Constraints from CFT three point functions.

Heraklion-Crete, 24-04-2014

work in progress with

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Introduction

Why interested in Conformal Field Theories (CFTs) ?

- CFTs serve as endpoints of renormalization group flows.
- Interesting low energy dynamics for several real world physical systems.

  e.g. *condensed matter, quark gluon plasma,*
  physics beyond the standard model, ...

- They are in principle more accessible to study.

  Non-perturbative techniques available:

  e.g. *conformal bootstrap techniques, AdS/CFT,...*

Introduction

Correlation functions of spin-ℓ primary operators with conformal dimension Δ are highly constrained:

$$\left\langle \mathcal{O}^{a_1\cdots a_\ell}(x_1)\mathcal{O}^{b_1\cdots b_\ell}(x_2)\right\rangle = C_{(\ell,\Delta)}\frac{I^{a_1b_1}(x_{12})\cdots I^{a_\ell b_\ell}(x_{12})}{x_{12}^{2\Delta}}$$

$$I^{ab}(x) = \eta^{ab} - 2\frac{x^a x^b}{x^2}$$

Unitarity implies $C_{(\ell,\Delta)} \geq 0$.

Choose a basis such that:

$C_{(\ell,\Delta)} = 1$ **and** $\langle \mathcal{O}(x_1)\mathcal{O}'(x_2)\rangle = 0$ **for** $\mathcal{O} \neq \mathcal{O}'$.

Introduction

Further constraints from the two-point functions of *descendants* **:**

$$\Delta \geq \frac{d}{2} - 1, \qquad l = 0$$
$$\Delta \geq l + d - 2, \qquad l \geq 1$$

Example:

A scalar field $\Phi(x)$ of dimension Δ in $d = 4$.

$$\langle \Phi(x)\Phi(0)\rangle = \frac{1}{x^{2\Delta}} \geq 0$$

Consider the two point function of the descendant $\partial_x^2\Phi(x)$.

$$0 \leq \partial_x^2\partial_x^2 \langle \Phi(x)\Phi(0)\rangle \sim C\left(\Delta^2 - 1\right)\frac{1}{x^{2\Delta+4}} \Rightarrow \Delta > 1$$

[Ferrara, Gatto, Grillo][Mack]

Introduction

Constraints from two-point functions.

What about three point functions?

A conformal field theory is characterized by:

- Its spectrum.

  A set of primary operators $\mathcal{O}^{(\ell,\Delta)}$ with conformal dimensions Δ.

- Three point functions of these operators, which are fixed by conformal incariance up to a few constant parameters.

  e.g.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}'(x_3)\rangle = \frac{\lambda_{\mathcal{O}\mathcal{O}\mathcal{O}'}}{x_{12}^{2\Delta-\Delta'}x_{23}^{\Delta}x_{31}^{\Delta}}, \qquad \Delta = [\mathcal{O}],\ \Delta' = [\mathcal{O}']$$

Introduction

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = C_T \frac{\mathcal{I}_{\mu\nu,\rho\sigma}(x)}{x^{2d}}$$

$$\langle T_{\mu\nu}(x_3)T_{\rho\sigma}(x_2)T_{\tau\kappa}(x_1)\rangle = \frac{\mathcal{A}\mathcal{J}_{\mu\nu\rho\sigma\tau\kappa}(x) + \mathcal{B}\mathcal{K}_{\mu\nu\rho\sigma\tau\kappa}(x) + \mathcal{C}\mathcal{M}_{\mu\nu\rho\sigma\tau\kappa}(x)}{x_{12}^d x_{13}^d x_{23}^d}$$

- **Ward Identites relate C_T with $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$**

$$C_T = \frac{(d-1)(d+2)\mathcal{A} - 2\mathcal{B} - 4(d+1)\mathcal{C}}{d(d+2)}$$

The coefficients of the conformal anomaly on a curved manifold c, a in four spatial dimensions, are directly related to $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ via $\mathrm{c} \propto C_T$ and $\mathrm{a} \propto \frac{13\mathcal{A}-2\mathcal{B}-40\mathcal{C}}{8}$.

Introduction

Non-perturbative approach widely used in recent years;

The Conformal Bootstrap:

Unitarity + Crossing Symmetry of four-point functions.

