New Skins for an Old Ceremony

The Conformal Bootstrap and the Ising Model

Sheer El-Showk
École Polytechnique & CEA Saclay

Based on:

arXiv:1203.6064 with M. Paulos, D. Poland, S. Rychkov,
D. Simmons-Duffin, A. Vichi

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May 16, 2013

Crete Center for Theoretical Physics

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Motivation & Approach

Why return to the bootstrap?

1. Conformal symmetry very powerful tool that *goes largely unused* in $D > 2$.
2. Completely non-perturbative tool to study field theories
   - Does not require SUSY, large N, or weak coupling.
3. In $D = 2$ conformal symmetry enhanced to *Virasoro* symmetry
   - Allows us to *completely solve* some CFTs ($c < 1$).
4. Long term hope: generalize this to $D > 2$?

Approach

- Use only "global" conformal group, valid in all D.
- Our previous result:
  - Constrained "landscape of CFTs" in $D = 2, 3$ using conformal bootstrap.
  - Certain CFTs (e.g. Ising model) sit at *boundary* of solution space.
- New result: "solve" spectrum & OPE of CFTs (in any D) on boundary.
  - Check against the $D = 2$ Ising model.
- The Future: Apply this to $D = 3$ Ising model?

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Outline

1. Motivation
2. The Ising Model
3. CFT Refresher
4. The Bootstrap & the *Extremal Functional Method*
5. Results: the 2d Ising model
6. The (Near) Future
7. Conclusions/Comments

The Ising model

The Ising Model

Basic Definition

- Lattice theory with nearest neighbor interactions

$$H = -J \sum_{<i,j>} s_i s_j$$

with $s_i = \pm 1$ (this is $O(N)$ model with $N = 1$).

Relevance

- Historical: 2d Ising model solved exactly. [Onsager, 1944].
- Relation to $\mathcal{E}$-expansion.
- "Simplest" CFT (universality class)
- Describes:
  1. Ferromagnetism
  2. Liquid-vapour transition
  3. ...

Continuum Limit

- To study fixed point can take continuum limit (and $\sigma(x) \in \mathbb{R}$)

$$H = \int d^D x \left[(\nabla \sigma(x))^2 + t\, \sigma(x)^2 + a\, \sigma(x)^4 \right]$$

- In $D < 4$ coefficient a is relevant and theory *flows* to a fixed point.

$\mathcal{E}$-expansion

Wilson-Fisher set $D = 4 - \mathcal{E}$ and study critical point perturbatively.

Setting $\mathcal{E} = 1$ can compute anomolous dimensions in $D = 3$:

$$[\sigma] = 0.5 \to 0.518 \ldots$$

$$[\epsilon] := [\sigma^2] = 1 \to 1.41 \ldots$$

$$[\epsilon'] := [\sigma^4] = 2 \to 3.8 \ldots$$

New Perspective

At fixed point conformal symmetry emerges:

- Strongly constrains data of theory.
- Can we use symmetry to fix e.g. $[\sigma], [\epsilon], [\epsilon'], \ldots$?
- Can we also fix interactions this way?

CFT Refresher

Conformal Symmetry in $D > 2$

Primary Operators

Conformal symmetry:

$$\underbrace{SO(1, D-1) \times \mathbb{R}^{1,D-1}}_{\textit{Poincare}} \quad + D \quad \text{(Dilatations)} + K_\mu \quad \text{(Special conformal)}$$

Highest weight representation built on primary operators $\mathcal{O}$:

$$\text{Primary operators:} \qquad K_\mu\, \mathcal{O}(0) = 0$$

$$\text{Descendents:} \qquad P_{\mu_1} \ldots P_{\mu_n}\, \mathcal{O}(0)$$

All dynamics of *descendants* fixed by those of primaries.

Clarifications vs 2D

- Primaries $\mathcal{O}$ called *quasi-primaries* in $D = 2$.
- Descendents are with respect to "small" conformal group: $L_0, L_{\pm 1}$.
- Example: Viraso descendents $L_{-2}\mathcal{O}$ are *primaries* in our language.

