)\frac{V_\phi}{V}\right]\langle(\phi^{(1)})^2\rangle\right\}$$

which also corresponds to the QFT result in curved space-time (Finelli, GM, Starobinsky, Vacca, Venturi (2009)).

Growth of quantum fluctuation: stochastic approach

Proceeding in the same way for the χ fluctuations one obtains the following first order stochastic equations

$$\frac{d\chi^{(1)}}{dt} = -\frac{1}{3H}\bar{V}_{\chi\chi}\chi^{(1)} + \frac{2}{3}\frac{H_\phi}{H^2}\bar{V}_\chi\varphi^{(1)} + f_\chi \,,$$

with solution

$$\chi^{(1)} = \bar{V}_\chi \int_{t_i}^{t} \left(\frac{2}{3}\frac{H_\phi}{H^2}\varphi^{(1)} + \frac{f_\chi}{\bar{V}_\chi} \right) d\tau \,,$$

while to second order we have

$$\begin{aligned}\frac{d\chi^{(2)}}{dt} \quad = \quad & -\frac{1}{3H}\bar{V}_{\chi\chi}\chi^{(2)} + \frac{2}{3}\frac{H_\phi}{H^2}\bar{V}_\chi\varphi^{(2)} - \frac{1}{3H}\bar{V}_{\chi\chi\chi}\chi^{(1)2} + \frac{4}{3}\frac{H_\phi}{H^2}\bar{V}_{\chi\chi}\phi^{(1)}\chi^{(1)} \\ & -\frac{2}{3}\bar{V}_\chi\left[-\frac{H_{\phi\phi}}{H^2} + 3\frac{H_\phi^2}{H^3} \right]\phi^{(1)2} + 2\frac{H_\phi}{H}\phi^{(1)} f_\chi\end{aligned}$$

Growth of quantum fluctuation: stochastic approach

The growth of the quantum χ fluctuations is then given by

$$\langle \chi^{(1)2} \rangle = \frac{\bar{V}_\chi^2}{4\pi^2} \int_{t_i}^{t} d\tau \left[\frac{H(\tau)^3}{\bar{V}_\chi(\tau)^2} - \frac{4}{9M_{pl}^2} \int_{t_i}^{\tau} d\eta \frac{\dot{H}(\tau)}{H(\tau)^3} \frac{\dot{H}(\eta)}{H(\eta)^3} \int_{t_i}^{\eta} d\sigma \frac{H(\sigma)^5}{\dot{H}(\sigma)} \right]$$

and

$$\langle \chi^{(2)} \rangle = \bar{V}_\chi \int_{t_i}^{t} d\tau \left[\frac{2}{3} \frac{H_\phi}{H^2} \langle \phi^{(2)} \rangle - \frac{1}{3H} \frac{\bar{V}_{\chi\chi\chi}}{\bar{V}_\chi} \langle \chi^{(1)2} \rangle + \frac{4}{3} \frac{H_\phi}{H^2} \frac{\bar{V}_{\chi\chi}}{\bar{V}_\chi} \langle \phi^{(1)} \chi^{(1)} \rangle + \frac{2}{3} \left(\frac{H_{\phi\phi}}{H^2} - 3 \frac{H_\phi^2}{H^3} \right) \langle \phi^{(1)2} \rangle \right]$$

where

$$\langle \phi^{(1)} \chi^{(1)} \rangle = -\frac{\bar{V}_\chi}{12\pi^2} \frac{\dot{\phi}}{HM_{pl}^2} \int_{t_i}^{t} d\tau \int_{\tau}^{t} d\eta \left[\frac{H(\tau)^5}{\dot{H}(\tau)} \frac{\dot{H}(\eta)}{H(\eta)^3} \right]$$

Outline

Two field quadratic model: physical bounds from stochastic approach

Let us consider the particular case $V(\phi) = \frac{m^2\phi^2}{2}$ and $\bar{V}(\chi) = \frac{m_\chi^2\chi^2}{2}$.
The background solution for the test field χ is

$$\chi^{(0)}(t) = \chi^{(0)}(t_i)\left(\frac{H(t)}{H(t_i)}\right)^{\frac{m_\chi^2}{m^2}} .$$

