Wavy line Wilson loops T-dual to short strings

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- Many known relations between closed and open strings:
  * in flat space propagation of a closed string may be viewed as describing also propagation of open string in periodic time;
  * an open string disc diagram for open strings may be viewed as a closed string emission by a D-brane into vacuum;
  * KLT relations express closed string scattering amplitudes in terms of open string amplitudes, etc.

- gauge-string duality: one example is relation between the coefficient of leading log S term in the large spin asymptotics of closed folded spinning string energy and the cusp anomalous dimension of the Wilson loop defined by light-like cusp

Also: KLT-like relation between correlator of closed string vertex operators at null separations and square of Wilson loop defined by corresponding null polygon

[Alday,Eden,Korchemsky,Maldacena, Sokatchev]

These are far-from-BPS configurations but there are other examples hint towards a possible closed-open string relation also for "short" strings or "small" deviations from BPS limit

straight line WL with a small cusp

[Drukker et al, Drukker,Forini ; Correa,Henn,Maldacena,Sever 2011]

$$W(C) = \frac{1}{N}\,\mathrm{tr}\,\mathrm{P}\exp\left[\oint_C d\tau\,\left(iA_\mu(x)\dot{x}^\mu + \Phi_i(x)\theta^i|\dot{x}|\right)\right]$$

Open string surface ending on straight line at the boundary [Maldacena 98]

$$x_0 = \tau \ , \qquad z = \sigma$$

BPS solution; Euclidean surface equivalent to circular Wilson loop surface, up to "conformal" anomaly

$$\langle W(\text{straight line})\rangle = 1$$

$$\langle W(\text{circle})\rangle = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right)$$

[Erickson,Semenoff,Zarembo 2000]

non-BPS generalisations? Minkowski signature case?

WL for straight line with a small cusp

[Drukker et al; Correa,Henn,Maldacena,Sever 2011]

$$\langle W_{\text{cusp}} \rangle = \exp\Big[-\Gamma_{\text{cusp}}(\phi,\lambda) \ln \Lambda + ...\Big] \,,$$

$$\Gamma_{\text{cusp}}(\phi,\lambda)_{\phi\to 0} = -B(\lambda)\,\phi^2 + \dots \,,$$

$$B(\lambda) = \frac{\sqrt{\lambda}}{4\pi^2}\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} = \frac{1}{4\pi^2}\Big(\sqrt{\lambda} - \frac{3}{2} + \frac{3}{8\sqrt{\lambda}} + ...\Big)$$

striking similarity between small angle coefficient B and slope function found by Basso: dimension of the sl(2) sector gauge-theory operator with spin S and twist J or energy of dual spinning string has the following expansion in small S limit [Basso 2011, Gromov 2012]

$$E^2 = J^2 + h(\lambda, J)\,\mathrm{S} + O(\mathrm{S}^2)\ ,$$

$$h = 2J + \bar{h}(\lambda, J)\ , \qquad \bar{h}(\sqrt{\lambda}, J) = 2\sqrt{\lambda}\,\frac{I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})}$$

For J=1 formal relation to B-coefficient:

$$h(\lambda, 1) = 2 + 8\pi^2 B(\lambda) = 2\sqrt{\lambda} - 1 + O(\frac{1}{\sqrt{\lambda}})\ .$$

Possible relation between small (nearly-point-like) closed strings in AdS and long open strings ending on the boundary along nearly-straight line ?

Another interpretation of $B(\lambda)$ as energy emitted by a slowly moving quark

[Correa et al; Fiol, Garolera, Lewkowycz, 2012]

$$E = 2\pi B(\lambda) \int d\tau \; \dot{v}^2 + \dots \,, \qquad \vec{v} = \dot{\vec{x}}(\tau) \ll 1$$

$\vec{x}(\tau)$ is a spatial deviation of the quark's trajectory from a straight line

he *Minkowski* open-string world surface ending on the quark's trajectory at the boundary
Poincare patch leads to the following expression for the classical ($\sqrt{\lambda} \gg 1$) string energy

$$E_0 = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \, \frac{\dot{v}^2 - [\vec{v} \times \dot{\vec{v}}]^2}{(1 - v^2)^3} \approx \frac{\sqrt{\lambda}}{2\pi} \int d\tau \; \dot{v}^2 + \dots$$

A. Mikhailov 03

wavy line WL [A. Mikhailov 2003]

energy radiated by moving charge = string end-point

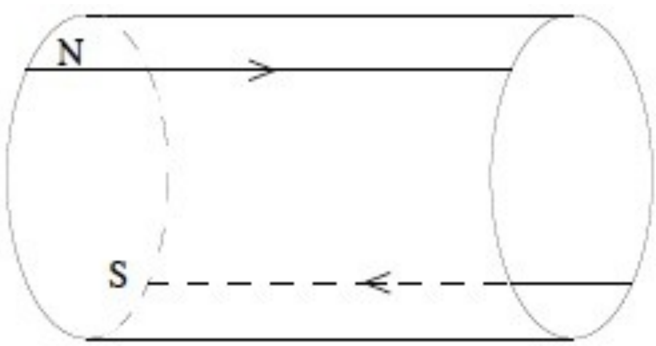


Figure 1: Timelike Wilson line.

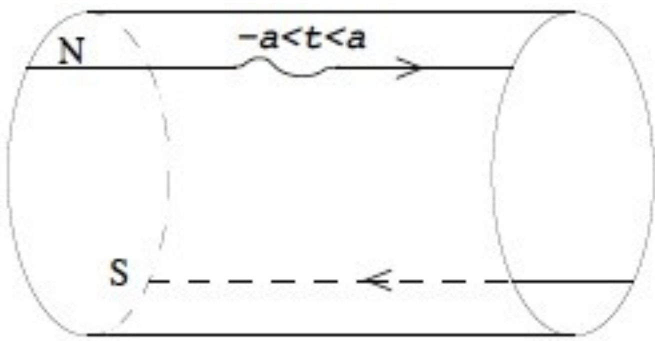


Figure 2: Moving one of the sources.

$$\Delta E = A \int_{-\infty}^{\infty} \frac{\ddot{\vec{x}}^{\,2} - [\dot{\vec{x}} \times \ddot{\vec{x}}]^2}{(1 - \dot{\vec{x}}^{\,2})^3} dt,$$

$$A = \frac{2}{3} e^2$$

Lienard formula

in Maxwell theory

Remarkably, same functional expression at weak coupling (in Maxwell theory) and at strong coupling (for classical string in AdS)

Energy of classical open string solution in AdS ending on a wavy line at the boundary:

$$E = \frac{\sqrt{\lambda}}{2\pi} \int dt \frac{\ddot{\vec{x}}^{\,2} - [\dot{\vec{x}} \times \ddot{\vec{x}}]^2}{(1 - \dot{\vec{x}}^{\,2})^3}$$

string tension as coefficient

$x_i(\tau)$ is transverse profile of boundary trajectory

Consider small transverse velocity: nearly-straight trajectory

$$ds^2 = z^{-2}(-dx_0^2 + dx_i^2 + dz^2)$$

static gauge

$$x_0 = \tau, \qquad z = \sigma \ ,$$

string (Nambu) action for small fluctuations in x_i-directions is

$$S = \frac{\sqrt{\lambda}}{2\pi} \int \frac{d\tau d\sigma}{\sigma^2} \Big[1 + \frac{1}{2}(\dot{x}_i^2 - x_i'^2) + O(\partial x^3)\Big]$$

solution uniquely determined by shape of the boundary curve and 3rd normal derivative

$$
\begin{aligned}
x_i(\tau,\sigma) &= \mathrm{x}_i(\tau,\sigma) - \sigma\partial_\sigma \mathrm{x}_i(\tau,\sigma) \\
&= \mathrm{x}_i^+(\tau+\sigma) + \mathrm{x}_i^-(\tau-\sigma) - \sigma\Big[\dot{\mathrm{x}}_i^+(\tau+\sigma) - \dot{\mathrm{x}}_i^-(\tau-\sigma)\Big] + O(\mathrm{x}^2)\ , \\
x_i(\tau,0) &\equiv \mathrm{x}_i(\tau) = \mathrm{x}_i^+(\tau) + \mathrm{x}_i^-(\tau)\ , \qquad \partial_\sigma^3 x_i(\tau,0) = -2\partial_\tau^3[\mathrm{x}_i^+(\tau) - \mathrm{x}_i^-(\tau)]
\end{aligned}
$$

$\mathrm{x}_i(\tau,\sigma) = \mathrm{x}_i^+(\tau+\sigma) + \mathrm{x}_i^-(\tau-\sigma)$ is generic flat-space solution (harmonic function)

string energy associated with translations in the $x_0 = \tau$ direction

$$
E_0 = \frac{\sqrt{\lambda}}{2\pi}\int_0^\infty \frac{d\sigma}{\sigma^2}\Big[1 + \frac{1}{2}(\dot{x}_i^2 + x_i'^2) + O(\partial x^3)\Big]
$$

renaming the integration variable $\sigma \to \tau$

$$
\frac{\sqrt{\lambda}}{2\pi}\int d\tau\ \ddot{\mathrm{x}}_i^2
$$

low-velocity limit of Lienard formula

Aim:

show that T-duality along AdS boundary directions relates world sheets of small closed strings in the bulk to open string world sheets ending on wavy lines at the boundary (representing small-velocity “quark” trajectories in dual gauge theory)

Starting observation:

T-duality along the 4 boundary directions of Poincare patch of AdS maps massless geodesic (point-like string surface) in the bulk of AdS into open-string surface ending on a straight line at the boundary (dual to straight-line WL)

massless scalar-scalar duality in 2d

$$\partial^a \partial_a \phi = 0 \ , \qquad \epsilon^{ab} \partial_b \phi = \tilde{\phi} \ , \qquad \partial^a \partial_a \tilde{\phi} = 0$$

T-duality: 2d scalar duality applied to isometric direction of a sigma model describing string propagation in curved space. It maps classical solutions of a sigma model into classical solutions of the dual sigma model.

