Superfluid Hydrodynamics, Thermal Partition Function and Lifshitz Scaling

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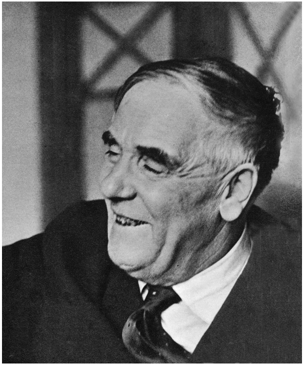
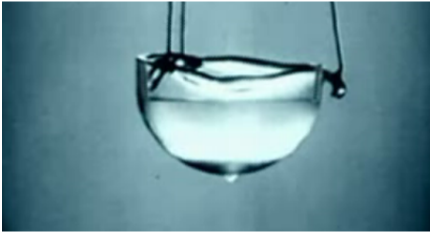
Outline

- Superfluids and Superconductors.
- Relativistic Superfluid Dynamics.
- Chiral Terms in Superfluids.
- Kubo Formulas From Equilibrium Partition Function.
- Lifshitz Scaling Symmetry.
- Quantum Critical Points.
- Experimental Implications.

Superfluidity and Superconductivity

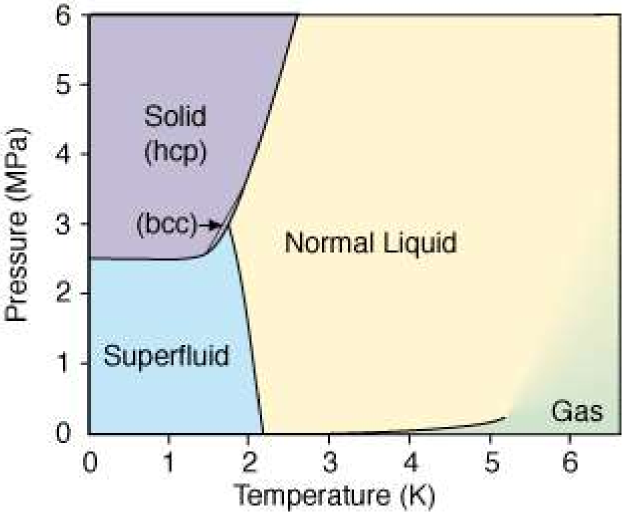
- **Pyotr Leonidovich Kapitsa - 1938**
  1978 - Nobel prize for discovery of superfluidity in ^4He





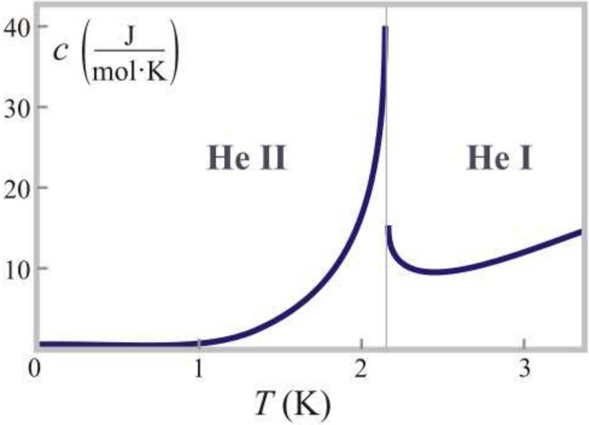
- Liquifies at $T \approx 4.2^\circ\mathrm{K}$
- At $T \approx 2.17^\circ\mathrm{K}$ - Second order phase transition.