Spectrum and OPE

CFT Background

CFT defined by specifying:

- Spectrum $\mathcal{S} = \{\mathcal{O}_i\}$ of primary operators dimensions, spins: (Δ_i, l_i)
- Operator Product Expansion (OPE)

$$\mathcal{O}_i(x) \cdot \mathcal{O}_j(0) \sim \sum_k C_{ij}^k D(x, \partial_x) \mathcal{O}_k(0)$$

$\mathcal{O}_i$ are primaries. Diff operator $D(x, \partial_x)$ encodes *descendent* contributions.

This data fixes all correlators in the CFT:

- 2-pt & 3-pt fixed:

$$\langle \mathcal{O}_i \mathcal{O}_j \rangle = \frac{\delta_{ij}}{x^{2\Delta_i}}, \qquad \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle \sim C_{ijk}$$

- Higher pt functions contain no new dynamical info:

$$\Big\langle \underbrace{\underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}_{\sum_k C_{12}^k D(x_{12},\partial_{x_2})\mathcal{O}_k(x_2)} \quad \underbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)}_{\sum_l C_{34}^l D(x_{34},\partial_{x_4})(x_3)\mathcal{O}_l(x_4)}}_{\sum_{k,l} C_{12}^k C_{34}^l D(x_{12},x_{34},\partial_{x_2},\partial_{x_4})\langle \mathcal{O}_k(x_2)\mathcal{O}_l(x_4)\rangle} \Big\rangle$$

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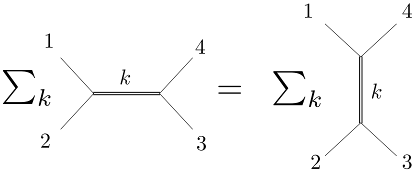
Crossing Symmetry

CFT Background

This procedure is not unique: $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$



Consistency requires equivalence of two different contractions

$$\sum_k C^k_{12} C^k_{34}\, G^{12;34}_{\Delta_k, l_k}(u,v) = \sum_k C^k_{14} C^k_{23}\, G^{14;23}_{\Delta_k, l_k}(u,v)$$

Functions $G^{ab;cd}_{\Delta_k, l_k}$ are *conformal blocks* (of “small” conformal group):

- Each G_{Δ_k, l_k} corresponds to one operator $\mathcal{O}_k$ in OPE.
- Entirely *kinematical*: all dynamical information is in C^k_{ij}.
- u, v are independent conformal cross-ratios: $u = \frac{x_{12}x_{34}}{x_{13}x_{24}}$, $v = \frac{x_{14}x_{23}}{x_{13}x_{24}}$
- Crossing symmetry give non-perturbative constraints on (Δ_k, C^k_{ij}).

CFT Background Recap

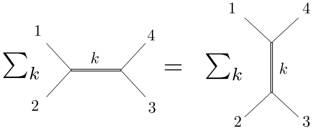
What have we learned so far:

1. CFTs completely specified by primary operator spectrum and OPE.

$$\{\Delta_i, l_i\},\ \{C_{ijk}\}\ \text{for all } \mathcal{O}_i$$

   This data allows us to compute all correlators.

2. Constrained by crossing symmetry



3. Crossing symmetry equations is *sum over primary operators* $\mathcal{O}_k$:

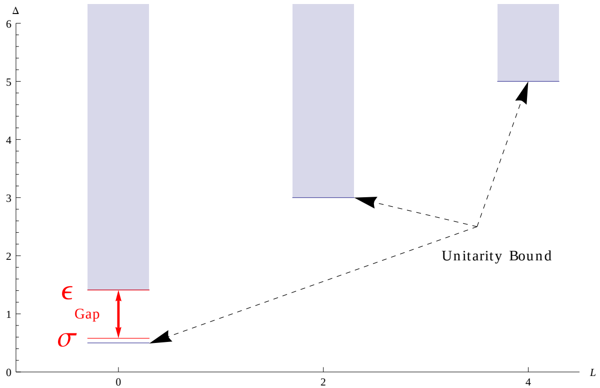
$$\sum_{\mathcal{O}_k} C_{12}^k C_{34}^k\, G_{\Delta_k,l_k}^{12;34}(u,v) = \sum_{\mathcal{O}_k} C_{14}^k C_{23}^k\, G_{\Delta_k,l_k}^{14;23}(u,v)$$