Implemented at quantum level via a path integral transformation. Relates, e.g., a theory on a circle R to theory on a circle $\tilde{\mathrm{R}} = \alpha' \tilde{\mathrm{R}}^{-1}$

After T-duality space-time symmetry of original sigma model in general becomes hidden (non-locally realised)

In general, such sigma model dualities do not respect manifest global symmetries of the theory, that is, symmetries "seen" by a point-like string. For example, the usual T-duality in x direction relates the models with target-space metrics $\mathrm{d}s^2 = \mathrm{d}r^2 + a^2(r)\mathrm{d}x^2$ and $\mathrm{d}s^2 = \mathrm{d}r^2 + a^{-2}(r)\mathrm{d}x^2$, so that in the case of S^2 when $a = \sin(r)$ and x is compact, the first model has $SO(3)$ symmetry while the second only $SO(2)$. The two sigma models are still classically equivalent, i.e. they share the same first-order formulation and integrable structure. Put differently, the corresponding $2d$ field theories (i.e. string models as opposed to their point-particle truncations) are, in a sense, equally symmetric.

AdS in Poincare coordinates:

$$ds^2 = \frac{1}{z^2}(dz^2 + dx^\mu dx_\mu)$$

T-duality along all x and inversion of z is a symmetry:

T- self-duality of AdS sigma model

$$d\tilde{s}^2 = \frac{1}{\tilde{z}^2}(d\tilde{z}^2 + d\tilde{x}^\mu d\tilde{x}_\mu)$$

- T-self-duality of AdS [Kallosh, AT 98]: formal T-duality along all boundary directions in Poincare coordinates maps AdS sigma model back into AdS sigma model if combined with a coordinate transf'n - inversion of radial direction z (which effectively interchanges boundary and horizon)

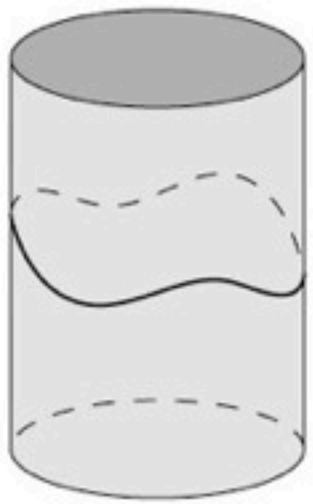
- used [Alday, Maldacena 07] to relate (imaginary, Euclidean) world-sheet solution that dominates semiclassical path integral for open-string ("gluon") scattering amplitudes to (real) solutions describing the corresponding null polygon Wilson loops in the dual momentum space.
  [Below we discuss case of Minkowski signature in both target space and world sheet, so that T-duality transformations will be mapping real solutions into real solutions]

- consistent with integrability: acts on the space of conserved charges mapping Noether charges into hidden charges and vice versa [Ricci,AT,Wolf 07] -- strong-coupling origin [AM] of dual conformal symmetry found in MHV amplitudes at weak coupling in SYM

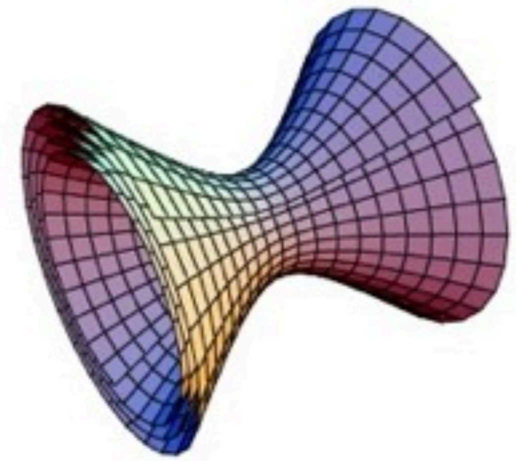
- AdS5 x S5 superstring generalisation: strong-coupling origin of dual superconformal symmetry [Berkovits, Maldacena 08; Beisert, Ricci, AT, Wolf 08]

Strings on $AdS_5 \times S^5$

IIB superstrings on the curved $AdS_5 \times S^5$ superspace



$\longrightarrow$



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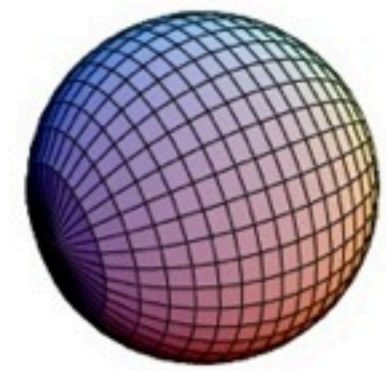


$\times$ fermions

Coset space

$$AdS_5 \times S^5 \times \text{fermi} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)} .$$

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$-Y_0^2 - Y_5^2 + Y_1^2 + \ldots + Y_4^2 = -1 \,, \qquad X_1^2 + \ldots + X_6^2 = 1$$

AdS coordinates

$$ds^2_{AdS_5} = dX_M dX^M \ , \qquad -X_M X^M = X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = 1$$

$$X_\pm = X_{-1} \pm X_4 \ , \qquad X_+ X_- - X_\mu X^\mu = 1 \ , \qquad X_\mu = (X_0, X_i)$$

$$X_0 + iX_{-1} = \cosh\rho \, e^{it}, \qquad X_r = \sinh\rho \, \hat{n}_r, \qquad r = 1, \ldots, 4 \ ,$$

$$ds^2 = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\Omega^2_{[3]} \ ,$$

$$Z = \frac{1}{X_-} \ , \qquad X_- = X_{-1} - X_4 = \cosh\rho \, \sin t - \sinh\rho \, \hat{n}_4 \ ,$$

$$\mathcal{X}_0 \equiv \mathcal{T} = \frac{X_0}{X_-} = \frac{\cosh\rho \, \cos t}{X_-} \ ,$$

$$\mathcal{X}_i = \frac{X_i}{X_-} = \frac{\sinh\rho \, \hat{n}_i}{X_-} \ , \qquad i = 1, 2, 3 \ ,$$

$$ds^2 = \frac{1}{Z^2}(dZ^2 + d\mathcal{X}_\mu d\mathcal{X}^\mu) \ , \qquad \mathcal{X}_\mu = (\mathcal{T}, \mathcal{X}_i), \qquad \mu = 0, 1, 2, 3$$

AdS string equations in conformal gauge

$$\bar{S} = \frac{1}{2}\int d\sigma d\tau \left(\partial_a X_+ \partial^a X_- - \partial_a X_\mu \partial^a X^\mu\right) + \frac{1}{2}\int d\sigma d\tau \Lambda \left(X_+ X_- - X_\mu X^\mu - 1\right)$$

$$\partial_a \partial^a X_\pm = \Lambda X_\pm \ , \qquad \partial_a \partial^a X^\mu = \Lambda X^\mu \ ,$$

$$\Lambda = \partial_a X_+ \partial^a X_- - \partial_a X_\mu \partial^a X^\mu \ , \qquad X_+ X_- - X_\mu X^\mu = 1 \ ,$$

$$\partial_\sigma X_+ \partial_\sigma X_- + \partial_\tau X_+ \partial_\tau X_- - \partial_\sigma X_\mu \partial_\sigma X^\mu - \partial_\tau X_\mu \partial_\tau X^\mu = 0 \ ,$$

$$\partial_\sigma X_+ \partial_\tau X_- + \partial_\tau X_+ \partial_\sigma X_- - 2\partial_\sigma X_\mu \partial_\tau X^\mu = 0 \ .$$