Example:

The four point function of scalar operators with the ope

$$\left\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \right\rangle =$$
$$= \sum_{\mathcal{O},\mathcal{O}'} \lambda_{\mathcal{O}} \lambda'_{\mathcal{O}} C_I(x_{12}, \partial_2) C_J(x_{34}, \partial_4) \left\langle \mathcal{O}^I(x_2) \mathcal{O}'^J(x_4) \right\rangle$$
$$= \sum \lambda_{\mathcal{O}}^2 \frac{g_{\ell,\Delta}(u,v)}{x_{12}^{2d} x_{34}^{2d}}$$

Introduction

- *The conformal block*:

$$g_{\ell,\Delta}(u,v) \equiv x_{12}^{2d} x_{34}^{2d} C_I(x_{12},\partial_2) C_J(x_{34},\partial_4) \left\langle \mathcal{O}^I(x_2) \mathcal{O}'^J(x_4) \right\rangle$$

- *The conformal cross ratios*

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

Symmetry under the exchange $x_1 \leftrightarrow x_3$ leads to the *crossing relation*:

$$\sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 g_{\ell,\Delta}(u,v) = \left(\frac{u}{v}\right)^d \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 g_{\ell,\Delta}(u,v)$$

Similar relations from the ope in different channels.

Introduction

The conformal bootstrap approach consists in solving these equations. It is a *powerful* but *numerical* technique.

- It is useful to gather all possible analytic results.

Constraints can be obtained analytically for a special class of three-point functions:

$$\left\langle \mathcal{O}^{(\ell)} T_{\mu\nu} \mathcal{O}^{(\ell)} \right\rangle$$

They are obtained from the requirement of positivity of the energy flux

$$\langle \mathcal{E}(\widehat{n}) \rangle \geq 0$$

Outline

- Energy flux operator; review.
- Non-conserved operators;

  Example: vector operator and constraints.
- Connection to Deep Inelastic Scattering (DIS):

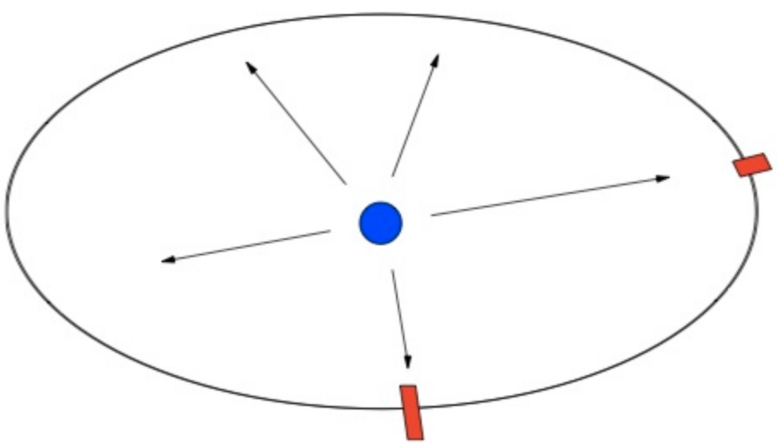
  Some puzzles.
- Conclusions and Open Questions

Energy Flux Operator

Definition: The energy flux operator $\mathcal{E}(\widehat{n})$ per unit angle measured through a very large sphere of radius r is

$$\mathcal{E}(\widehat{n}) = \lim_{r\to\infty} r^{d-2} \int dt\, \widehat{n}^i\, T^0_i(t, r\widehat{n}^i)$$

n^i is a unit vector specifying the position on S^{d-2} where energy measurements may take place. Integrating over all angles yields the total energy flux at large distances.



Energy Flux Operator

Consider the normalized energy flux one-point function

$$\frac{\langle \Psi | \mathcal{E}(\widehat{n}) | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

There are several possibilities for the states $|\Psi\rangle$:

- Conserved currents

$$|\Psi\rangle = \int d^d x e^{-iqx} J_\mu \epsilon^\mu \, |0\rangle$$

$$|\Psi\rangle = \int d^d x e^{-iqx} T_{\mu\nu} \epsilon^{\mu\nu} \, |0\rangle$$

- Generic primary

$$|\Psi\rangle = \int d^d x e^{-iqx} \mathcal{O}_{a_1 \cdots a_l} \epsilon^{a_1 \cdots a_l} \, |0\rangle$$

Energy Flux Operator

- Rotational symmetry fixes the form of the energy flux one–point function up to two independent parameters.