- **Phase diagram of ^4He**



- Remains liquid at absolute zero
- Condensate of atoms in ground state - Collective mode

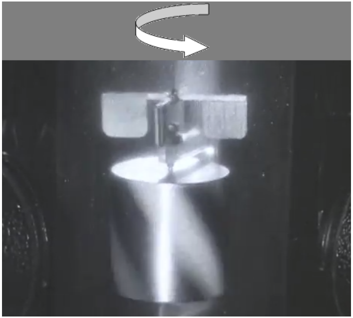
- **The λ point**



- Heat conductivity increase by a factor of 10^6
- Large part of atoms in ground state - Condensate - collective mode

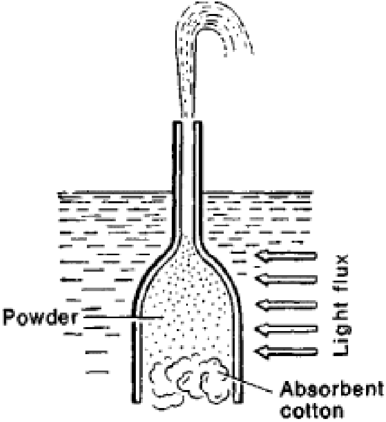
The Two Fluid Picture

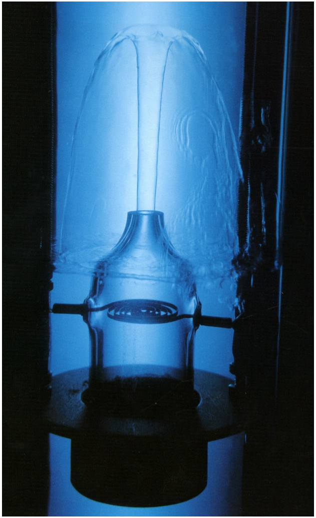
- **Viscosity vs. no viscosity**



- Effective picture - two fluid flows $\vec{v}_n$, $\vec{v}_s$
- Due to Landau

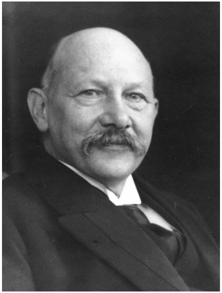
- **Superfluid component carries no entropy**

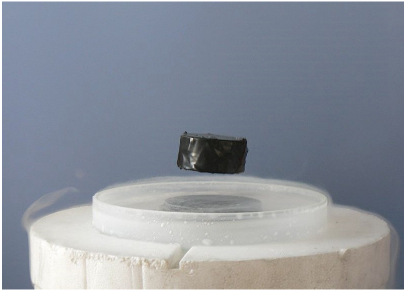




Superconductivity

- **Heike Kamerlingh Onnes - 1911**
  1913 - Nobel prize for liquifying helium





- Zero resistance to DC current.
- Meissner effect (1933).
- Condensation of Cooper pairs.
- Two fluid picture - paired electrons form superfluid.

Relativistic Superfluids

- Spontaneous symmetry breaking
- S-wave superfluid - condensate - complex scalar operator
- Phase of scalar ϕ - Goldstone mode - participates in the hydrodynamics.
- Fluid variables:
  T - temperature,
  μ - chemical potential,
  u^μ - normal fluid 4-velocity,
  $\xi^\mu = -\partial_\mu \phi$ - Goldstone phase gradient,
  $u_s^\mu = -\xi^\mu / \xi$ - Superfluid velocity.
- Superconductors - broken gauge symmetry - $\xi^\mu \equiv -\partial_\mu \phi + \mathcal{A}_\mu$.

Relativistic Superfluids

- New thermodynamic parameters ξ^μ.
- Thermodynamic relations:

$$\varepsilon_n + P = sT + q_n\mu$$
$$dP = s dT + q_n d\mu + \frac{f}{2} d\xi^2$$

- Stress tensor and current:

$$\begin{aligned} T^{\mu\nu} &= \varepsilon_n u^\mu u^\nu + P(\eta^{\mu\nu} + u^\mu u^\nu) + \varepsilon_s u_s^\mu u_s^\nu + \pi^{\mu\nu} \\ J^\mu &= q_n u^\mu + q_s u_s^\mu + j_{diss}^\mu \end{aligned}$$