This remains a test field for the whole duration of the inflation era if

$$\chi^{(0)}(t_i)^2 \ll \left[1 + \frac{\alpha}{9}\frac{m^2}{H^2}\right]^{-1} \frac{1}{\alpha}\left(\frac{H}{H_i}\right)^{2-2\alpha} 6\frac{H_i^2}{m^2}M_{pl}^2$$

for any value of H (where $\alpha = \frac{m_\chi^2}{m^2}$).
For the case for $\alpha \ll 1$ at the end of inflation ($H \simeq m$) one has

$$\chi^{(0)}(t_i)^2 \ll \frac{6}{\alpha}M_{pl}^2$$

Two field quadratic model: physical bounds from stochastic approach

We then have

$$\langle \chi^{(1)2} \rangle = \frac{3H^{2\alpha}}{8\pi^2 m^2(2-\alpha)}(H_0^{4-2\alpha} - H^{4-2\alpha}) + \\ -\frac{\alpha^2}{48\pi^2}\frac{\chi^{(0)}(t_i)^2}{M_{pl}^2}\left(\frac{H}{H_i}\right)^{2\alpha}\frac{1}{H^4}\left(H^2 - H_i^2\right)^3 .$$

The term dependent from the background value $\chi(0)$ will be negligible with respect to the leading value of the first one, for $\alpha < 2$, if

$$\chi^{(0)}(t_i)^2 \ll \frac{18}{2-\alpha}\frac{1}{\alpha^2}\frac{M_{pl}^2 m^2}{H_i^2} .$$

This condition is different, and can be stronger, with respect to the previous one.

If we consider the particular case of $\alpha \ll 1$ and require that the previous condition implies this one, we obtain

$$\alpha \ll \frac{3}{2}\frac{m^2}{H_i^2} .$$

Two field quadratic model: physical bounds from stochastic approach

Analogously we can evaluate the growth of the quantum second order χ field

$$\langle \chi^{(2)} \rangle = \frac{\alpha}{4\pi^2} \frac{\chi^{(0)}(t_i)}{M_{pl}^2} \left(\frac{H}{H_i} \right)^{\alpha} \left[-\frac{H_i^6}{H^4} \frac{1-\alpha/2}{6} + \frac{H_i^4}{H^2} \frac{1-\alpha}{4} + H_i^2 \frac{\alpha}{4} - H^2 \frac{1+\alpha}{12} \right] ,$$

which at leading order gives

$$\langle \chi^{(2)} \rangle = -\frac{\alpha}{24\pi^2} \frac{\chi^{(0)}(t_i)}{M_{pl}^2} \left(\frac{H_i}{H} \right)^{4-\alpha} H_i^2 \left(1 - \frac{\alpha}{2} \right) .$$

Two field quadratic model: physical bounds from stochastic approach

Bound on cosmological perturbation theory

The quantum growth of the gauge invariant inflaton fluctuations is given, at first order, by

$$\langle(\phi^{(1)})^2\rangle = \frac{H_0^6 - H^6}{8\pi^2 m^2 H^2} \,,$$

and, to second order, by

$$\langle\phi^{(2)}\rangle = \frac{\dot{\phi}}{16\pi^2 m^2 H M_{pl}^2}\left[\frac{H_0^6 - H^6}{H^2} - 3\left(H_0^4 - H^4\right)\right] \,,$$

Let us study the validity of the perturbative expansion by considering the ratios

$$\frac{\langle\phi^{(2)}\rangle}{\sqrt{\langle(\phi^{(1)})^2\rangle}} \quad , \quad \frac{\sqrt{\langle(\phi^{(1)})^2\rangle}}{\phi^{(0)}}$$

Two field quadratic model: physical bounds from stochastic approach

We then obtain

$$\frac{\langle \delta\phi^{(2)} \rangle}{\sqrt{\langle (\phi^{(1)})^2 \rangle}} = -\frac{1}{4\pi\sqrt{3}}\frac{1}{M_{pl}}\frac{\frac{1}{H^2}\left(H_0^6 - H^6\right) - 3\left(H_0^4 - H^4\right)}{\left(H_0^6 - H^6\right)^{1/2}}$$

$$\frac{\sqrt{\langle (\phi^{(1)})^2 \rangle}}{\phi^{(0)}} = \frac{1}{4\pi\sqrt{3}}\frac{1}{M_{pl}}\frac{1}{H^2}\left(H_0^6 - H^6\right)^{1/2} .$$

The two ratio are the same at the leading order toward the end of inflation.

