THE NON-GAUSSIAN SKY

Alex Kehagias

NTU Athens

Heraklion, March 2013

Work in collaboration with A. Riotto:

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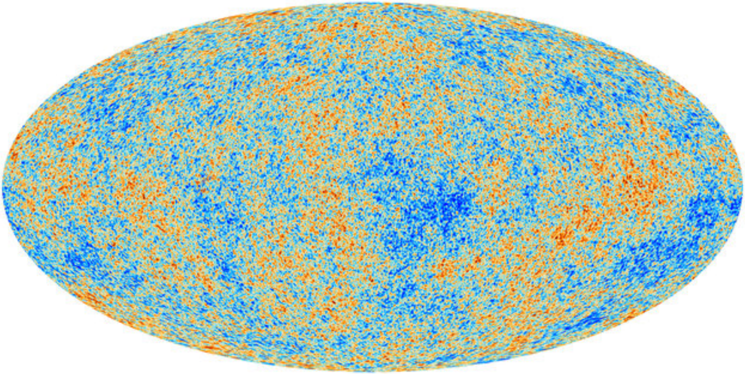
1 Introduction

The latest cosmological data from Planck



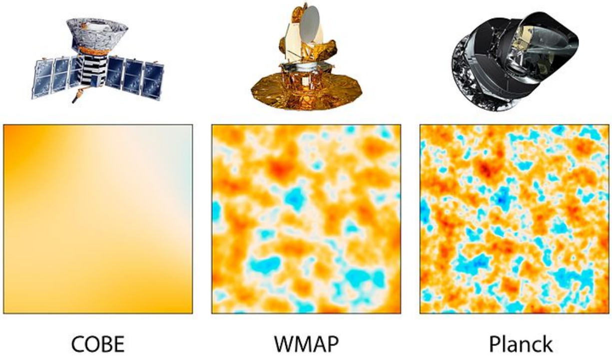
The CMB sky as seen from Planck mission

agree impressively well with a Universe which at large scales is :

- homogeneous,
- isotropic
- spatially flat,

(well described by a FRW spatially flat geometry).

The Planck CMB pattern as compared to the corresponding pattern of COBE and WMAP



A theoretical puzzle:

A flat FRW Universe is *extremely fine tuned* solution in GR.

Many attempts have been put forward to solve this puzzle.

However, the most developed and yet simple idea still remains *Inflation*. Inflation solves homogeneity, isotropy and flatness problems in one go just by postulating a rapid expansion of the early time Universe post Big Bang.

A phenomenological implementation of Inflation: "slow rolling" scalar field

the Inflaton

Homogeneous and isotropic Universe is described by the FRW metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \tag{1}$$

whereas the gravitational dynamics is governed by Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2}$$

Einstein equations are written for the FRW cosmology

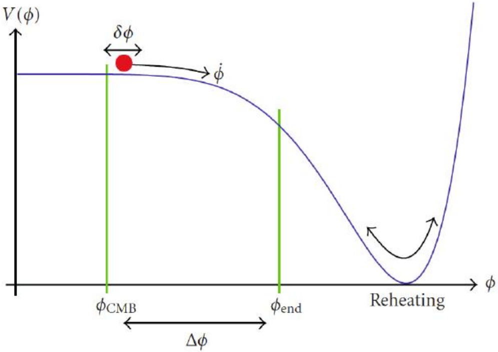
$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) \\ H^2 &= \left(\frac{\dot{a}}{\dot{a}}\right)^2 = \frac{8\pi G}{3}\rho \end{aligned} \tag{3}$$

from where the conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{4}$$

follows.