4. $G_{\Delta_k,l_k}(u,v)$ encode contribution of primary $\mathcal{O}_k$ and its decendents.

How Strong is Crossing Symmetry?

Constraining the spectrum

Figure : A Putative Spectrum in $D = 3$



- Unitarity implies:

$$\Delta \geq \frac{D-2}{2} \quad (l = 0),$$
$$\Delta \geq l + D - 2 \quad (l \geq 0)$$

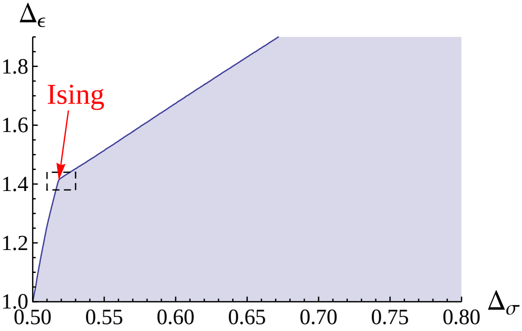
- "Carve" landscape of CFTs by imposing gap in scalar sector.
- Fix lightest scalar: σ.
- Vary next scalar: ϵ.
- Spectrum otherwise *unconstrained*: allow any other operators.

Constraining Spectrum using Crossing Symmetry

Is crossing symmetry consistent with a gap?

σ four-point function: $\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle$

Crossing symmetric values of σ-ϵ



Blue = solution may exists.

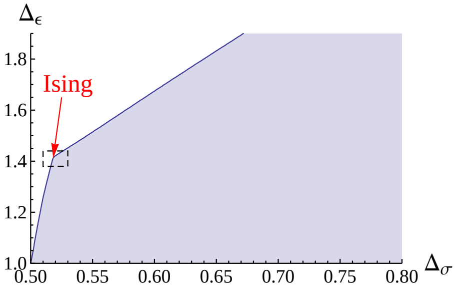
White = No solution exists.

- Certain values of σ, ϵ inconsistent with crossing symmetry.
- Solutions to crossing:
  1. white region $\Rightarrow$ 0 solutions.
  2. blue region $\Rightarrow$ ∞ solutions.
  3. boundary $\Rightarrow$ 1 solution (unique)!
- Ising model special in two ways:
  1. On boundary of allowed region.
  2. At kink in boundary curve.

Solving CFTs on the boundary via Crossing

Summarize Our Approach

Use the uniqueness of the boundary solution to compute OPE & spectrum of a putative CFTs at any point on the boundary.



Implementing Crossing Symmetry

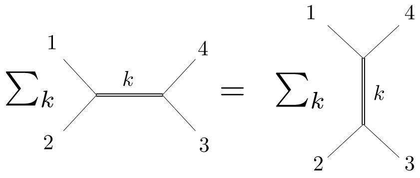
Crossing Symmetry Nuts and Bolts

So how do we enforce crossing symmetry in practice?

Consider four *identical* scalars: $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ $\quad \dim(\phi) = \Delta_\phi$

Recall crossing symmetry constraint:

$$\sum_{\mathcal{O}_k}(C^k_{\phi\phi})^2\, G^{12;34}_{\Delta_k,l_k}(x) = \sum_{\mathcal{O}_k}(C^k_{\phi\phi})^2\, G^{14;23}_{\Delta_k,l_k}(x)$$



Crossing Symmetry Nuts and Bolts

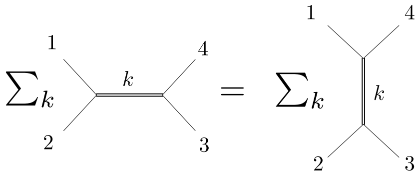
Bootstrap

So how do we enforce crossing symmetry in practice?