When the perturbative expansion breaks down? When $\frac{\sqrt{\langle (\phi^{(1)})^2 \rangle}}{\phi^{(0)}} \sim 1$!

To study this ratio we use the variable $\tilde{N}$ defined as the number of e-folds away from the maximum value $N_{max} = N_0 = \log \frac{a_{max}}{a(t_i)} = \frac{3}{2}\frac{H_0^2}{m^2}$

$$N_{max} - \tilde{N} = \log \frac{a(t)}{a(t_i)} \rightarrow \tilde{N} = \frac{3}{2}\frac{H^2}{m^2}$$

Two field quadratic model: physical bounds from stochastic approach

To leading order

$$\frac{\sqrt{\langle(\phi^{(1)})^2\rangle}}{\phi^{(0)}} = -\frac{\sqrt{2}}{12\pi}\frac{m}{M_{pl}}\frac{N_{max}}{\tilde{N}}\left(N_{max}-\tilde{N}\right)^{1/2}$$

On requiring that the absolute value of this ratio be less then one we obtain, under the condition $768\pi^2\frac{M_{pl}^2}{H_0^2} >> 1$, the following approximate constraint

$$\tilde{N} \geq \frac{3\sqrt{3}}{24\pi}\frac{H_0^3}{M_{pl}m^2}\,.$$

Therefore, the perturbative expansion is valid for all the duration of inflation only if

$$H_0 < \left(\frac{12\pi}{\sqrt{3}}M_{pl}m^2\right)^{1/3}\,,$$

namely, for $M_{pl} = 10^5 m$, if

$$H_0 < 129.596m$$

In agreement with previous investigations (Finelli, GM, Vacca, Venturi (2006)).

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Test fields vs inflaton fluctuations

Let us consider three different test scalar fields with a small effective mass and a zero homogeneous expectation value on different inflationary backgrounds driven by an inflaton with potential $V(\phi)$ in the slow-roll approximation and in the UCG.

The stochastic growth of such test scalar field will be described, in this simplified case, starting from the result given before in the particular limit $\chi^{(0)} = 0$.

A. Test scalar field with a constant mass m_χ

The stochastic equation is:

$$\frac{d\langle\chi^{(1)\,2}\rangle}{dt} + \frac{2m_\chi^2}{3H(t)}\langle\chi^{(1)\,2}\rangle = \frac{H^3(t)}{4\pi^2} .$$

Its general solution is

$$\langle\chi^{(1)\,2}\rangle = \left(\int^N dn \frac{H^2(n)}{4\pi^2}\, e^{\int^n \frac{2m_\chi^2}{3H^2(\tilde{n})} d\tilde{n}}\right) e^{-\int^N \frac{2m_\chi^2}{3H^2(n)} dn} ,$$

where we have assumed $\langle\chi^{(1)\,2}\rangle(N_i) = 0$.

Test fields vs inflaton fluctuations

B. Test scalar field with $m_\chi^2 = cH^2$

If $|c| \ll 1$ the stochastic equation is:

$$\frac{d\langle\chi^{(1)\,2}\rangle}{dt} + \frac{2c}{3}H(t)\langle\chi^{(1)\,2}\rangle = \frac{H^3(t)}{4\pi^2} .$$

Its general solution is

$$\langle\chi^{(1)\,2}\rangle = \left(\int^N dn \frac{H^2(n)}{4\pi^2} e^{\frac{2}{3}cn}\right) e^{-\frac{2}{3}cN} ,$$

where we have assumed $\langle\chi^{(1)\,2}\rangle(N_i) = 0$.