$$\langle \mathcal{E}(\hat{n}) \rangle_{T_{ij}} = \frac{\langle \epsilon^*_{ik} T_{ik} \mathcal{E}(\hat{n}) \epsilon_{lj} T_{lj} \rangle}{\langle \epsilon^*_{ik} T_{ik} \epsilon_{lj} T_{lj} \rangle} =$$

$$= \frac{E}{\Omega_{d-2}} \left[1 + t_2 \left(\frac{\epsilon^*_{il} \epsilon_{lj} n_i n_j}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{1}{d-1} \right) + t_4 \left(\frac{|\epsilon_{ij} n_i n_j|^2}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{2}{d^2-1} \right) \right]$$

Here t_2, t_4 are arbitrary constants. By construction, they can be related to $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

[Hofman, Maldacena]

Energy Flux Operator

Demand positivity of the energy flux one point function, *i.e.*, $\langle \mathcal{E}(\hat{n}) \rangle \geq 0$.

The positivity of the energy flux imposes constraints on t_2, t_4:

$$C_G(\mathcal{A},\mathcal{B},\mathcal{C}) \equiv 1 - \frac{1}{d-1} t_2 - \frac{2}{d^2-1} t_4 \geq 0$$

$$C_V(\mathcal{A},\mathcal{B},\mathcal{C}) \equiv 1 - \frac{1}{d-1} t_2 - \frac{2}{d^2-1} t_4 + \frac{t_2}{2} \geq 0$$

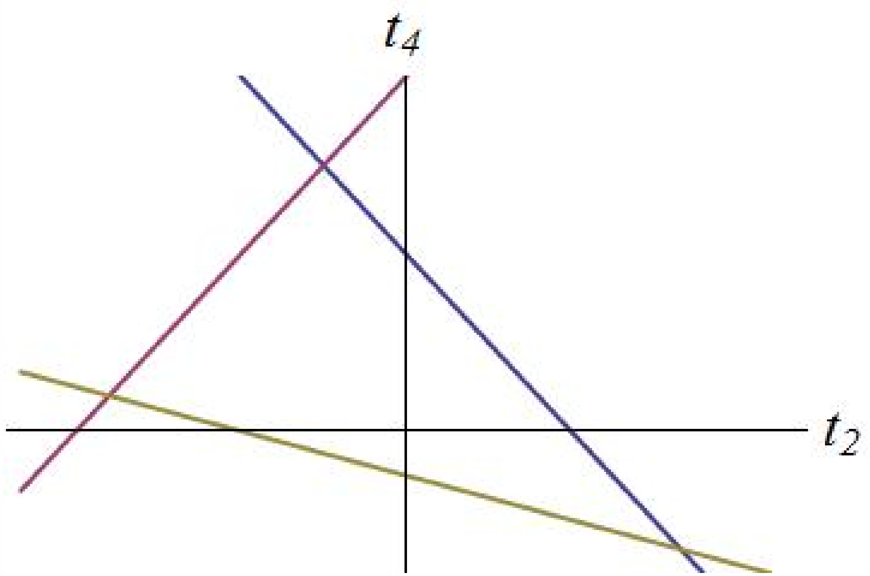
$$C_S(\mathcal{A},\mathcal{B},\mathcal{C}) \equiv 1 - \frac{1}{d-1} t_2 - \frac{2}{d^2-1} t_4 + \frac{d-2}{d-1}(t_2 + t_4) \geq 0$$

Constraints are saturated by free theories of bosons, fermions and $\left(\frac{d}{2}-1\right)$-form fields in even dimensions.

[Hofman, Maldacena][Zhiboedov]

Bounds on t_2, t_4.

Parameter space t_2, t_4 of a consistent CFT. Values outside the triangle are forbidden.



Example: CFT $d = 4$ dimensions

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$$

Bounds for generic operators

Natural generalization; non-conserved currents.

- ope coefficients depend on marginal couplings - non-perturbative results.

- Ising model;

  Indications it contains an infinite number of *almost* conserved currents.

  [Komagordski, Zhiboedov]

  [Fitzpatrick, Kaplan, Poland, Simmons-Duffin]

The simplest case to consider: vector operator.