Consider four *identical* scalars: $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \qquad \dim(\phi) = \Delta_\phi$

Move everything to LHS:

$$\sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2 \, G^{12;34}_{\Delta_k,l_k}(x) - \sum_{\mathcal{O}_k} (C^k_{\phi\phi})^2 \, G^{14;23}_{\Delta_k,l_k}(x) = 0$$



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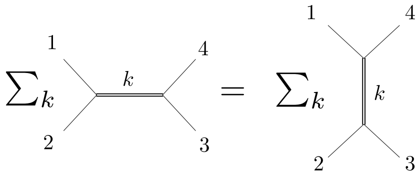
Express as sum of functions with positive coefficients:

$$\sum_{\mathcal{O}_k} \underbrace{(C^k_{\phi\phi})^2}_{p_k} \underbrace{[G^{12;34}_{\Delta_k,l_k}(x) - G^{14;23}_{\Delta_k,l_l}(x)]}_{F_k(x)} = 0$$



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Functions $F_k(x)$ are formally infinite dimensional vectors.

$$p_1 \underbrace{(F_1, F_1', F_1'', \dots)}_{\vec{v}_1} + p_2 \underbrace{(F_2, F_2', F_2'', \dots)}_{\vec{v}_2} + p_3 \underbrace{(F_3, F_3', F_3'', \dots)}_{\vec{v}_3} + \cdots = \vec{0}$$

1. Each vector $\vec{v}_k$ represents the contribution of an operator $\mathcal{O}_k$.
2. If $\{\vec{v}_1, \vec{v}_2, \dots\}$ span a positive cone there is no solution.
3. Efficient numerical methods to check if vectors $\vec{v}_k$ span a cone.
4. When cone “unfolds” solution is unique!

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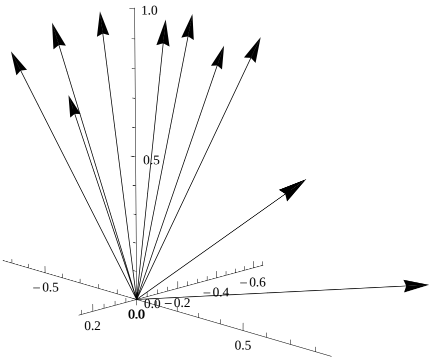
Cones in Derivative Space

$\boxed{\text{Vectors form cone} \Rightarrow \text{no solution.}}$

No Solutions to Crossing



1. Axes are derivatives:

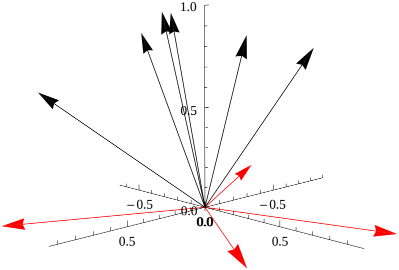
$$F'_{\Delta,l}(x), F''_{\Delta,l}(x), F'''_{\Delta,l}(x)$$

2. Vectors represents operators
3. All operators lie *inside* half-space.
4. $\vec{0}$ not in positive cone.

Cones in Derivative Space

Cone "unfolds" giving unique solution.

Unique Solution to Crossing



1. Axes are derivatives:

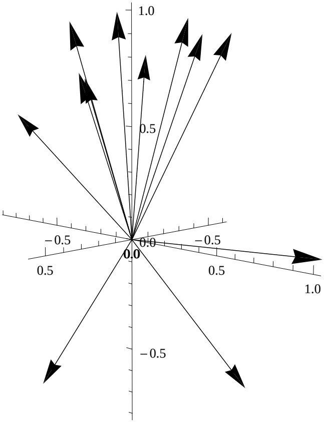
$$F'_{\Delta,l}(x), F''_{\Delta,l}(x), F'''_{\Delta,l}(x)$$

2. Vectors represents operators
3. Boundary of cone (red) spans a plane.
4. $\vec{0}$ in span of red vectors.

Cones in Derivative Space

As more operators added solutions no longer unique.