Test fields vs inflaton fluctuations

C. Non-minimally coupled test scalar field

The stochastic equation is:

$$\frac{d\langle\chi^{(1)\,2}\rangle}{dt} + 4\xi H(t)(2-\epsilon)\langle\chi^{(1)\,2}\rangle = \frac{H^3(t)}{4\pi^2}\,,$$

where ξ is the non-minimal coupling to the Ricci scalar R and we assume that $|\xi| \ll 1$ (however, ξN may be large).
The term in the action proportional to $\xi\chi^2 R$ gives an effective time dependent mass for χ: $\quad m_\chi^2 = 6\xi H^2(2-\epsilon) \quad$ where $\quad \epsilon = -\frac{\dot{H}}{H^2}$.

Its general solution is

$$\langle\chi^{(1)\,2}\rangle = \left(\int^N dn \frac{H^{2+4\xi}(n)}{4\pi^2 H_i^{4\xi}} e^{8\xi n}\right)\left(\frac{H_i}{H(N)}\right)^{4\xi} e^{-8\xi N}\,,$$

where we have assumed $\langle\chi^{(1)\,2}\rangle(N_i) = 0$.

Test fields vs inflaton fluctuations

The results from these test fields should be compared with the growth of the first order gauge-invariant Sasaki-Mukhanov variable.
In the UCG this can be described as

$$\frac{d\langle\phi^{(1)\,2}\rangle}{dN} + 2\left(\eta - 2\epsilon\right)\langle\phi^{(1)\,2}\rangle = \frac{H^2(t)}{4\pi^2}\,,$$

where

$$\epsilon = \frac{M_{\rm pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 \quad , \quad \eta = M_{\rm pl}^2\frac{V_{\phi\phi}}{V}\,.$$

The general solution is given by

$$\langle\phi^{(1)\,2}\rangle = \frac{\epsilon(N)}{4\pi^2}\int^N dn\frac{H^2(n)}{\epsilon(n)}\,,$$

where we have assumed $\langle\phi^{(1)\,2}\rangle(N_i) = 0$.

Test fields vs inflaton fluctuations

In function of the scalar spectral index n_s and of the tensor-to-scalar ratio r

$$n_s = 1 - 6\epsilon + 2\eta \quad , \quad r = 16\epsilon$$

one obtains

$$\frac{d\langle\phi^{(1)\,2}\rangle_{\rm REN}}{dN} + \left(n_s - 1 + \frac{r}{8}\right)\langle\phi^{(1)\,2}\rangle_{\rm REN} = \frac{H^2(t)}{4\pi^2} \, .$$

Power-law inflation, for which $n_s - 1 = -r/8$ holds, lies at the threshold between two opposite behaviours.

$\langle\phi^{(1)\,2}\rangle$ has the same eq. for a moduli with $m_\chi^2 = cH^2$ and $c = 3(n_s - 1 + r/8)/2$

$$\Downarrow$$

Below the power-law inflation line inflaton fluctuations behave as a moduli with negative c.

Inflationary Zoo

Inflationary zoo

- Case 1: $V(\phi) = \frac{m^2}{2}\phi^2$ Chaotic quadratic inflation.
- Case 2: $V(\phi) = V_0 e^{-\frac{1}{M_{pl}}\sqrt{\frac{2}{p}}\phi}$ Power-law inflation.
- Case 3: $V(\phi) = V_0 - \frac{M^2}{2}\phi^2$ Small field inflation model.
- Case 4: $V(\phi) = V_0 + \frac{M^2}{2}\phi^2$ (approximation for) Hybrid inflation.

Chaotic quadratic inflation

- GI Test Field A: $\langle\chi^{(1)\,2}\rangle = \frac{3H^{2\frac{m_\chi^2}{m^2}}}{8\pi^2(2m^2-m_\chi^2)}\left(H_i^{4-2\frac{m_\chi^2}{m^2}} - H^{4-2\frac{m_\chi^2}{m^2}}\right)$
- GI Test Field B: $\langle\chi^{(1)\,2}\rangle = \frac{m^2}{6\pi^2}\left[\left(1-e^{-\frac{2}{3}cN}\right)\left(\frac{9}{4c^2}+\frac{3}{2c}N_T\right)-\frac{3}{2c}N\right]$,

  where $N_T = \frac{3H_i^2}{2m^2}$.
- GI Test Field C:

$$\langle\chi^{(1)\,2}\rangle \simeq \frac{m^2}{6\pi^2}\frac{e^{8\xi(N_T-N)}}{(N_T-N)^{2\xi}}\Big[(N_T-N)^{2+2\xi}E_{-1-2\xi}\left(8\xi(N_T-N)\right) -(N_T-N_i)^{2+2\xi}E_{-1-2\xi}\left(8\xi(N_T-N_i)\right)\Big].$$