Inflation is driven by a scalar field ϕ with a generic potential of the form



Dymanics is described by the Lagrangian

$$\mathcal{L} = \frac{1}{16\pi G}R - \frac{1}{2}\partial\phi^2 - V(\phi) \tag{5}$$

with corresponding energy density and pressure

$$\rho = \frac{1}{2}\dot{\phi} + V(\phi) \tag{6}$$

$$p = \frac{1}{2}\dot{\phi} - V(\phi) \tag{7}$$

When potential energy dominates kinetic energy

$$\frac{1}{2}\dot{\phi} << V(\phi) \tag{8}$$

we get an equation of state

$$p \approx -\rho \ , \qquad a \approx e^{Ht} \ , \quad H = \text{const.} \tag{9}$$

This is an almost de Sitter background, specified by the slow-roll parameters

$$\epsilon = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 , \quad \eta = M_P^2\left(\frac{V''}{V}\right) \tag{10}$$

In the quasi-de Sitter phase

$$\epsilon << 1, \quad \eta << 1 \tag{11}$$

An important quantity is the number of e-folds

$$N = \log\frac{a_f}{a_i} = \int_{t_i}^{t_f} H dt \tag{12}$$

which, in terms of the scalar is written as

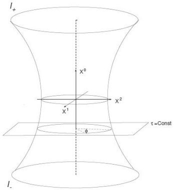
$$N = \int_{\phi_i}^{\phi_f} \frac{8\pi G}{3}\frac{V}{V'} d\phi \tag{13}$$

The four-dimensional

de Sitter spacetime

of radius H^{-1} is described

by the hyperboloid defined by



$$\eta_{AB} X^A X^B = -X_0^2 + X_i^2 + X_5^2 = \frac{1}{H^2} \quad (i = 1, 2, 3), (14)$$

embedded in 5D Minkowski spacetime $\mathbb{M}^{1,4}$ with coordinates X^A and flat metric $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1)$. A particular parametrization of the de Sitter hyperboloid is provided by

$$X^0 = \frac{1}{2H}\left(H\eta - \frac{1}{H\eta}\right) - \frac{1}{2}\frac{x^2}{\eta},$$
$$X^i = \frac{x^i}{H\eta},$$
$$X^5 = -\frac{1}{2H}\left(H\eta + \frac{1}{H\eta}\right) + \frac{1}{2}\frac{x^2}{\eta}, \qquad (15)$$

which may easily be checked that satisfies Eq. (14). The de Sitter metric is the induced metric on the hyperboloid from the five-dimensional ambient Minkowski spacetime

$$\mathrm{d}s_5^2 = \eta_{AB} dX^A dX^B. \qquad (16)$$

For the particular parametrization (15), for example, we find

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2\right). \tag{17}$$

The group $SO(1,4)$ acts linearly on $\mathbb{M}^{1,4}$. Its generators are

$$J_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} \quad A, B = (0,1,2,3,5) \tag{18}$$

and satisfy the $SO(1,4)$ algebra

$$[J_{AB}, J_{CD}] = \eta_{AD} J_{BC} - \eta_{AC} J_{BD} + \eta_{BC} J_{AD} - \eta_{BD} J_{AC}. \tag{19}$$

We may split these generators as

$$J_{ij}, \quad P_0 = J_{05}\,, \quad \Pi_i^+ = J_{i5} + J_{0i}\,, \quad \Pi_i^- = J_{i5} - J_{0i}, \tag{20}$$

which act on the de Sitter hyperboloid as

$$\begin{aligned}
J_{ij} &= x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \\
P_0 &= \eta \frac{\partial}{\partial \eta} + x^i \frac{\partial}{\partial x^i}, \\
\Pi_i^- &= -2H\eta x^i \frac{\partial}{\partial \eta} + H\left(x^2 \delta_{ij} - 2x_i x_j\right) \frac{\partial}{\partial x_j} - H\eta^2 \frac{\partial}{\partial x_i}, \\
\Pi_i^+ &= \frac{1}{H} \frac{\partial}{\partial x_i}
\end{aligned} \tag{21}$$

They satisfy the commutator relations

$$
\begin{aligned}
&[J_{ij}, J_{kl}] = \delta_{il} J_{jk} - \delta_{ik} J_{jl} + \delta_{jk} J_{il} - \delta_{jl} J_{ik}, \\
&[J_{ij}, \Pi_k^{\pm}] = \delta_{ik} \Pi_j^{\pm} - \delta_{jk} \Pi_i^{\pm}, \\
&[\Pi_k^{\pm}, P_0] = \mp \Pi_k^{\pm}, \\
&[\Pi_i^{-}, \Pi_j^{+}] = 2J_{ij} + 2\delta_{ij} P_0.
\end{aligned}
\tag{22}
$$

This is the $SO(1,4)$ algebra written in a strange base.