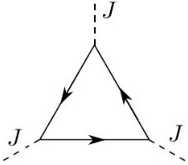
- Josephson relation:

$$u^\mu \xi_\mu = \mu + \mu_{diss}.$$

- Hydrodynamic equations - conservation law:

$$\partial_\mu T^{\mu\nu} = \mathcal{F}^{\nu\mu} J_\mu$$
$$\partial_\mu J^\mu = C E_\mu B^\mu$$
$$\partial_\mu \xi_\nu - \partial_\nu \xi_\mu = \mathcal{F}_{\mu\nu}$$

- Electric field: $E^\mu = \mathcal{F}^{\mu\nu} u_\nu$
- Magnetic field: $B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \mathcal{F}_{\rho\sigma}$
- C - triangular anomaly of three currents.



"Chiral" anomaly

- Goal: constrain expressions for $\pi^{\mu\nu}$, j^μ_{diss}, μ_{diss}.

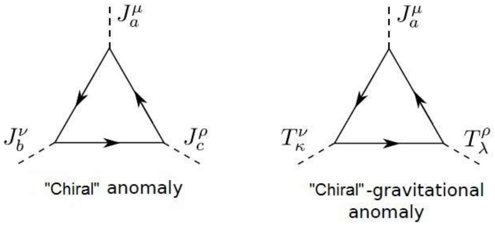
Chiral Effects in Superfluid

- Local second law - Constrain current and conductivities

$$
\begin{aligned}
J^\mu &= q_n u^\mu + q_s u_s^\mu + \sigma E^\mu \\
&\qquad + B^\mu (C\mu + 2Tg_1) + \omega^\mu (C\mu^2 + 4g_1 \mu T - 2g_2 T^2) + \dots \\
\pi^{\mu\nu} &= \eta \sigma_{\mu\nu} + \zeta \partial_\mu u^\mu + \dots
\end{aligned}
$$

- $g_1 = g_1(T, \mu, \xi^2); \qquad g_2 = g_2(T, \mu, \xi^2).$
- Vorticity: $\omega^\mu = \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma$
- Shear tensor: $\sigma_{\mu\nu} = P_\mu^\rho P_\nu^\sigma \partial_{(\rho} u_{\sigma)} - \frac{1}{3} P_{\mu\nu} \partial_\rho u^\rho.$
- Comparison to normal fluid

- C - triangular anomaly of three currents.
- In the normal fluid g_2 integration constant - related to mixed chiral-gravitational anomaly. [Yarom: 1207.5824]



$$\partial_\mu J^\mu \sim \epsilon^{\mu\nu\rho\sigma}\left(\frac{C}{8}F_{\mu\nu}F_{\rho\sigma} + \frac{\beta}{32\pi^2}\mathcal{R}^{\alpha}_{\beta\mu\nu}\mathcal{R}^{\beta}_{\alpha\rho\sigma}\right).$$

- Numerical evidence that at $T \to 0$ one restore the normal fluid values [Amado: 1401.5795]
- Chiral effects - ^3He, Neutron stars.

Equilibrium Partition Function

- Alternative method to derive hydrodynamic current
- Minwalla et al. - 1203.3544, Yarom et al. - 1203.3556
- Consider equilibrated fluid on a curved manifold with non-trivial gauge fields

$$\begin{aligned} ds^2 &= -e^{2\sigma(\vec{x})}\left(dt + a_i(\vec{x})dx^i\right)^2 + g_{ij}(\vec{x})dx^i dx^j, \\ \mathcal{A} &= \mathcal{A}_0(\vec{x})dx^0 + \mathcal{A}_i(\vec{x})dx^i, \end{aligned}$$

- KK invariant gauge field:

$$\begin{aligned} A_0 &\equiv \mathcal{A}_0 + \mu_0 \ , \\ A_i &\equiv \mathcal{A}_i - A_0 a_i \ , \end{aligned}$$