Bounds for a vector operator

Generic form of the energy flux for vector operators

$$\langle \mathcal{E} \rangle = \frac{q^0}{\Omega_{d-2}} \left[1 + a_2 \left(\frac{|\vec{\epsilon}.\hat{n}|^2}{\vec{\epsilon}^2 + g(\Delta)(\epsilon^0)^2} - \frac{1}{d-1} \frac{\vec{\epsilon}^2}{\vec{\epsilon}^2 + g(\Delta)(\epsilon^0)^2} \right) + \right.$$

$$\left. + a_4 \frac{\epsilon^0 (\vec{\epsilon}.\hat{n}) + c.c.}{\vec{\epsilon}^2 + g(\Delta)(\epsilon^0)^2} \right]$$

- **The denominator is fixed by the two point function:**

$$\langle (\epsilon^*.\mathcal{O}(-q)) (\mathcal{O}(q).\epsilon) \rangle \propto \left[(\epsilon^*.\epsilon) - 2 \frac{\Delta - \frac{d}{2}}{\Delta - 1} \frac{(\epsilon^*.q)(\epsilon.q)}{q^2} \right] (q^2)^{\Delta - \frac{d}{2}}$$

- $g(\Delta) = \frac{\Delta + 1 - d}{\Delta - 1} \geq 0$, **saturated at the unitarity bound.**
- **Additional parameter due to non-conservation,** a_4.

Bounds for a vector operator

Polarization choices:

- $\vec{\epsilon}.\hat{n} = 0$

$$a_2 \leq d - 1$$

Constraint identical to the case of conserved current.

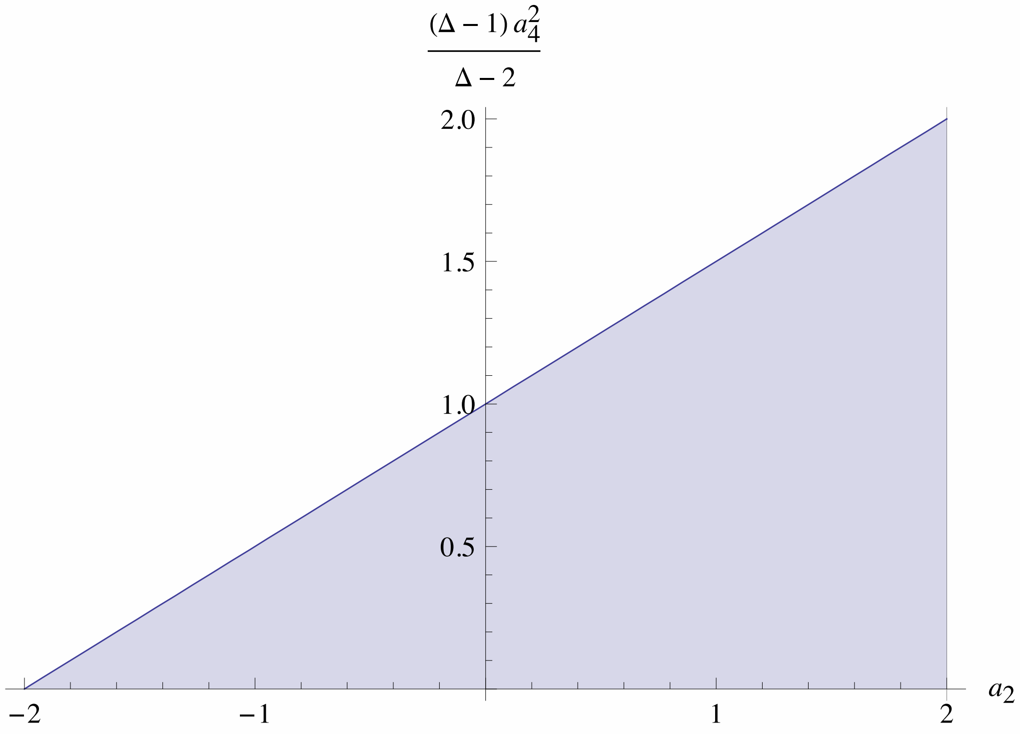
- $\vec{\epsilon}.\hat{n} \neq 0$.

Set $\hat{n} = (0, 0, 1)$ **and** $\epsilon^\mu = (\epsilon^0, 0, 0, 1)$. **Choose** ϵ^0 **which minimizes the energy flux** $\Rightarrow$ $\epsilon^0 = f(a_2, a_4)$**.**

$$\frac{a_4^2}{g(\Delta)} \leq 1 + \frac{d-2}{d-1} a_2$$

Bounds for a vector operator

Bounds on a_2, a_4 from the positivity of the energy flux.