More standard generators are

$$L_{ij} = iJ_{ij}\,, \quad D = -iP_0\,, \quad P_i = -i\Pi_i^+\,, \quad K_i = i\Pi_i^-\,, \tag{23}$$

we get

$$\begin{aligned} P_i &= -\frac{i}{H}\partial_i, \\ D &= -i\left(\eta\frac{\partial}{\partial\eta} + x^i\partial_i\right), \\ K_i &= -2iHx_i\left(\eta\frac{\partial}{\partial\eta} + x^i\partial_i\right) - iH(-\eta^2 + x^2)\partial_i, \\ L_{ij} &= i\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right). \end{aligned} \tag{24}$$

These are also the Killing vectors of de Sitter spacetime.

They generate space translations (P_i), dilitations (D), special conformal transformations (K_i) and space rotations (L_{ij}). They satisfy the conformal algebra in its standard form

$$[D, P_i] = iP_i, \tag{25}$$

$$[D, K_i] = -iK_i, \tag{26}$$

$$[K_i, P_j] = 2i\Big(\delta_{ij} D - L_{ij}\Big) \tag{27}$$

$$[L_{ij}, P_k] = i\Big(\delta_{jk} P_i - \delta_{ik} P_j\Big), \tag{28}$$

$$[L_{ij}, K_k] = i\Big(\delta_{jk} K_i - \delta_{ik} K_j\Big), \tag{29}$$

$$[L_{ij}, D] = 0, \tag{30}$$

$$[L_{ij}, L_{kl}] = i\Big(\delta_{il} L_{jk} - \delta_{ik} L_{jl} + \delta_{jk} L_{il} - \delta_{jl} L_{ik}\Big). \tag{31}$$

The de Sitter algebra $SO(1,4)$ has two Casimir invariants

$$\mathcal{C}_1 = -\frac{1}{2} J_{AB} J^{AB} \,, \tag{32}$$

$$\mathcal{C}_2 = W_A W^A \,, \qquad W^A = \epsilon^{ABCDE} J_{BC} J_{DE} . \tag{33}$$

Using Eqs. (20) and (23), we find that

$$\mathcal{C}_1 = D^2 + \frac{1}{2}\{P_i, K_i\} + \frac{1}{2} L_{ij} L^{ij}, \tag{34}$$

which turns out to be, in the explicit representation Eq. (24),

$$H^{-2}\mathcal{C}_1 = -\frac{\partial^2}{\partial \eta^2} - \frac{2}{\eta}\frac{\partial}{\partial \eta} + \nabla^2 . \tag{35}$$

As a result, $\mathcal{C}_1$ is the Laplace operator on the de Sitter hyperboloid and for a scalar field $\phi(x)$ we have

$$\mathcal{C}_1\phi(x) = \frac{m^2}{H^2}\phi(x). \tag{36}$$

Super horizon scales: Let us now consider the case $H\eta \ll 1$. The parametrization (15) turns out then to be

$$\begin{aligned} X^0 &= -\frac{1}{2H^2\eta} - \frac{1}{2}\frac{x^2}{\eta}, \\ X^i &= \frac{x^i}{H\eta}, \\ X^5 &= -\frac{1}{2H^2\eta} + \frac{1}{2}\frac{x^2}{\eta} \end{aligned} \tag{37}$$

We may easily check that the hyperboloid has been degenerated to the hypercone

$$-X_0^2 + X_i^2 + X_5^2 = 0. \tag{38}$$

We identify points $X^A \equiv \lambda X^A$ (which turns the cone (38) into a projective space). As a result, η in the denominator of the X^A can be ignored due to projectivity condition. Then, on the cone, the conformal group acts linearly, whereas induces the (non-linear) conformal transformations