- Local temperature $T(\vec{x}) = T_0 e^{-\sigma}$
- Local chemical potential $\mu(\vec{x}) = A_0 e^{-\sigma}$

Equilibrium Partition Function

- Build the most general equilibrium partition function [effective action]

$$
\begin{aligned}
S &= S_0 + S_1 \ , \\
S_0 &= \int d^3x \frac{1}{T} P(T, \mu, \hat{\zeta}^2) \ , \\
S_1 &= \int d^3x \, \hat{\zeta} \cdot (g_1 \, \partial \times A + T g_2 \, \partial \times a) \\
&\quad + C \int d^3x \, A \cdot \left(\frac{\mu}{3T} \partial \times A + \frac{\mu^2}{6T} \partial \times a \right) + \dots
\end{aligned}
$$

- $\hat{\zeta} \equiv -\partial_i \phi + A_i$, transverse [spatial] part of goldstone field
- Differentiate with respect to the gauge field to obtain the current
- Advantage - algebraic rather then differential
- Disadvantage - only captures equilibrium properties

Linear Response Theory

- Relates transport coefficients to retarded correlation function of stress tensors and currents in terms of **Kubo formulas**
- Allow for a microscopic calculation e.g. Feynmann diagrams
- Deriving Kubo formulas - normally requires to solve the the equations of motion for a particular source of perturbation
- Alternative shorter algebraic method - from variations of the equilibrium partition function
- Reproduces known Kubo formulas for various fluid cases
- New Kubo formulas for superfluids

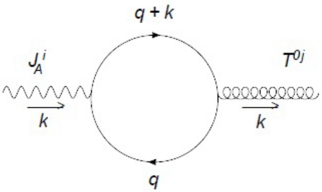
Results

- Kubo formulas -

$$g_1(T,\mu,\zeta^2) = -\lim_{k_n\to 0}\sum_{ij}\frac{i}{4Tk_n}\epsilon_{ijn}\langle J^i(k_n)J^j(-k_n)\rangle_{\substack{\omega=0\\ k\parallel\zeta}} - \frac{C}{2}\left(\frac{\mu}{T}\right)\ ,$$

$$g_2(T,\mu,\zeta^2) = \lim_{k_n\to 0}\sum_{ij}\frac{i}{2T^2k_n}\epsilon_{ijn}\left[\langle J^iT^{0j}\rangle - \mu\langle J^iJ^j\rangle\right]\Bigg|_{\substack{\omega=0\\ k\parallel\zeta}} - \frac{C}{2}\left(\frac{\mu}{T}\right)^2$$

- Spatial superfluid velocity - new thermal parameter - ζ^i.
- Similar role to chemical potential in the spatial direction.
- Substitution rules in propagators - $q^\mu \to (i\omega_n+\mu, \vec{q}+\vec{\zeta})$.



- Holographic calculation also possible.

Lifshitz Superfluids - Quantum Critical Points

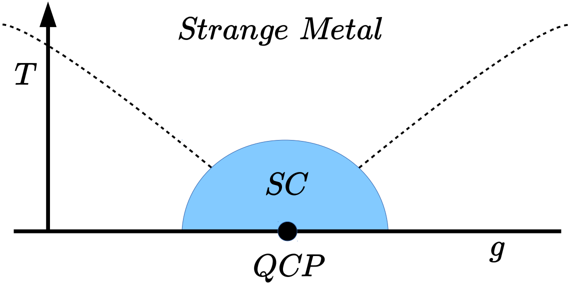
- Anisotropic Weyl - Lifshitz scaling symmetry:

$$t \to \Omega^z t$$
$$x^i \to \Omega x^i$$

  z - dynamical critical exponent
- Must be accompanied by broken boost invariance
- Phase transition at zero temperature
  - Driven by quantum fluctuations
  - Quantum tuning parameter [B, doping, pressure]
- First and Second order transition
- Infinite correlation length - scale invariance
- Hydrodynamic regime - $l_c \gg L \gg l_T$

Quantum Criticality

- Influence of quantum critical point felt way above $T = 0$.