T-duality in AdS : 2d duality + inversion of Z

$$(Z, \mathcal{X}_\mu) \to (\tilde{Z}, \tilde{\mathcal{X}}_\mu) \text{ where } \partial_a \tilde{\mathcal{X}}_\mu = \epsilon_{ab} Z^{-2} \partial^b \mathcal{X}_\mu$$

$$\partial_\tau \tilde{\mathcal{X}}_\mu = -\frac{1}{Z^2}\partial_\sigma \mathcal{X}_\mu \ , \qquad \partial_\sigma \tilde{\mathcal{X}}_\mu = -\frac{1}{Z^2}\partial_\tau \mathcal{X}_\mu \ , \qquad \tilde{Z} = \frac{1}{Z} = X_-$$

same eqn's in $\quad ds^2 = \tilde{Z}^{-2}(d\tilde{\mathcal{X}}^2 + d\tilde{Z}^2)$

T-duality relations may be written as:

$$\partial^a \tilde{\mathcal{X}}_\mu = -\epsilon^{ab}\, j_{b,-\mu} \ ,$$
$$j_{a,MN} \equiv X_M \partial_a X_N - X_N \partial_a X_M \ , \qquad \partial^a j_{a,MN} = 0 \ ,$$

$j_{a,MN}$ is the $SO(4,2)$ Noether current associated to the AdS_5 space symmetry

the corresponding charge

$$J_{MN} = \int d\sigma \ j_{\tau,MN}$$

Thus the particular $SO(4,2)$ angular momentum components $J_{-\mu}$ get the interpretation of the "winding numbers" in the dual $\tilde{\mathcal{X}}_\mu$ directions, $\int d\sigma\, \partial_\sigma \tilde{\mathcal{X}}_\mu = J_{-\mu}$.

[cf. closed string energy as period in dual time coordinate]

Digression: T-duality in non-compact directions (subtle for closed strings with w-sheet as 2d cylinder)

consider only one bosonic $\mathcal{X}_0 = \mathcal{T}$

$$S = \int d\sigma d\tau \Big[g_{\mathcal{T}\mathcal{T}} (\partial_\tau \mathcal{T} \partial_\tau \mathcal{T} - \partial_\sigma \mathcal{T} \partial_\sigma \mathcal{T}) + 2 g_{\mathcal{T}i} (\partial_\tau \mathcal{T} \partial_\tau \mathcal{X}^i - \partial_\sigma \mathcal{T} \partial_\sigma \mathcal{X}^i) + g_{ij} (\partial_\tau \mathcal{X}^i \partial_\tau \mathcal{X}^j - \partial_\sigma \mathcal{X}^i \partial_\sigma \mathcal{X}^j) \Big] ,$$

Gauging as usual the shifts in $\mathcal{T}$

$$S = \int d\sigma d\tau \Big[g_{\mathcal{T}\mathcal{T}} (A_\tau A_\tau - A_\sigma A_\sigma) + 2 g_{\mathcal{T}i} (A_\tau \partial_\tau \mathcal{X}^i - A_\sigma \partial_\sigma \mathcal{X}^i) + g_{ij} (\partial_\tau \mathcal{X}^i \partial_\tau \mathcal{X}^j - \partial_\sigma \mathcal{X}^i \partial_\sigma \mathcal{X}^j) + \tilde{\mathcal{T}} (\partial_\sigma A_\tau - \partial_\tau A_\sigma) \Big]$$

theory defined on the cylinder

$$\mathcal{T} = \xi_0(\tau) + \sum_{n \neq 0} e^{in\sigma} \xi_n(\tau) , \qquad \tilde{\mathcal{T}} = \tilde{\xi}_0(\tau) + \sum_{n \neq 0} e^{in\sigma} \tilde{\xi}_n(\tau) ,$$

$$A_\sigma = \xi_{\sigma 0}(\tau) + \sum_{n \neq 0} e^{in\sigma} \xi_{\sigma n}(\tau) , \qquad A_\tau = \xi_{\tau 0}(\tau) + \sum_{n \neq 0} e^{in\sigma} \xi_{\tau n}(\tau)$$

Ignoring first the zero modes, let us check that the path integral over $\xi_{\sigma n}(\tau)$, $\xi_{\tau n}(\tau)$ and $\tilde{\xi}_n(\tau)$ is equivalent to the path integral over $\xi_n(\tau)$. Indeed, the action has a term

$$\int d\tau \sum_{n\neq 0} \tilde{\xi}_{-n}(in\xi_{\tau n} - \partial_\tau \xi_{\sigma n})$$

The path integral over $\tilde{\xi}_n$ enforces the relation $\xi_{\tau n} = \frac{1}{in}\partial_\tau \xi_{\sigma n}$ which means that the only independent variables remaining are $\xi_{\sigma n}$. Then defining $\xi_n = \frac{1}{in}\xi_{\sigma n}$ accomplishes the task.

zero modes present problems:

$\xi_{\tau 0}$ which should be equal to $\partial_\tau \xi_0$ does not appear in the last term

$$S_A = \int d\tau d\sigma\, \tilde{\mathcal{T}}(\partial_\sigma A_\tau - \partial_\tau A_\sigma)$$

To enforce the required relation we may add an extra term to the action variable $\mu_0(\tau)$

$$S_0 = -\int d\tau\ \mu_0(\tau)(\xi_{\tau 0} - \partial_\tau \xi_0) = -\frac{1}{2\pi}\int d\tau d\sigma\ \mu_0(\tau) A_\tau + \int d\tau\ \mu_0(\tau)\partial_\tau \xi_0$$

Integrating by parts in the last term

$$S_0 = -\frac{1}{2\pi}\int d\tau d\sigma\ \mu_0(\tau)A_\tau - \int d\tau\ \partial_\tau\mu_0(\tau)\xi_0(\tau) + \mu_0(\tau_f)\xi_0(\tau_f) - \mu_0(\tau_i)\xi_0(\tau_i)$$

$$S_A + S_0 = -\int d\tau d\sigma\left[(\partial_\sigma\tilde{\mathcal{T}} + \bar{\mu}_0)A_\tau - \partial_\tau\tilde{\mathcal{T}}A_\sigma\right] + 2\pi\bar{\mu}_0[\xi_0(\tau_f) - \xi_0(\tau_i)]\ , \qquad \bar{\mu} = \frac{1}{2\pi}\mu_0$$

In the T-dual action we should replace $\partial_\sigma\tilde{\mathcal{T}} \to \partial_\sigma\tilde{\mathcal{T}} + \bar{\mu}_0$.

If we Fourier transform over the center of mass positions $\xi_0(\tau_f)$ and $\xi_0(\tau_i)$ we get that the initial and final momenta in the $\mathcal{T}$ direction should be equal to μ_0. Therefore, instead of integrating over μ_0 we can fix μ_0 to be equal to the value of the corresponding conserved momentum, namely, the energy conjugate to $\mathcal{T} = \mathcal{X}_0$.

Up to now we discussed the theory defined on the cylinder with all the functions periodic in σ. However, we may formally define the T-dual coordinate $\tilde{\mathcal{T}}$ as

$$\tilde{\mathcal{T}} = \bar{\mu}_0\sigma + \sum_{n\neq 0}\tilde{\xi}_n(\tau)e^{in\sigma}$$

Including the linear term in σ is allowed as $\tilde{\mathcal{T}}$ enters in the action through its derivatives which are all periodic. A problem, however, arises if we insert vertices with momenta in the direction of $\tilde{\mathcal{T}}$: the presence of a factor $\exp(i\tilde{E}\tilde{\mathcal{T}})$ will break the periodicity in σ.

Equivalently, we may consider the coordinate $\tilde{\mathcal{T}}$ to be periodic with period determined by the energy, $\tilde{\mathcal{T}} \equiv \tilde{\mathcal{T}} + 2\pi E$. Physically, the period changes with the energy so it should not be considered a property of the dual space-time but of the configuration that we are considering. The T-dual string surface is then an open string world sheet which is periodic along $\tilde{\mathcal{T}}$.