$x_i \to x'_i$ with

$$x'_i = a_i + M_i^j x_j, \quad x'_i = \lambda x_i, \tag{39}$$

$$x'_i = \frac{x_i + b_i x^2}{1 + 2 b_i x_i + b^2 x^2}. \tag{40}$$

on Euclidean $\mathbb{R}^3$ with coordinates x^i. They correspond to translations and rotations (P_i, L_{ij}), dilations (D) and special conformal transformations (K_i), respectively, acting now on the constant time hypersurfaces of de Sitter spacetime. Special conformal transformations can be written in terms of inversion

$$x_i \to x'_i = \frac{x_i}{x^2} \tag{41}$$

as inversion$\times$translation$\times$inversion.

The representations of the $SO(1,4)$ algebra are constructed by employing the method of induced representations. Let us consider the stability subgroup at $x^i = 0$ which is the group G generated by (L_{ij}, D, K_i). It is easy to see from the conformal algebra, that P_i and K_i are actually raising and lowering operators for the dilation operator D. Therefore there should be states which will be annihilated by K_i. Every irreducible representation will then be specified by an irreducible representation of the rotational group $SO(3)$ (*i.e.* its spin) and a definite conformal dimension annihilated by K_i.

Representations $\phi_s(\vec{0})$ of the stability group at $\vec{x} = \vec{0}$ with spin s and dimension Δ are specified by

$$\begin{aligned} [L_{ij}, \phi_s(\vec{0})] &= \Sigma_{ij}^{(s)} \phi_s(\vec{0}), \\ [D, \phi_s(\vec{0})] &= -i\Delta \phi_s(\vec{0}), \\ [K_i, \phi_s(\vec{0})] &= 0, \end{aligned} \tag{42}$$

where $\Sigma_{ij}^{(s)}$ is a spin-s representation of $SO(3)$. Those representations $\phi_s(\vec{0})$ that satisfy the relations (42) are primary fields. Once the primary fields are known, all other fields, the descendants, are constructed by taking derivatives of the primaries $\partial_i \cdots \partial_j \phi_s(\vec{0})$.

In particular, we have for scalars

$$[\mathcal{C}_1, \phi(\vec{0})] = -\Delta(\Delta - 3)\phi(\vec{0}), \tag{43}$$

which implies that their masses are

$$m^2 = -\Delta(\Delta - 3)H^2. \tag{44}$$

It can be shown that the scalar representations of the de Sitter group $SO(1,4)$ actually splits into three distinct series:

1) the principal series with masses $m^2 \geq 9H^2/4$,

2) the complementary series, $0 < m^2 < 9H^2/4$ and

3)the discrete series. Only the principal representations survive the Winger-Inonü contraction ($H \to 0$) to the Poincáre group.

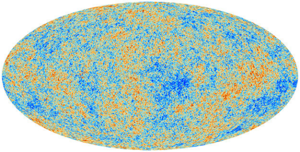
What we have learned up to now:

1) The Universe undergone an inflationary phase driven by the inflaton

2) During this phase, space time is almost de Sitter

3) At superhorizon scales ($H\eta << 1$) the theory should exhibit 3D conformal symmetry at equal time hypersurfaces (dS/CFT Correspondence)

The CMB sky is not exactly isotropic. Density anisotropies at the time of recombination are imprinted as temperature anisotropies in the CMB today.



The anisotropies are divided into two types:

1) primary anisotropy, due to effects which occur at the last scattering surface and before and

2) secondary anisotropy, due to effects such as interactions of the background radiation with hot gas or gravitational potentials, which occur between the last scattering surface and the observer.

The structure

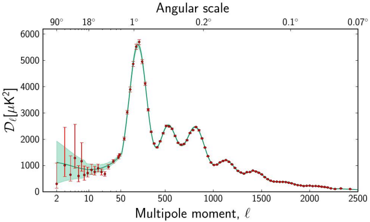
of the cosmic microwave

background anisotropies is

principally determined by two

effects: acoustic oscillations

and diffusion (Silk) damping.