- Example: anti-ferromagnetic $\rightarrow$ heavy fermion metal transition.
- Strange metal behavior $\rho \sim T$ ($\sim T^2$ in normal metals)
- Characteristic of high T_c superconductors in the non-superconducting regime.

Hydrodynamics with broken boost invariance

- Under lorentz transformations

$$\delta \mathcal{L} = T_{\mu\nu}\omega^{\mu\nu}$$

  $\omega^{\mu\nu}$ antisymmetric parameter of Lorentz transformation

- Asymmetric stress tensor in time direction.
- Assumption - fluid can be described using former variables.
- No need of external time vector [phonons]
- Fluid velocity in the local rest frame points in the time direction
- Antisymmetric part of stress tensor:

$$T^{[\mu\nu]} = u^{[\mu}V_A^{\nu]}$$

Constitutive relations

- Stress-tensor:

$$T^{\mu\nu} = (\varepsilon_n + p)u^\mu u^\nu + p\eta^{\mu\nu} + \varepsilon_s u_s^\mu u_s^\nu + \pi^{(\mu\nu)} + \pi_A^{[\mu\nu]} \ .$$

- Choice of frame - removing a redundancy by shift of thermal variables:

$$u^\mu \to u^\mu + \delta u^\mu; \quad T \to T + \delta T; \quad \mu \to \mu + \delta\mu \ .$$

- Clark Putterman frame - no current corrections,

$$j^\mu_{diss} = 0$$

$$\pi^{\mu\nu} u_\mu u_\nu = 0$$

- Decompose:

$$\pi^{(\mu\nu)} = (Q^\mu u^\nu + Q^\nu u^\mu) + \Pi P^{\mu\nu} + \Pi_t^{\mu\nu} \ ,$$

where

$$Q^\mu u_\mu = 0, \quad \Pi_t^{\mu\nu} u_\nu = 0, \quad \Pi_t^{\mu\nu} P_{\mu\nu} = 0 \ .$$

Q^μ represent the heat flow.

Entropy Increase

- Entropy current:

$$J_s^\mu = su^\mu - \frac{u_\nu}{T}\pi^{\mu\nu} + \frac{f}{T}\mu_{diss}\zeta^\mu \ .$$

- Entropy production rate

$$\partial_\mu J_s^\mu = -\frac{[\Pi(\partial_\mu u^\mu) + \Pi_t^{\mu\nu}\sigma_{\mu\nu}]}{T} - \frac{Q_\mu}{T}\left[a^\mu + P^{\mu\nu}\frac{\partial_\nu T}{T}\right] + \mu_{diss}P^{\mu\nu}\partial_\mu\left(\frac{f\zeta_\nu}{T}\right) - \frac{V_{A\,\mu}}{2T}\left[a^\mu - P^{\mu\nu}\frac{\partial_\nu T}{T}\right] \ ,$$

- Has to be positive sum of quadratic forms.
- Constraint dissipative corrections: $\Pi_t^{\mu\nu}, \Pi, Q^\mu, V_{A\,\mu}, \mu_{diss}$
- New vector - acceleration $a^\mu \equiv u^\nu\partial_\nu u^\mu$. Two projections - in the direction and in the transverse direction to the superfluid velocity

- Number of transport terms in a superfluid

	T − preserving	T − breaking
non − Lifshitz	14	7
Lifshitz	22	13

- More detailed results in the NR limit
- Only included parity preserving effects