Which theory corresponds to the cusp?

Bounds for generic operators

- What about other operators?
  - For spin $s \geq 2$ the number of structures in the three point function increases linearly with spin:

$$\#\ of\ possible\ structures = 3s$$

[Zhiboedov]

  - Ward Identities lead to $3s - 1$ structures.
  - The number of independent parameters in the energy flux expression grows at least like s^2.

    1-1 correspondence with the parameters in the three point function is lost.

    e.g.: For $s = 2$ there are *five* independent parameters in the ope but *seven* in the energy flux.

[work in progress]

Positivity of the energy flux

The positivity of the energy flux is a reasonable assumption - is there a proof?

- Proof known for free theories.
- Remarkable evidence from holography:

Bounds realized earlier holographically.

The arena: Black holes in Lovelock gravity, a special class of higher derivative theories, $a \neq c$.

The principle: Causality of the retarded propagator at finite temperature T in the limit of large momenta, *i.e.*, $\omega,\ |q| >> T$.

[Brigante, Liu, Myers, Shenker, Yaida]

[Myers, Buchel] [Hofman]

Positivity of the energy flux

The energy flux positivity constraints are related to causality in the gravity language. Can we see something similar in field theory?

Guide from the AdS/CFT analysis:

- Consider the Fourier transform of the two–point function of the stress energy tensor at finite temperature.
- Focus on large momenta, small temperatures $\frac{k}{T} \gg 1$.
- Three independent polarizations; each polarization yields a different set of constraints.

Positivity of the energy flux

- **How do we compute the two–point function of the stress-energy tensor in an arbitrary CFT at finite temperature?**

In the regime of small temperatures use the OPE

$$T_{\mu\nu}(x)T_{\rho\sigma}(0) \sim \frac{\mathcal{I}_{\mu\nu,\rho\sigma}}{x^{2d}} + \cdots + \mathcal{A}_{\mu\nu\rho\sigma\kappa\tau}(x)T^{\kappa\tau}(0) + \cdots$$

Several operators appear in the ope.

Focus on $\mathcal{A}_{\mu\nu\rho\sigma\kappa\tau}(x)$ which is related to the three point function of the stress energy tensor.

Explicit expression for $\mathcal{A}_{\mu\nu\rho\sigma\kappa\tau}(x)$ exists [Osborn, Petkou].

Positivity of the energy flux

- **Take expectation value and Fourier transform**

$$\mathcal{G}_{\mu\nu,\rho\sigma}(\omega,\, q) = \int d^4x\, \langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle\, e^{i\omega t - iqx_3}$$

Note: $\langle T_{00}\rangle = 3\langle T_{ii}\rangle \equiv 3\tilde{c}T^4$

- **What about the three independent polarizations?**

Generic form of thermal correlator

$$\mathcal{G}_{\mu\nu,\rho\sigma}(\omega, q) = S_{\mu\nu,\rho\sigma}\mathbf{G_V}(\omega, \mathbf{q}) + Q_{\mu\nu,\rho\sigma}\mathbf{G_S}(\omega, \mathbf{q}) + L_{\mu\nu,\rho\sigma}\mathbf{G_T}(\omega, \mathbf{q})$$

$S_{\mu\nu,\rho\sigma}$, $Q_{\mu\nu,\rho\sigma}$, $L_{\mu\nu,\rho\sigma}$ **are completely fixed by symmetry**

[Kapusta, Kovtun, Starinets].

Positivity of the energy flux

Example:

Transverse polarization when $\omega,\, q >> |k| >> T$.

$$\mathbf{G_T}(\omega, q) = \cdots + C_G(\mathcal{A},\mathcal{B},\mathcal{C})\,\frac{\omega^2}{k^2}\,\tilde{c}T^4 + \cdots$$

Note: **The ope coefficient precisely matches the quantity the energy flux constrains:**

$$C_G(\mathcal{A},\mathcal{B},\mathcal{C}) = \left(1 - \frac{t_2}{3} - \frac{t_4}{15}\right)$$

The same is true for the other two polarizations.

Can we find a separate argument for positivity?

Positivity of the energy flux

Consider the spectral density $\rho(\omega, q) \equiv -\mathrm{Im} G_T \geq 0$.

Focus on the regime ω, $q >> |k| >> T$.