- GI Inflaton fluctuation:

$$\langle\phi^{(1)\,2}\rangle = \frac{H_i^6-H^6}{8\pi^2m^2H^2}.$$

Chaotic quadratic inflation

Chaotic quadratic inflation

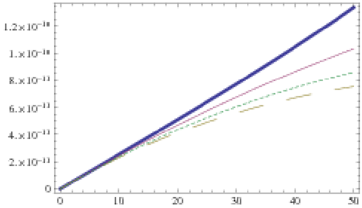

Figure : Evolution of the mean square quantum fluctuations (in units of $m_{\rm pl}^2$) versus the number of e-folds N for the quadratic chaotic model. For the inflationary background we have chosen the inflationary trajectory with $m = 10^{-6}\, m_{\rm pl}$ and $H_i = 10\, m$. The mean square gauge invariant inflaton fluctuation (thick line) dominates over those of gauge invariant test fields ($m_\chi \simeq 0.3m$ is the solid line, $c = 0.02$ is the dashed line, $\xi = 0.001$ is the dotted line).

Power-law inflation model

- GI Test Field A:

$$\langle\chi^{(1)\,2}\rangle = \frac{p}{8\pi^2}H_i^2\exp\left(-\frac{p}{3}\frac{m_\chi^2}{H^2}\right)\left[-\exp\left(\frac{p}{3}\frac{m_\chi^2}{H^2}\right)\frac{H^2}{H_i^2} + \frac{p}{3}\frac{m_\chi^2}{H_i^2}Ei\left(\frac{p}{3}\frac{m_\chi^2}{H^2}\right) + \exp\left(\frac{p}{3}\frac{m_\chi^2}{H_i^2}\right) - \frac{p}{3}\frac{m_\chi^2}{H_i^2}Ei\left(\frac{p}{3}\frac{m_\chi^2}{H_i^2}\right)\right],$$

  where E_i is the exponential integral function.

- GI Test Field B: $\langle\chi^{(1)\,2}\rangle = \frac{p}{8\pi^2}H_i^2\left(c\frac{p}{3}-1\right)^{-1}\left(e^{-2\frac{N}{p}} - e^{-\frac{2}{3}cN}\right)$.

- GI Test Field C:

$$\langle\chi^{(1)\,2}\rangle = \frac{p}{8\pi^2}H_i^2\left(-2\xi - 1 + 4p\xi\right)^{-1}\left(e^{-2\frac{N}{p}} - e^{\xi N\left(\frac{4}{p}-8\right)}\right).$$

- GI Inflaton fluctuation: $\langle\phi^{(1)\,2}\rangle = \frac{p}{8\pi^2}(H_i^2 - H^2)$.

Power-law inflation model

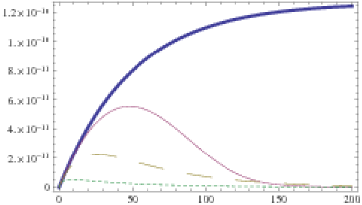

Figure : Evolution of the mean square quantum fluctuations (in units of $m_{\rm pl}^2$) versus the number of e-folds N for the exponential potential. For the inflationary background we have chosen the inflationary trajectory with $p = 100$ and $t_i = 10^7\, m_{\rm pl}^{-1}$. The mean square gauge invariant inflaton fluctuation (thick line) dominates over those of gauge invariant test fields ($m_\chi = 10^{-6}\, m_{\rm pl}$ is the solid line, $c = 0.1$ is the dashed line, $\xi = 0.05$ is the dotted line).