T-duality between closed string and open string world sheets will be understood in this sense

T-duality relates one AdS string solution to another (is a map on a set of conserved charges)

Remarkable example:

massless geodesic in AdS is mapped to surface ending on straight line along the boundary (BPS to BPS)

$$t = t(\tau), \ \ \rho = \rho(\tau), \ \ \sinh\rho = \kappa\tau, \ \ \tan t = \kappa\tau, \ \ \cos t = \tfrac{1}{\sqrt{1+\kappa^2\tau^2}}$$

$$Z = \mathcal{T} = \frac{1}{\kappa\tau} \ , \qquad \qquad \tilde{Z} = \kappa\sigma \ , \qquad \tilde{\mathcal{T}} = \kappa\tau$$

Here interchanged τ and σ after duality to have

new time coordinate $\sim \tau$

(σ runs along open string ending at the boundary)

Other examples:

Massive geodesic (or massless BMN geodesic) $t = \kappa\tau,\ \rho = 0$

$$Z = \frac{1}{\sin\kappa\tau}\ , \qquad \mathcal{T} = \cot\kappa\tau\ , \qquad \tilde{Z} = \sin\kappa\sigma\ , \qquad \tilde{\mathcal{T}} = \kappa\tau\ .$$

again interchanged τ and σ after T-duality

dual surface is folded (goes up and down)

Infinite straight static string in AdS
(special zero spin limit of folded spinning string)

$$t = \kappa\tau,\ \ \rho = \rho(\sigma),\ \tanh\tfrac{\rho}{2} = \tan\tfrac{\kappa\sigma}{2},\ \cosh\rho = \frac{1}{\cos\kappa\sigma}$$

$$Z = \frac{\cos\kappa\sigma}{\sin\kappa\tau}\ , \qquad \mathcal{T} = \cot\kappa\tau\ , \qquad \tilde{Z} = \frac{\sin\kappa\sigma}{\cos\kappa\tau}\ , \qquad \tilde{\mathcal{T}} = \tan\kappa\tau$$

To interpret T-dual world surface as that of an open string stretched inside AdS and ending at the boundary at a point (“quark") one needs to:

(i) Formally *“decompactify”* the σ direction. Starting with a periodic solution for a small closed string (defined on a 2d cylinder) and applying the T-duality (in a particular gauge, see below) we will get a σ-periodic open-string world sheet, apart from the $\tilde{\mathcal{T}}$ direction which gets a term linear in σ. world-sheet theory will still be periodic
since $\tilde{\mathcal{T}}$ enters only through its derivatives (assuming there are no vertex operator insertions depending on $\tilde{\mathcal{T}}$).

(ii) *Interchange the τ and σ coordinates*

(a familiar operation in the usual flat-space closed– open string duality relation)

Recall definition of AdS coordinates

$$ds^2_{_{AdS_5}} = dX_M dX^M \ , \qquad -X_M X^M = X_{-1}^2 + X_0^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = 1$$

$$X_\pm = X_{-1} \pm X_4 \ , \qquad X_+ X_- - X_\mu X^\mu = 1 \ , \qquad X_\mu = (X_0, X_i)$$

$$Z = \frac{1}{X_-} \ , \qquad X_- = X_{-1} - X_4 = \cosh\rho \, \sin t - \sinh\rho \, \hat{n}_4 \ ,$$

$$\mathcal{X}_0 \equiv \mathcal{T} = \frac{X_0}{X_-} = \frac{\cosh\rho \, \cos t}{X_-} \ ,$$

$$\mathcal{X}_i = \frac{X_i}{X_-} = \frac{\sinh\rho \, \hat{n}_i}{X_-} \ , \qquad i = 1, 2, 3 \ ,$$

$$ds^2 = \frac{1}{Z^2}(dZ^2 + d\mathcal{X}_\mu d\mathcal{X}^\mu) \ , \qquad \mathcal{X}_\mu = (\mathcal{T}, \mathcal{X}_i), \qquad \mu = 0, 1, 2, 3$$

Apply T-duality to small closed strings
at the "center" of AdS

(i) consider small region (small strings)

$$X_0 \sim 1 + \epsilon^2 \,, \qquad\qquad X_{M\neq 0} \sim \epsilon \,.$$

(ii) do large boost in hyperbolic plane

$$X_- \to \frac{1}{\epsilon} X_-, \qquad X_+ \to \epsilon X_+ \,, \qquad \epsilon \to 0$$

to make small string at the center move fast towards the boundary following nearly-massless geodesic: boost exposes near BPS nature of nearly point-like small string

After T-duality: get open string extended along $\tilde{T}, \tilde{Z}$
with small fluctuations in $\tilde{X}_{i=1,2,3}$

T-duality relation between short closed strings and long wavy open strings

$$\rho = \epsilon \bar{\rho}, \qquad t = \epsilon y_0$$

$$\begin{aligned} ds^2 &= -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_{[3]}^2 \\ &\simeq \epsilon^2 \left(-dy_0^2 + dy_r dy_r\right) , \qquad y_r = \bar{\rho} \hat{n}_r \end{aligned}$$

small string eqns are as in flat space: conformal gauge + l.c.

$$y_- \equiv y_0 - y_4 = \tau$$

$$y_i(\sigma,\tau) = \mathrm{y}_i(\sigma,\tau) \equiv \mathrm{y}_i^+(\sigma+\tau) + \mathrm{y}_i^-(\sigma-\tau) , \qquad i = 1,2,3$$

$$y_i(\sigma,0) = \mathrm{y}_i(\sigma) = \mathrm{y}_i^+(\sigma) + \mathrm{y}_i^-(\sigma)$$

is T-dual to open string ending at the boundary along trajectory (after tau-sigma interchange)

$$\mathcal{X}_0 = \tau , \qquad \mathcal{X}_i = \mathrm{x}_i(\tau) = \int^\tau d\tau' \, \mathrm{y}_i(\tau')$$

Choose period of sigma as:

$$\sigma \equiv \sigma + \Sigma \qquad \bar{E} = \bar{P}_- = \int d\sigma\, \partial_\tau X_- = \epsilon\Sigma$$

The periodicity of X_+ in σ implies

$$0 = \epsilon \int d\sigma\, \partial_\sigma X_+ = 2 \int d\sigma\, \partial_\sigma X_i \partial_\tau X_i = 2\epsilon^2 \int d\sigma\, \left[(\partial_\tau \mathrm{y}_i^+)^2 - (\partial_\tau \mathrm{y}_i^-)^2\right]$$

$$X_i = \epsilon y_i = \epsilon \mathrm{y}_i\,, \qquad \mathrm{y}_i(\sigma,\tau) = \mathrm{y}_i^+(\sigma+\tau) + \mathrm{y}_i^-(\sigma-\tau)\,,$$

small string limit in embedding and in Poincare coordinates:

$$X_0 = 1\,, \qquad X_i = \epsilon\bar{\rho}\,\hat{n}_i = \epsilon y_i, \qquad X_\pm = \epsilon(y_0 \pm \bar{\rho}\,\hat{n}_4) = \epsilon(y_0 \pm y_4)$$

$$Z = \frac{1}{X_-}\,, \qquad \mathcal{X}_\mu = \frac{X_\mu}{X_-}\,, \qquad \mu = 0, \cdots, 3$$

$$Z = \frac{1}{\epsilon\,(y_0 - y_4)}\,, \qquad \mathcal{X}_0 \equiv \mathcal{T} = \frac{1}{\epsilon\,(y_0 - y_4)}\,, \qquad \mathcal{X}_i = \frac{y_i}{y_0 - y_4}$$

after the boost:

$$Z = \mathcal{T} = \frac{1}{y_-} \, , \qquad \mathcal{X}_i = \frac{\epsilon \, y_i}{y_-} \, , \qquad y_- \equiv y_0 - y_4$$

get boosted small string solution in Poincare patch

Rescaling and choosing l.c. gauge gives leading-order solution in Poincare coordinates:

$$Z = \mathcal{T} = \frac{1}{\tau} \, , \qquad \mathcal{X}_i = \frac{y_i(\sigma, \tau)}{\tau} \, , \qquad y_i = \mathrm{y}_i(\sigma, \tau)$$

Applying T-duality:

$$\begin{aligned} \tilde{Z} &= \frac{1}{Z} = \tau \, , \\ \partial_\sigma \tilde{\mathcal{T}} &= -\frac{1}{Z^2} \partial_\tau \mathcal{T} = 1 \, , \\ \partial_\sigma \tilde{\mathcal{X}}_i &= -\frac{1}{Z^2} \partial_\tau \mathcal{X}_i = -\tau^2 \partial_\tau \frac{y_i}{\tau} = y_i - \tau \partial_\tau y_i \end{aligned}$$

Get leading-order solution in the dual AdS Poincare patch (removing tildes)

$$Z(\sigma,\tau) = \tau\,, \qquad \mathcal{T}(\sigma,\tau) = \sigma\,,$$

$$\mathcal{X}_i(\sigma,\tau) = \int^{\sigma} d\sigma' \left[y_i(\sigma',\tau) - \tau\partial_\tau y_i(\sigma',\tau)\right]$$

$y_i = \mathrm{y}_i$ is the harmonic function

This surface ends at the boundary Z=0 at $\tau = 0$ along

$$\mathcal{T} = \sigma\,, \qquad \mathcal{X}_i(\sigma) = \int^{\sigma} d\sigma'\, y_i(\sigma',\tau=0) = \int^{\sigma} d\sigma' \left[\mathrm{y}_i^+(\sigma') + \mathrm{y}_i^-(\sigma')\right]$$

third normal derivative

$$\partial_\tau^3\, \mathcal{X}_i(\sigma,0) = -2\int^{\sigma} d\sigma'\, \partial_\tau^3 y_i(\sigma',\tau=0) = -2\int^{\sigma} d\sigma'\, \partial_{\sigma'}^3 \left[\mathrm{y}_i^+(\sigma') - \mathrm{y}_i^-(\sigma')\right]$$

Compare to "wavy-line"open string solution of Mikhailov

after interchanging τ and σ

$$Z = \sigma \,, \qquad \mathcal{T} = \tau \,, \qquad \mathcal{X}_i = x_i(\tau,\sigma) \,,$$

$$x_i(\tau,\sigma) = \epsilon \int^{\tau} d\tau' \left(\mathrm{y}_i^+(\tau+\sigma) + \mathrm{y}_i^-(\tau-\sigma) - \sigma[\dot{\mathrm{y}}_i^+(\tau+\sigma) - \dot{\mathrm{y}}_i^-(\tau-\sigma)] \right)$$

equivalent to open string solution if

$$\mathrm{y}_i^{\pm}(\tau) = \dot{\mathrm{x}}_i^{\pm}(\tau)$$

original small-string profile (of flat-space solution) is mapped into velocity - derivative of open-string end-point coordinate at the boundary: expansion in small string size corresponds to small velocity expansion for the wavy line

Since interchanged τ and σ world surface here is periodic in τ with period E equal to small closed string energy

This proves that an arbitrary flat-space solution for a small closed string maps under “T-duality plus boost" into linearized solution for an open string ending at the boundary along wavy-line small-velocity trajectory.