Does the stress energy tensor give the leading contribution?

What about other operators in the OPE?

- Higher spin operators: subleading for ω, $q >> |k| >> T$.
- Relevant operators: unitarity $\Rightarrow$ *scalars, vectors*.
  - *Vectors*

    Odd-spin operators do not contribute in the ope of two identical operators.

[Costa, Penedones, Poland, Rychkov].

Positivity of the energy flux

- **Focus on scalars.**
  - *Scalar Operators* $\langle O \rangle \sim T^{\Delta}$
    Contribution to the OPE proportional to $T^{\Delta}(k^2)^{2-\frac{\Delta}{2}}$
    Note: **Non-singular when** $\Delta \leq 4$**.**

Consider $\rho(\omega, q) \equiv -\mathrm{Im} G_T$ **in the regime** $\omega,\ q >> |k| >> T$**.**

The dominant contribution comes from $T_{\mu\nu}$

$$\rho(\omega, q) \simeq C_G\, \omega^2 \delta\left(k^2\right) \tilde{c} T^4 \geq 0$$

which implies

$$C_G \geq 0$$

[Parnachev, MK.]

Positivity of the energy flux

BUT...

$k^2 = 0$ **outside the validity of the OPE regime!**

Yet, the Hofman-Maldacena constraints are constraints on the OPE coefficients.

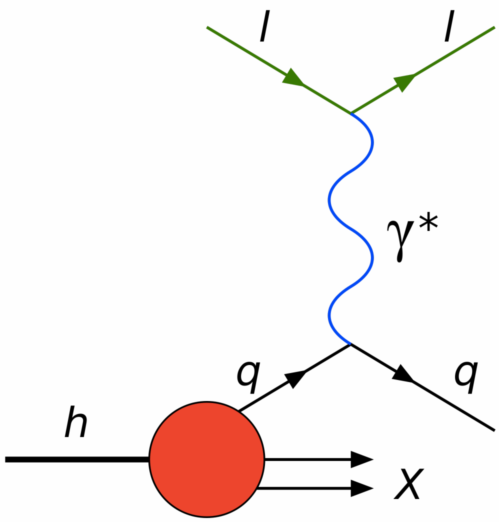
CAN WE FIX THIS?

There is a standard way to obtain results from the ope in the physical region.

Deep inelastic scattering (DIS) sum rules.

Deep Inelastic Scattering

Consider a CFT perturbed by a relevant operator which flows to a gapped phase. Bombard the state $|h\rangle$, lightest in the theory, with virtual gravitons γ^*.



The optical theorem relates the cross section to the imaginary part of the forward scattering amplitude $\mathrm{Im}\mathcal{A}$.

Deep Inelastic Scattering

The amplitude in our setup is

$$\mathcal{A} = \int d^d x \, e^{-ikx} \langle P | \epsilon^{*\mu\nu} T_{\mu\nu}(x) \epsilon^{\rho\sigma} T_{\rho\sigma}(0) | P \rangle$$

where P_μ, k_μ are the momenta of h and γ^*.

The amplitude depends on two kinematical invariants

$$k^2, \qquad x \equiv \frac{k^2}{2k.P}$$

For h the lightest particle, $\mathcal{A}(k^2, x)$ has a branch cut in the complex x–plane for $-1 \leq x \leq 1$.

Deep Inelastic Scattering

For spacelike $k^2 \to \infty$ we evaluate $\mathcal{A}(k^2, x)$ using the ope.

Example:

Transverse polarization $\epsilon_{\mu\nu} P^\mu = \epsilon_{\mu\nu} k^\mu = 0$

$$\mathcal{A} \simeq (\epsilon^* . \epsilon) \sum_{s=0,2,\cdots} C_s \tilde{c}_s x^{-s} k^{d-\tau_s}$$

with $C_s, \tilde{c}_s$ are related to the ope coefficient and the expectation value of spin-**s** operator in the ope.

The leading contribution for each spin $s \neq 0$ comes from the smallest *twist* τ_s operator. These are the conserved currents with $\tau_s = d - 2$.