The non-relativistic limit

- Fluid variables:
  - ρ_n, ρ_s - mass densities,
  - $\vec{v}_n, \vec{v}_s$ - velocities,
  - $\vec{\omega} = \vec{v}_s - \vec{v}_n$ - counterflow
  - projector $P_w^{ij} = \delta^{ij} - \frac{w^i w^j}{w^2}$.
- expansion in powers of c:
  - Expand thermal parameters:
    - $u^\mu = (1, \frac{\vec{v}_n}{c})$
    - $\xi^\mu = -c(1, \frac{\vec{v}_s}{c})$
    - $\mu_{rel} = c + \frac{1}{c}(\mu + \omega^2/2)$
    - $\epsilon_n = \rho_n c^2 + U_n - \rho_n \frac{v_n^2}{2}$
  - Expand constitutive relations

$$\pi^{\mu\nu} = \sum_n \frac{1}{c^n} \pi^{\mu\nu}_{(n)}$$

- equations of motion:
  - mass conservation:

$$\partial_t(\rho_n + \rho_s) + \partial_i(\rho_n v_n^i + \rho_s v_s^i) = 0$$

  - Navier-Stokes:

$$\partial_t(\rho_n v_n^i + \rho_s v_s^i) + \partial_k(\rho_n v_n^i v_n^k + \rho_s v_s^i v_s^k) + \partial^i p + \nu^i = 0$$

  - Energy conservation:

$$\partial_t E + \partial_i[Q^i + Q'^i] + \nu_e = 0$$

- Dissipative corrections:

$$
\begin{aligned}
Q'^i &\sim \pi^{i0}_{(1)} = \\
&= \omega^i (Q_1 \omega^j \partial_j T + Q_2 \omega^j D_t v_{n\,j}) + Q_3 P^{ij}_\omega \partial_j T + Q_4 P^{ij}_\omega D_t v_{n\,j} + \dots \\
\nu^i &= \partial_t \pi^{0i}_{(-1)} + \partial_k \pi^{(ki)}_{T\,(0)} + \dots \\
\pi^{ij}_{T\,(0)} &= -\eta \sigma^{ij} - \zeta \partial_k v^k \delta_{ij} - t_1 \partial_k (\rho_s \omega^k) \\
&\quad - w^i w^j (t_2 \omega^i \partial_i T + t_3 \omega^i D_t v_i) + \dots \\
\pi^{0i}_{(-1)} &= -\omega^i (A_1 \omega^j \partial_j T + A_2 \omega^j D_t v_j) - A_3 P^{ij}_\omega \partial_j T - A_4 P^{ij}_\omega D_t v_j + \dots
\end{aligned}
$$

- scaling of transport coefficients

$$
\sim T^{\frac{\Delta}{z}} F\left(\frac{\mu}{T^{\frac{2(z-1)}{z}}}, \frac{w^2}{T^{\frac{2(z-1)}{z}}} \right) \, .
$$

$$
[A_1] = d + 2 - 3z; \quad [A_2] = d - 4(z-1); \quad [A_3] = d - z; \quad [A_4] = d - 2(z-1)
$$
$$
[Q_1] = d - z; \quad [Q_2] = d - 2(z-1); \quad [Q_3] = z + d - 2; \quad [Q_4] = d
$$

Experimental Implications

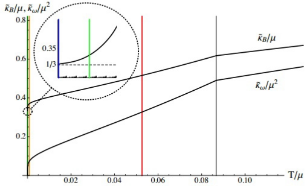
- Part of heat flow proportional to acceleration
- Can be related to chemical potential:

$$D_t v_n^i \simeq \partial^i \left(\mu + \frac{w^2}{2} \right) + \dots$$

- Anisotropy between the direction of the counterflow and the transverse direction.
- Hard to disentangle from effect of shear viscosity
- In superconductors alternating current create phase between normal and super components
- Perhaps some unusual frequency dependence would reveal the effect in superconductors

Outlook

- evaluate g_1 and g_2 using the Kubo formulas
  - Weakly coupled field theory
  - holographic model
  - Perhaps explain temperature dependence of Amado [1401.5795]:



- Lifshitz Superfluid
  - Suggest a measurement in superconductors.
  - Include parity violating effects.
  - Construct holographic model


Thank you