As deviation from the straight line is small, this may be interpreted as near BPS configuration. Same applies to original small string: large boost makes its world surface close to that of a null geodesic in AdS.

String energy is given in terms of its profile by Lienard formula: T-duality “demystifies” it: energy depends on first derivative of string coordinate which is mapped to first derivative of velocity of boundary trajectory

Remarks:

Above relation is valid in small-fluctuation (small velocity) approximation. Possible to extend it beyond leading order and also generalize to non-zero S5 momentum

It is reasonable to expect that this closed-open string relation extends beyond the classical approximation, i.e. applies to all orders in the large string tension expansion.

This is suggested by the fact that the world-sheet theory is the same, the difference is in the interpretation of its fields (which are related by T-duality).

Area of the open string surface

(in general, determines expectation value of Wilson loop for small deformation of straight line at strong coupling)

in Minkowski signature: specify boundary shape and also 3rd normal derivative (or left+right parts of function y)

$$S = \frac{\sqrt{\lambda}}{2\pi}\bar{S} \ , \qquad \bar{S} = \int \frac{d\tau d\sigma}{2\sigma^2}\left[(\partial_\tau x_i)^2 - (\partial_\sigma x_i)^2\right]$$

$$\bar{S} = \int \frac{d\tau}{2\sigma^2}\,(x_i\partial_\sigma x_i)\Big|_{\sigma=\varepsilon}^{\infty} = -\frac{1}{2\varepsilon^2}\int d\tau\,(x_i\partial_\sigma x_i)\Big|_{\sigma=\varepsilon}$$

$$x_i\Big|_{\sigma=0} = \mathrm{x}_i\Big|_{\sigma=0} = \mathrm{x}_i(\tau) \ , \quad \partial_\sigma x_i\Big|_{\sigma=0} = 0 \ , \quad \partial_\sigma^2 x_i\Big|_{\sigma=0} = -\partial_\sigma^2 \mathrm{x}_i\Big|_{\sigma=0} = -\partial_\tau^2 \mathrm{x}_i(\tau)$$

$$\partial_\sigma^3 x_i\Big|_{\sigma=0} = -2\partial_\sigma^3 \mathrm{x}_i\Big|_{\sigma=0}, \qquad \partial_\tau^k x_i\Big|_{\sigma=0} = \partial_\tau^k \mathrm{x}_i(\tau) \ .$$

$$\bar{S} = -\frac{1}{2\varepsilon}\int d\tau\,[\partial_\tau \mathrm{x}_i(\tau)]^2 - \frac{1}{4}\int d\tau\,\mathrm{x}_i(\tau)(\partial_\sigma^3 x_i)(\tau,0)$$

Divergent term is correction to length of WL

Finite part:

$$\bar{S}_{\rm fin} = -\frac{1}{4}\int d\tau\, x_i(\tau,0)(\partial_\sigma^3 x_i)(\tau,0)$$

$$\bar{S}_{\rm fin} = \frac{1}{2}\int d\tau\,(\mathrm{x}_i^+ + \mathrm{x}_i^-)(\partial_\tau^3\mathrm{x}_i^+ - \partial_\tau^3\mathrm{x}_i^-) = \frac{1}{2}\int d\tau\; v_i(a_i^- - a_i^+)$$

$$v_i = \partial_\tau \mathrm{x}_i(\tau,0), \qquad a_i^\pm = \partial_\tau^2 \mathrm{x}_i^\pm(\tau)$$

Energy of the open string ending on a wavy line

Let us now consider the energy of the open string that ends on the boundary at a point which is fluctuating around the straight Wilson line $\mathcal{T} = \tau$ with the transverse coordinates $\mathrm{x}_i = \mathrm{x}_i(\tau)$ changing slowly. This end point may be referred to as a "quark". The energy is time dependent because the work needed to keep the quark on its trajectory is radiated as waves into the attached string. There may be also waves that arrive from the string and are absorbed by the quark. (Mikhailov studied case of pure emission)

energy of the string at a time $\mathcal{T}$

$$
\begin{aligned}
E(\mathcal{T}) &= \frac{\sqrt{\lambda}}{2\pi}\bar{E}(\mathcal{T}) \, , \\
\bar{E}(\mathcal{T}) &= \frac{1}{\varepsilon}\frac{1}{\sqrt{1-v^2}} - \frac{v\cdot a}{(1-v^2)^2} + \int_{-\infty}^{\mathcal{T}} d\tau \, \frac{(1-v^2)\, a^2 + (v\cdot a)^2}{(1-v^2)^3}
\end{aligned}
$$

$$v_i = \partial_\tau \mathrm{x}_i(\tau), \; a_i = \partial_\tau^2 \mathrm{x}_i(\tau)$$

first two terms come from the regularization near the boundary, i.e. at $Z = \sigma = \varepsilon \to 0$.

The last term is the accumulated energy that the quark radiates into the string

Leading small velocity approximation:

$$\bar{E}(\mathcal{T}) = \frac{1}{\varepsilon}(1 + \frac{1}{2}v^2) - v \cdot a + \int_{-\infty}^{\mathcal{T}} d\tau \; a^2(\tau)$$

$$\frac{\partial \bar{E}}{\partial \mathcal{T}} = \frac{1}{\varepsilon} v \cdot a - \partial_{\mathcal{T}}(v \cdot a) + a^2 \; .$$

Same expression can be explicitly derived from small-wave solution:

$$\bar{E}(\tau) = \bar{E}^{\mathrm{op}} = \int_{\varepsilon}^{\infty} \frac{d\sigma}{\sigma^2}\Big[1 + \frac{1}{2}(\partial_\sigma x_i)^2 + \frac{1}{2}(\partial_\tau x_i)^2\Big]$$

$$\partial_\tau \bar{E} = \int_{\varepsilon}^{\infty} \frac{d\sigma}{\sigma^2}(\partial_\sigma x_i \partial^2_{\sigma\tau} x_i + \partial_\tau x_i \partial^2_\tau x_i)$$

Using the equations of motion

$$\partial^2_\tau x_i - \partial^2_\sigma x_i + \frac{2}{\sigma}\partial_\sigma x_i = 0$$

$$\partial_\tau \bar{E} = -\left.\frac{\partial_\sigma x_i \partial_\tau x_i}{\sigma^2}\right|_{\sigma=\varepsilon}$$

$$\partial_\sigma x_i(\tau,\varepsilon) = -\varepsilon \partial_\tau^2 x_i(\tau,0) + \frac{1}{2}\varepsilon^2 \partial_\sigma^3 x_i(\tau,0)$$

$$\partial_\tau \bar{E} = \left(\frac{1}{\varepsilon}\partial_\tau x_i \partial_\tau^2 x_i - \frac{1}{2}\partial_\tau x_i \partial_\sigma^3 x_i\right)(\tau,0) = \frac{1}{\varepsilon} v_i a_i - \frac{1}{2} v_i \partial_\sigma^3 x_i(\tau,0)$$

$$x_i(\tau,0) = \mathrm{x}_i^+(\tau) + \mathrm{x}_i^-(\tau)\ , \qquad \partial_\sigma^3 x_i(\tau,0) = -2[\partial_\tau^3 \mathrm{x}_i^+(\tau) - \partial_\tau^3 \mathrm{x}_i^-(\tau)]$$

$$\partial_\tau \bar{E} = \frac{1}{\varepsilon} v_i a_i + v_i(\dot{a}_i^+ - \dot{a}_i^-) = \frac{1}{\varepsilon} v \cdot a + \partial_\tau \Big[v \cdot (a^+ - a^-) \Big] + a^{-2} - a^{+2}$$

$$v_i = \partial_\tau \mathrm{x}_i = \partial_\tau \mathrm{x}_i^+ + \partial_\tau \mathrm{x}_i^-\ , \qquad a_i = a_i^+ + a_i^-\ , \qquad a_i^\pm = \partial_\tau^2 \mathrm{x}_i^\pm$$

case of pure emission described by the special solution with $\mathrm{x}_i^+ = 0$ (i.e. with $a = a^-$)

when there are both the left and the right moving waves,

$\bar{E}$ increases when the quark radiates the energy into the string (the term $(a^-)^2 > 0$) and decreases when the quark absorbs the energy from the string (the term $-(a^+)^2 < 0$).