Many Solutions to Crossing



1. Axes are derivatives:

$$F'_{\Delta,l}(x), F''_{\Delta,l}(x), F'''_{\Delta,l}(x)$$

2. Vectors represents operators
3. Vectors span full space.
4. Many ways to form $\vec{0}$.

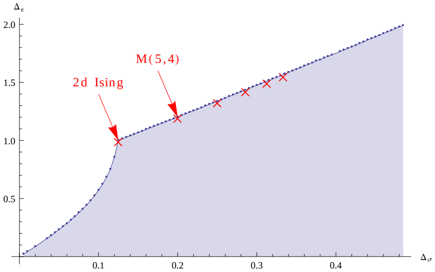
Spectrum and OPE from crossing?

Checking the extremal functional method

How Powerful is Crossing Symmetry?

To check our technique lets apply to 2d Ising model.

- Same plot in 2d.
- Completely solvable theory.
  - Using Virasoro symmetry can compute full spectrum & OPE.



Can we reproduce using crossing symmetry & only “global” conformal group?

Crossing Symmetry vs. Exact Results

Exact (Virasoro) results compared to unique solution at “kink” on boundary:

Spin 0

L	Bootstrap Δ	Virasoro Δ	Δ Error (in %)	Bootstrap OPE	Virasoro OPE	OPE Error (in %)
0	1.	1	0.0000106812	0.500001	0.5	0.000140121
0	4.00145	4	0.03625	0.0156159	0.015625	0.0582036
0	8.035	8	0.4375	0.00019183	0.000219727	12.6962
0	12.175	12	1.45833	3.99524×10^{-6}	6.81196×10^{-6}	41.3496

Mileage from Crossing Symmetry?

- 12 OPE coefficients to $< 1\%$ error.
- Spectrum better:
  1. In 2d Ising expect operators at $L, L+1, L+4$.
  2. We find this structure up to $L = 20$
     $\sim$ 38 operator dimensions $< 1\%$ error!

What about 3d Ising Model?

Current "State-of-the-Art"

3d Ising model

Using $\mathcal{E}$-expansion, Monte Carlo and other techniques find partial spectrum:

Field:	σ	ϵ	ϵ'	$T_{\mu\nu}$	$C_{\mu\nu\rho\lambda}$
Dim (Δ):	0.5182(3)	1.413(1)	3.84(4)	3	5.0208(12)
Spin (l):	0	0	0	2	4

Only 5 operators and no OPE coefficients known for 3d Ising!

Lots of room for improvement!

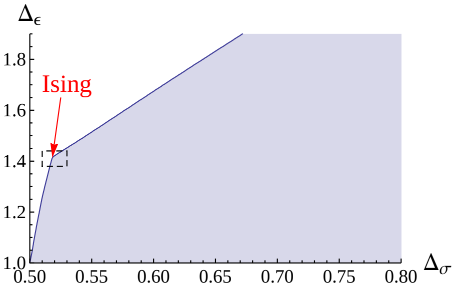
Our Goal

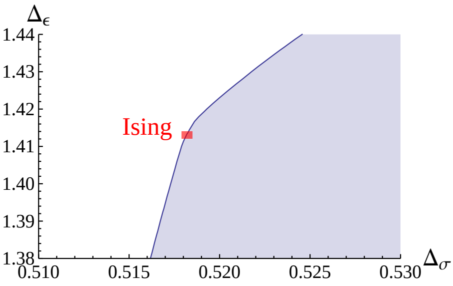
Compute these anomolous dimensions (and many more) and OPE coefficients using the bootstrap applied along the boundary curve.