Small field inflation and Hybrid inflation models

- GI Test Field A:

$$\langle \chi^{(1)\,2} \rangle \simeq \frac{3H_0^4}{8\pi^2 m_\chi^2}\left(1 - e^{-\frac{2m_\chi^2}{3H_0^2}N}\right)$$

- GI Test Field B:

$$\langle \chi^{(1)\,2} \rangle = \frac{3H_0^2}{8\pi^2 c}\left(1 - e^{-\frac{2}{3}cN}\right).$$

- GI Test Field C:

$$\langle \chi^{(1)\,2} \rangle = \frac{H_0^2}{32\pi^2 \xi}\left(1 - e^{-8\xi N}\right).$$

- GI Inflaton fluctuation:

$$\langle \phi^{(1)\,2} \rangle \simeq \pm \frac{4V_0^2(1-y) + 3M^4\phi_i^2 y\left(4M_{pl}^2(N-N_i) + \phi_i^2(1-y)\right) \pm y(1-y^2)\frac{M^6}{4V_0}}{96\pi^2 M^2 M_{\rm pl}^4\left(1 \pm y\frac{M^2\phi_i^2}{2V_0}\right)^2},$$

where we have set $y = y(N) = e^{\mp\frac{2M^2M_{\rm pl}^2}{V_0}(N-N_i)}$.

Small field inflation

Small field inflation model

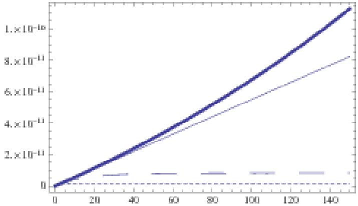

Figure : Evolution of the mean square quantum fluctuations (in units of $m_{\rm pl}^2$) versus the number of e-folds N for the small field inflationary model. For the inflationary background we have chosen $V_0 = 2.6 \times 10^{-12} m_{\rm pl}^4$, $M = 0.85 \times 10^{-6} m_{\rm pl}$ and $\phi_i = 0.3\, m_{\rm pl}$ as parameters. The mean square gauge invariant inflaton fluctuation (thick line) dominates over those of gauge invariant test fields ($m_\chi = 10^{-2} H_0$ is the solid line, $c = 0.1$ is the dashed line, $\xi = 0.05$ is the dotted line).

Hybrid inflation

Hybrid inflation model

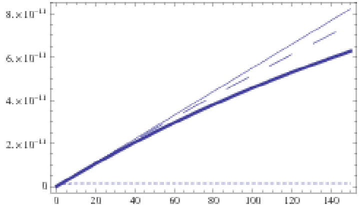

Figure : Evolution of the mean square quantum fluctuations (in units of $m_{\rm pl}^2$) versus the number of e-folds N for the hybrid model. For the inflationary background we have chosen $V_0 = 2.6 \times 10^{-12} m_{\rm pl}^4$, $M = 1.8 \times 10^{-6} m_{\rm pl}$ and $\phi_i = 0.3\, m_{\rm pl}$ as parameters. In this case the mean square of gauge invariant moduli can dominate over the mean square of gauge invariant inflaton fluctuation (thick line): the parameters chosen are $m_\chi = 10^{-2} H_0$ (solid line), $c = 0.002$ (dashed line), $\xi = 0.05$ (dotted line).

Conclusions

- Using the field theory results as a guideline, we have shown that the stochastic equations for the gauge invariant variable associated with any scalar fluctuations are naturally formulated as a flow in terms of the number of e-folds N.
- We have given some interesting bound on the validity of the test field approximation and of the cosmological perturbation theory for quadratic models.
- For most of the inflationary models $\langle Q^{(1)\,2} \rangle$ dominates over $\langle Q_\chi^{(1)\,2} \rangle$, if the moduli has a non-negative effective mass.
- Hybrid inflationary models can be an exception: $\langle Q_\chi^{(1)\,2} \rangle$ can dominate over $\langle Q^{(1)\,2} \rangle$ on choosing parameters appropriately.
- The understanding of inflaton dynamics including metric fluctuations is more important than the moduli problem in most of the inflationary models.

THANKS FOR THE ATTENTION!