Energy can be expressed in terms of the variables of the T-dual short string

$$\mathrm{y}^{\pm}(\tau) = \partial_\tau \mathrm{x}^{\pm}(\tau)$$

given a small closed string data $\mathrm{y}_i^{\pm}(\sigma)$ and replacing $\sigma \leftrightarrow \tau$

we get the corresponding wavy line data $\mathrm{x}_i^{\pm}(\tau)$

when the open-string world sheet is periodic in τ with period Σ

total energy change of the open string should be computed over one period, giving

$$\Delta \bar{E}^{\rm op} = \bar{E}^{\rm op}(\Sigma) - \bar{E}^{\rm op}(0) = \int_0^\Sigma d\tau \ (a^{-2} - a^{+2}) = \int_0^\Sigma d\tau \ \left[(\partial_\tau \mathrm{y}_i^-)^2 - (\partial_\tau \mathrm{y}_i^+)^2\right] \ ,$$

i.e. it vanishes once one uses the closed-string data for $\mathrm{y}^{\pm}$ satisfying the level matching condition

the energy of the closed string computed directly

$$\bar{E}^{\rm cl} = \int_0^\Sigma d\tau \ \left[(\partial_\tau \mathrm{y}_i^-)^2 + (\partial_\tau \mathrm{y}_i^+)^2\right] \ ,$$

in the dual (open-string) language

$$\bar{E}^{\rm cl} = \int_0^\Sigma d\tau \ (a^{-2} + a^{+2})$$

The energy of the open string ending on the boundary is not conserved (we need do work on the quark to move it along a specified trajectory). The total energy change, however, for *these* solutions is:

$$\Delta\bar{E} = \bar{E}(+\infty) - \bar{E}(-\infty) = \int_{-\infty}^{+\infty} d\tau \ \left(a^{-2} - a^{+2}\right)$$

$$= \int_{-\infty}^{+\infty} d\tau \ \left[(\partial_\tau \mathrm{y}_i^-)^2 - (\partial_\tau \mathrm{y}_i^+)^2\right] = 0$$

It vanishes from the level matching condition for the closed string. However if we add the left and right movers we get the energy of the closed string:

$$\bar{E}^{\mathrm{cl}} = \int_{-\infty}^{+\infty} d\tau \ \left(a^{-2} + a^{+2}\right)$$

The spin is also not conserved.

$$\partial_\tau \bar{\mathrm{S}}_{ij} = (v_i^+ a_j^+ - v_j^+ a_i^+) - (v_i^- a_j^- - v_j^- a_i^-)$$

The velocity and acceleration are evaluated at the boundary. The reason is that the spin defines a conserved current, any increase in spin is due to a flux from the boundary. The total spin absorbed by the open string is precisely equal to the spin of the original closed string.

$$\Delta \bar{\mathrm{S}}_{ij} = \int d\tau \left[(v_i^+ a_j^+ - v_j^+ a_i^+) - (v_i^- a_j^- - v_j^- a_i^-)\right] = \mathrm{S}_{ij}^+ - \mathrm{S}_{ij}^-$$

Comments on interpretation of T-duality relation

The Poincare patch of Minkowski-signature AdS space is the near horizon limit of extremal D3 brane geometry: horizon at $Z = \infty$

In global AdS coordinates horizon is a light-like surface $X_- = 0$

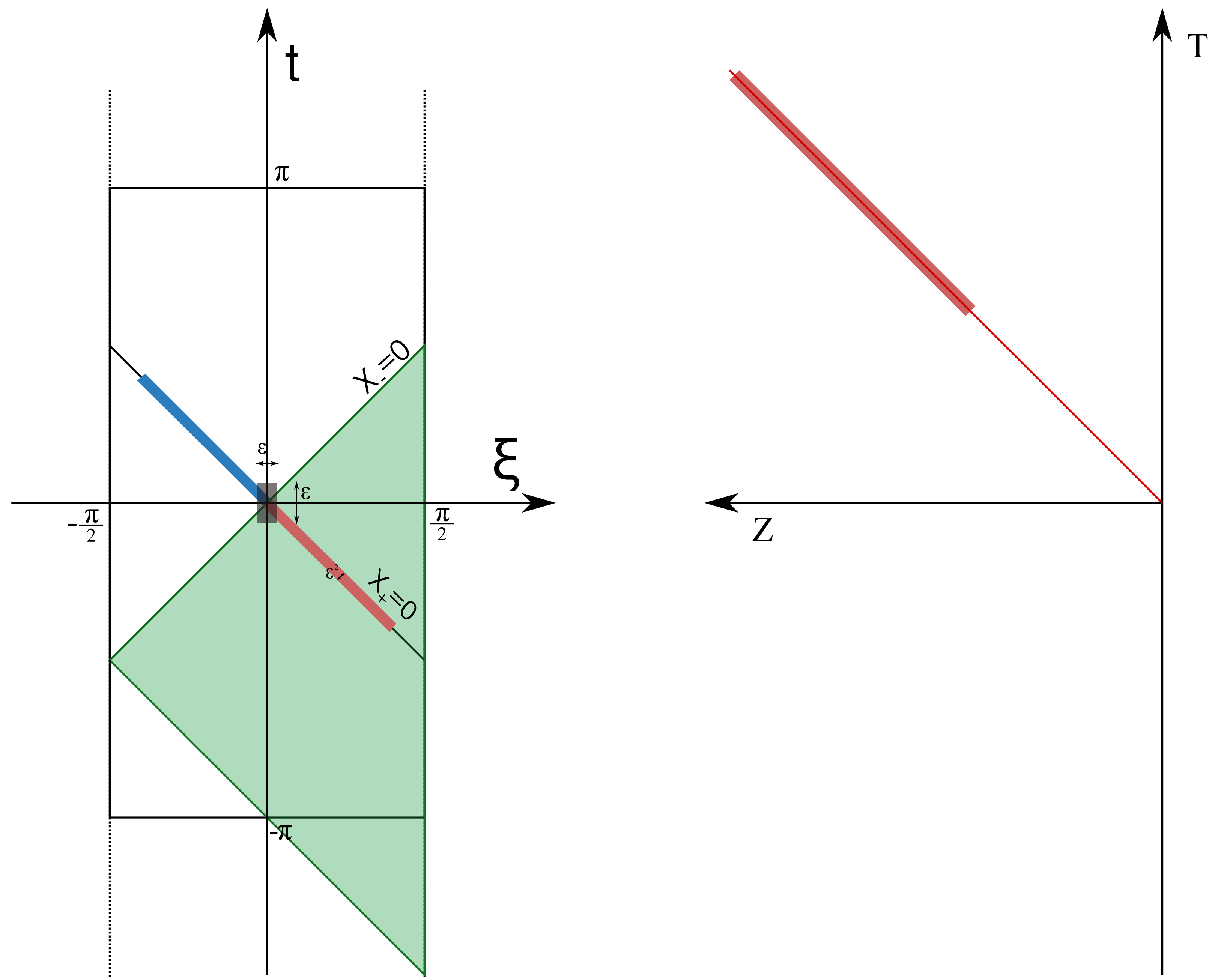
To recall:
Small string in the bulk after a boost falls into the horizon at the point $X_0 = 1 \quad X_\pm = X_i = 0$

After T-duality, the small string falling into the horizon becomes effectively an open string with the boundary end-point that may be interpreted in the dual field theory as a heavy quark moving along a prescribed trajectory.

The quark's trajectory is periodic with period = closed-string energy.
The velocity of the quark is directly determined by the profile of the closed string when falling through the horizon. The quark absorbs and radiates the energy and the spin.

The energy is expressed in terms of third derivative of the string position and is given by the momentum density of the string.