Deep Inelastic Scattering

In the limit of large momentum

$$\oint dx\; x\mathcal{A}(k^2, x) \simeq C_G\, \tilde{c}\, k^{d-2}(\epsilon^*.\epsilon)$$

Note: **Positivity of the energy of** $|h\rangle$ **implies that** $\tilde{c} \geq 0$**.**

With the standard contour trick we can relate the ope region to the physical region

$$\oint dx\; x\mathcal{A}(k^2, x) = \int_0^1 dx x\, \mathrm{Im}\mathcal{A}(k^2, x) \geq 0$$

which leads to the energy-flux constraint $C_G \geq 0$**.**

Deep Inelastic Scattering

- What about the other polarizations/constraints?

Recall energy flux constraints - three distinct cases.

$$(\epsilon^*_{\mu\nu}\epsilon^{\mu\nu}), \quad (\epsilon^*_{\mu\nu}\epsilon^{\nu}_{\rho}\widehat{\mathbf{n}}^{\rho}\widehat{\mathbf{n}}^{\mu}), \quad (\epsilon^*_{\mu\nu}\widehat{\mathbf{n}}^{\mu}\widehat{\mathbf{n}}^{\nu})(\epsilon_{\mu\nu}\widehat{\mathbf{n}}^{\mu}\widehat{\mathbf{n}}^{\nu})$$

But the expectation value only supplies two momenta

$$\langle P|T_{\mu\nu}|P\rangle \propto P_\mu P_\nu$$

to contract with $\epsilon^{\mu\nu}$.

- Important to allow for $\epsilon_{\mu\nu}k^\mu \neq 0$.

Deep Inelastic Scattering

Take the Fourier transform of the ope coefficient

$$T_{\mu\nu}(-k)T_{\rho\sigma}(k) \sim \cdots + \mathcal{A}_{\mu\nu\rho\sigma\kappa\lambda}T^{\kappa\lambda} + \cdots$$

Impose Ward Identities

$$k^{\mu}\mathcal{A}_{\mu\nu\rho\sigma\alpha\beta} = k_{\nu}\mathcal{E}_{\rho\sigma,\alpha\beta} + k_{\rho}\mathcal{E}_{\nu\sigma,\alpha\beta} + k_{\sigma}\mathcal{E}_{\rho\nu,\alpha\beta}$$

$\mathcal{E}_{\rho\sigma,\alpha\beta}$ **is the projector onto traceless tensors**

$$\mathcal{E}_{\rho\sigma\alpha\beta} \equiv \frac{1}{2}(\eta_{\rho\alpha}\eta_{\sigma\beta} + \eta_{\rho\beta}\eta_{\sigma\alpha}) - \frac{1}{4}\eta_{\rho\sigma}\eta_{\alpha\beta}$$

It turns out that structures in the ope coefficient which depend on the momentum are necessary to satisfy the Ward Identity. These structures will be absent had we imposed $k^{\mu}\mathcal{A}_{\mu\nu\rho\sigma\alpha\beta} = 0$.

! DIS and Hofman-Maldacena constraints agree !

Deep Inelastic Scattering

More puzzles...

Consider operators other than conserved currents.

Example: Scalar operator $\mathcal{O}$ of dimension $\Delta_{\mathcal{O}}$.

In the large momentum limit

$$0 \leq \int_0^1 dx\, x\, \mathrm{Im}\mathcal{A}(k^2, x) \propto \tilde{c}\, \Gamma[d + 1 - \Delta_{\mathcal{O}}]\, (k^2)^{-d+\Delta_{\mathcal{O}}}$$

But $\Gamma[d + 1 - \Delta_{\mathcal{O}}]$ has alternating sign for $d + 1 \leq \Delta_{\mathcal{O}}$.

Similar results for other operators.

- Correspondence with the energy flux goes through for special values of $\Delta_{\mathcal{O}}$.

Deep Inelastic Scattering

- What goes wrong?

Basic assumption:

The amplitude is bounded polynomially

$$\lim_{x \to 0} \mathcal{A}(k^2, x) \leq x^{-N+1}$$

The contour manipulation which leads to the positivity of the ope coefficient can be trusted only for operators of spin $s \geq N$. For $s = 2$ we need $N \leq 2$.

Is the energy flux telling us that this is not a justified assumption ? − possibly in combination with $\Delta_{\mathcal{O}}$.

Open Questions

- Understand what DIS is telling us.

  Can we use this as an estimate of polynomial boundedness of the amplitude?

- Obtain more constraints.
- Identify constraints holographically.
- Combine with conformal bootstrap techniques for even more stringent constraints.
- Identify the theory which saturates the bounds, if it exists.