Spectrum of the 3d Ising Model

Computing 3d Spectrum from Boundary Functional?

A first problem: what point on the boundary? what is correct value of σ?





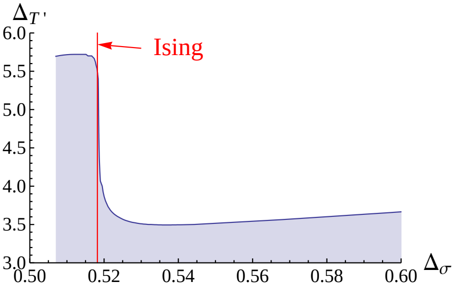
1. In $D = 2$ we know σ by other means.
2. "Kink" is not so sharp when we zoom in.
3. Gets sharper as we increase number of constraints
   $\Rightarrow$ should taylor expand to higher order!

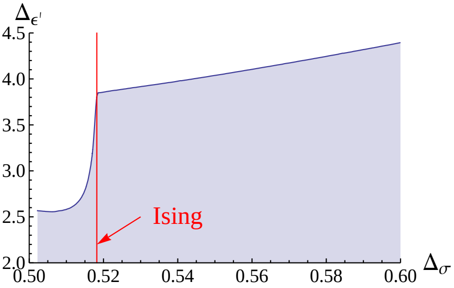
Origin of the Kink

Re-arrangment of spectrum?

Spectrum near the kink undergoes rapid re-arrangement.

Plots for next Scalar and Spin 2 Field





1. "Kink" in (ϵ, σ) plot due to rapid rearrangement of *higher dim spectrum*.
2. Important to determine σ to high precision.
3. Does this hint at some analytic structure we can use?

The Future

What's left to do?

Honing in on the Ising model?

- Fix dimension of σ in 3d Ising using "kink" or other features.
- Use boundary functional to compute spectrum, OPE for 3d Ising.
- Compare with lattice or experiment!
- Additional constraints: add another correlator $\langle \sigma\sigma\epsilon\epsilon \rangle$.
- Study spectrum, OPE as a function of spacetime dimension.

Exploring the technology

- How specific is this structure to Ising model?
- Can we impose more constraints and find new "kinks" for other CFTs?
- Can any CFT be "solved" by imposing a few constraints (gaps) and then solving crossing symmetry?
- What about SCFTs? Need to know structure of supersymmetric conformal blocks.

The Future

What's left to do?

More Questions/Thoughts

- Technology still begin refined $\Rightarrow$ lots to do!
- Why is Ising model on boundary? Why at a "kink"?
- Do these features have physical meanings or artifacts of method?
- Only just begun to take advantage of conformal symmetry in $D > 2$.

AdS/CFT Applications

- Generalized Free Field CFTs are dual to free ($N \sim \infty$) fields in AdS [Heemskerk et al, SE and Papadodimas]
- Higher spin GFFs are "multi-particle states" in bulk:

$$\mathcal{O} \sim \phi \partial_{\{\mu_1} \ldots \partial_{\mu_n\}} \phi$$

with $\Delta_{\mathcal{O}} = n + 2\Delta_\phi$ and $\Delta_\phi > \frac{D-2}{2}$.

- Tentative result: Bound on gap for *any* spins is *saturated* by GFFs.
- If true then: leading $1/N^2$ always negative!

The Future

What's left to do?

More Questions/Thoughts

- Technology still begin refined $\Rightarrow$ lots to do!
- Why is Ising model on boundary? Why at a "kink"?
- Do these features have physical meanings or artifacts of method?
- Only just begun to take advantage of conformal symmetry in $D > 2$.

AdS/CFT Applications

- Generalized Free Field CFTs are dual to free ($N \sim \infty$) fields in AdS [Heemskerk et al, SE and Papadodimas]
- Higher spin GFFs are "multi-particle states" in bulk:

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Thanks