Consider the case of a small string



$$ds^2 = \frac{1}{\cos^2\xi}\left(-dt^2 + d\xi^2 + \sin^2\xi\ d\Omega^2_{[3]}\right)\ , \qquad \cos\xi = \frac{1}{\cosh\rho}$$

Recall that null geodesic is mapped by T-duality to open string world sheet ending at a line

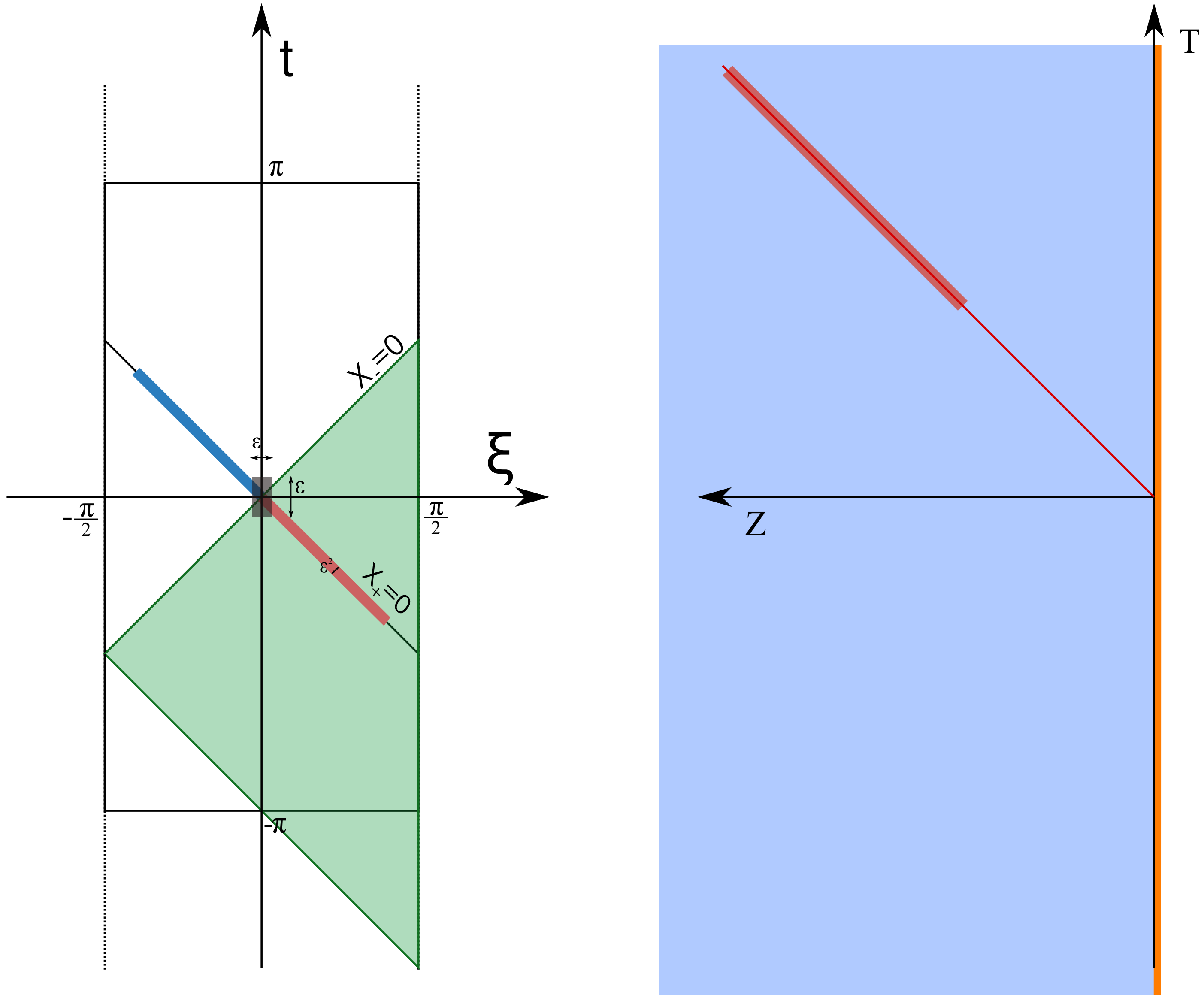
Simple but useful example:

$$T = \frac{1}{\tau}, \quad Z = \frac{1}{\tau}$$

$$\partial_\sigma \left(\tau^2 \partial_\sigma \frac{1}{\tau} \right) = \partial_\tau \left(\tau^2 \partial_\tau \frac{1}{\tau} \right)$$

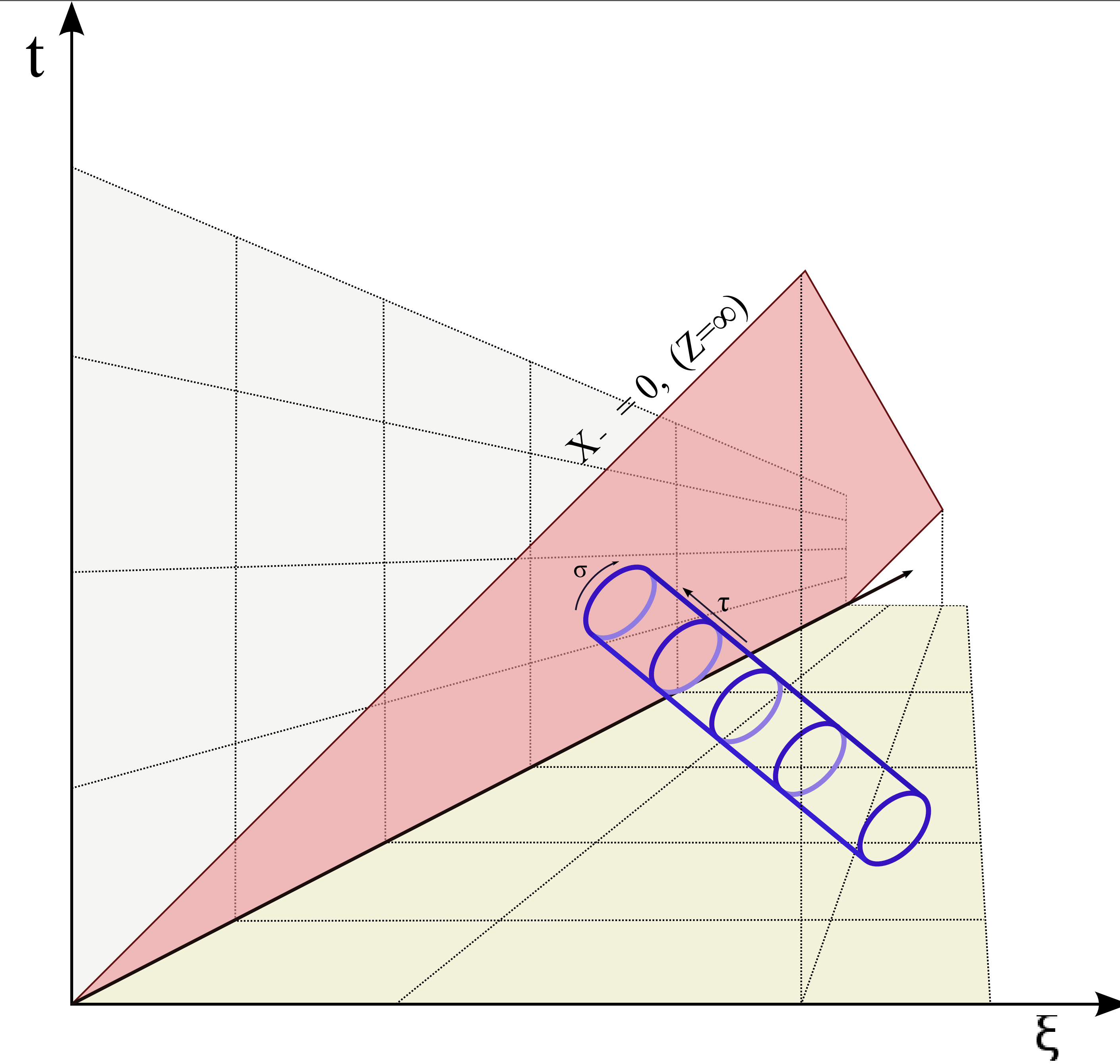
$$\tilde{T} = \sigma, \quad \tilde{Z} = \tau$$

$$\partial_\sigma \left(\frac{1}{\tau^2} \partial_\sigma \sigma \right) = \partial_\tau \left(\frac{1}{\tau^2} \partial_\tau \sigma \right)$$



A point-like string becomes a straight WL of the T-dual theory.