The acoustic oscillations arise because of a conflict in the photonbaryon plasma in the early universe which gives the microwave background its characteristic peak structure. The peaks correspond, roughly, to resonances in which the photons decouple when a particular mode is at its peak amplitude.

The peaks contain interesting physical signatures. The angular scale of the first peak determines the curvature of the universe (but not the topology of the universe). The next peak - ratio of the odd peaks to the even peaks - determines the reduced baryon density. The third peak can be used to get information about the dark matter density. The locations of the peaks also give important information about the nature of the primordial density perturbations.

There are two fundamental types of density perturbations: **adiabatic** and **isocurvature**. A general density perturbation is a mixture of both, and different theories that try to explain the primordial density perturbation spectrum predict different mixtures.

Types of density perturbations:

1) **Adiabatic density perturbations:**

The fractional additional density of each type of particle (baryons, photons ...) is the same.

Cosmic inflation predicts that the primordial perturbations are adiabatic.

2) **Isocurvature density perturbations:**

In each place the sum (over different types of particle) of the fractional additional densities is zero.

Cosmic strings would produce mostly isocurvature primordial perturbations.

The CMB spectrum can distinguish between these two because these two types of perturbations produce different peak locations. Isocurvature density perturbations produce a series of peaks whose angular scales (ℓ-values of the peaks) are roughly in the ratio 1:3:5:..., while adiabatic density perturbations produce peaks whose locations are in the ratio 1:2:3:... Observations are consistent with the primordial density perturbations being entirely adiabatic, providing key support for inflation, and ruling out many models of structure formation involving, for example, cosmic strings.

Detection of non-adiabatic fluctuations immediately rule out single-field inflation models.

What is measured today is that

$$\frac{|\frac{\delta\rho_c}{\rho_c} - \frac{3\delta\rho_\gamma}{4\rho_\gamma}|}{\frac{1}{2}|\frac{\delta\rho_c}{\rho_c} - \frac{3\delta\rho_\gamma}{\rho_\gamma}|} < 0.09 \quad (95\% CL) \tag{45}$$

Are there other quantities which allow us discriminate between various inflationary models, consistent though with Planck data?

It is customary in cosmology to express the observables such as temperature or polarization anisotropies or large scale distribution of galaxies in terms of curvature perturbations in the uniform density gauge denoted by ζ. There is a formalism (called δN-formalism), which relates ζ with the perturbations δN in the number of e-folds N,arising from the perturbation of the initial scalar field ϕ_{in} in flat gauge

$$\zeta(x, t) = \delta N(x, t) \tag{46}$$

But what is the origin of the cosmological perturbations?

Is it the scalar field that drives inflation (inflaton), or

Is it another (scalar) field (curvaton)?

1st possibility:

Density perturbation are generated by the inflaton. In this case we get

$$\zeta = \delta N = \frac{\delta N}{\delta \phi}\delta\phi = \frac{H}{\dot{\phi}}\delta\phi\Big|_{k=aH} \tag{47}$$

with power spectrum

$$P_{\zeta} = \frac{1}{2}\left(\frac{H}{2\pi M_P \epsilon^{1/2}}\right)^2 \left(\frac{k}{aH}\right)^{n_s-1} \tag{48}$$

and spectral index

$$n_s = 1 + 2\eta - 6\epsilon \tag{49}$$

2nd possibility:

perturbations are generated by fields other that the inflaton

$$\sigma \neq \phi \tag{50}$$

These are the curvaton models.

Both types of models of the 1st and 2nd possibility predict:

- negligible tensor modes (i.e., $r = 16\epsilon$ for the inflaton)
- almost scale invariant spectrum (spectral index close to unity)

Quest: How to discriminate inflationary models?

Scalar spectral index of curvature perturbations and the tensor-to-scalar amplitude ratio is not enough to distinguish between inflationary models that are degenerate on the basis of their power spectra alone.

Basic assumption in deriving the spectrum of perturbations is that they are Gaussian.