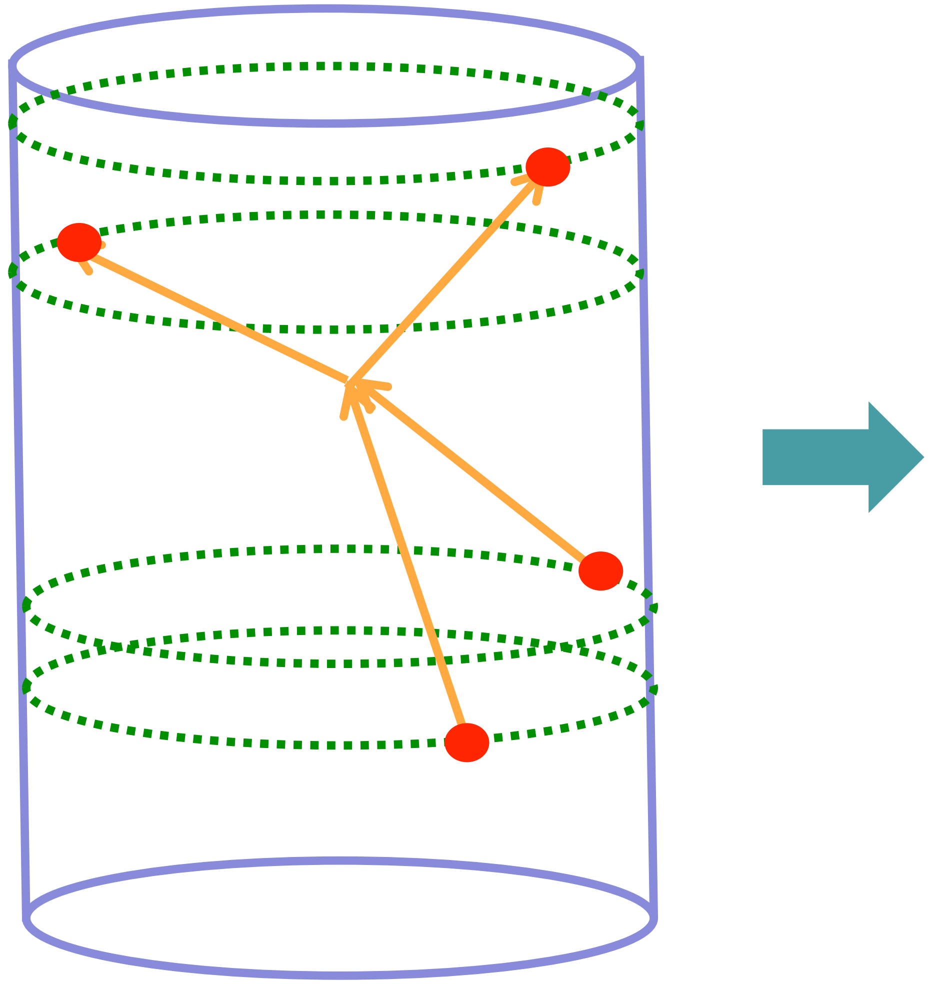
The state in which string falls through horizon is given by its shape and momentum density and maps directly to properties of dual WL



Small string falling through horizon

The “crossing of the horizon" picture is relevant because the shape of the string at the horizon is what maps to the Wilson loop shape: the surface **X_ =0** maps to boundary.

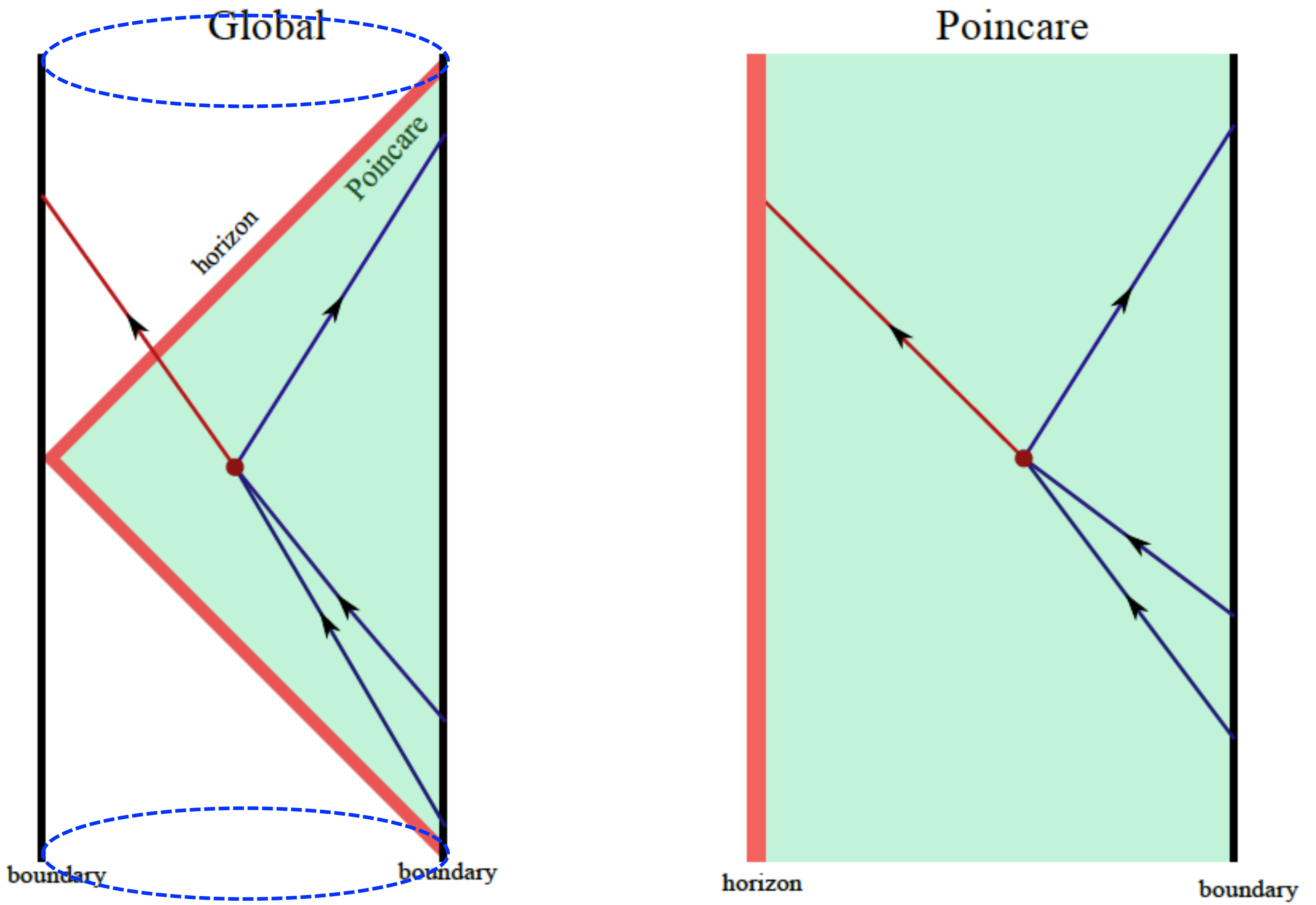
Consider the computation of a correlation function for the field theory living on S^3



Should map to a correlation function of the theory in flat space.

However, Poincare coordinates only cover part of the space and also of the boundary.

A string can ”escape” through the horizon!.



The disappearing string should be represented by an operator in the field theory. Which operator?

T-duality is a kind of “position - momentum" (cf. amplitudes- null polygonal WL relation) Here degrees of freedom (strings) near horizon are those of dual (“momentum”) theory. Suggests that string that falls through the horizon should correspond to an insertion of a Wilson Loop operator in the T-dual theory.

A precise map for short strings: for closed string that crosses horizon insert Wilson loop with shape determined by profile of closed string.

In Euclidean AdS/CFT correspondence, N = 4 Euclidean SYM correlators can be computed as string theory correlators in Euclidean AdS for strings that emerge from boundary by the insertion of appropriate vertex operators. If consider Minkowski signature case in global coordinates same applies. It may happen that one of the local operators inserted in not on the boundary patch - one of the closed strings falls into the Poincare horizon. Then the world sheet has a boundary: the world sheet has three operator insertions plus a boundary at the horizon.

Suggested picture: the conditions at such boundary can be expressed by assuming that the world sheet ends at a wavy line (defining dual WL) in the T-dual version of the AdS space. The boundary theory amplitude is given by “mixed" correlator - of three local operator insertions and a non-local Wilson loop of the T-dual (“momentum") theory.

Comments:

Equivalence via T-duality of small (near-flat) closed string solution and Mikhailov's small-velocity wavy-line open string solution in AdS.
Suggests existence of a map between state of a closed string falling into Poincare horizon and a Wilson loop of T-dual boundary theory. Profile of Wilson loop is determined by shape of the closed string.

May allow to define correlation functions of local operators where one of the corresponding strings falls into horizon: string that crosses horizon is replaced by a Wilson loop.
Dynamics of closed strings in small near-flat-space region in the bulk of AdS may be expressed in terms of correlation functions of certain Wilson loops in gauge theory?

T-duality relation suggests that one should be able to translate results known about Wilson loops into statements about closed strings crossing horizon.
Exact in string tension result for the wavy-line open string energy should have its counterpart on the small closed string side -- providing link to the slope function ?

Problem: describe the T-duality map at quantum string level?
Algebra of oscillators of the closed string should map to the algebra of the corresponding deformation operators of the Wilson loop.
Also, partition functions of the T-dual world-sheet theories should map to each other: compute 1-loop correction - B-function vs slope function?!