(Gaussian $\Longleftrightarrow$ free non-interacting fields, collection of harmonic oscillators. No mode-mode coupling)

We should go beyond the linear theory.

Non-Gaussianity goes beyond the linear theory. Primordial NG is one of the most informative finger prints of the origin of structure in the Universe, probing physics at extremely high energy scales inaccessible to laboratory experiments. Possible departures from a purely Gaussian distribution of the CMB anisotropies provide powerful observational access to this extreme physics

Primordial NG in single-field slow-roll models of inflation is suppressed by the slow-roll parameter

$$f_{NL} \sim \mathcal{O}(\epsilon, \eta) \sim 10^{-2} \tag{51}$$

Similarly, in the squeezed limit

$$f_{NL} \sim n_s - 1 \,, \qquad k_1 << k_2, k_3 \tag{52}$$

Therefore if NG is observed in this configuration, all single field models are ruled out.

In multifield models, the NG of the curvature perturbation is sourced by light fields other than the inflaton. By the δN formalism, the comoving curvature perturbation ζ on a uniform energy density hypersurface at time t_f is, on sufficiently large scales,

$$\zeta(t_\mathrm{f}, \vec{x}) = N_I \sigma^I + \frac{1}{2} N_{IJ} \sigma^I \sigma^J + \cdots \,, \tag{53}$$

where N_I and N_{IJ} are the first and second derivative, respectively, of the number of e-folds

$$N(t_\mathrm{f}, t_*, \vec{x}) = \int_{t_*}^{t_\mathrm{f}} \mathrm{d}t \, H(t, \vec{x}). \tag{54}$$

with respect to the field σ^I.

From the expansion (53) one can read off the n-point correlators. For instance, the three- and four-point correlators of the comoving curvature perturbation, the so-called bispectrum and trispectrum respectively,

$$B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \tag{55}$$

$$T_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle \tag{56}$$

is given by

$$B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = N_I N_J N_K B^{IJK}_{\vec{k}_1 \vec{k}_2 \vec{k}_3} + N_I N_{JK} N_L \left(P^{IK}_{\vec{k}_1} P^{JL}_{\vec{k}_2} + 2\,\mathrm{perm.} \right) \tag{57}$$

$$
\begin{aligned}
T_\zeta(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = {} & N_I N_J N_K N_L T^{IJKL}_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} \\
& + N_{IJ} N_K N_L N_M \left(P^{IK}_{\vec{k}_1} B^{JLM}_{\vec{k}_{12}\vec{k}_3\vec{k}_4} + 11 \text{ perm.} \right) \\
& + N_{IJ} N_{KL} N_M N_N \left(P^{JL}_{\vec{k}_{12}} P^{IM}_{\vec{k}_1} P^{KN}_{\vec{k}_3} + 11 \text{ perm.} \right) \\
& + N_{IJK} N_L N_M N_N \left(P^{IL}_{\vec{k}_1} P^{JM}_{\vec{k}_2} P^{KN}_{\vec{k}_3} + 3 \text{ perm.} \right),
\end{aligned}
$$

where

$$
\begin{aligned}
\langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \rangle &= (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2) P^{IJ}_{\vec{k}_1}, \\
\langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \sigma^K_{\vec{k}_3} \rangle &= (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B^{IJK}_{\vec{k}_1\vec{k}_2\vec{k}_3}, \\
\langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \sigma^J_{\vec{k}_3} \sigma^L_{\vec{k}_4} \rangle &= (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 \vec{k}_4) T^{IJKL}_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4}, \qquad (58)
\end{aligned}
$$

We see that the three-point correlator of ζ is the sum of two pieces. One, proportional to the three-point correlator of the σ^I fields, is model-dependent and present when the fields σ^I are intrinsically NG. The second one is universal and is generated when the modes of the fluctuations are superhorizon and is present even if the σ^I fields are gaussian. Even though the intrinsically NG contributions to the *n*-point correlators are model-dependent, their forms are dictated by the conformal symmetry of the de Sitter stage (although their amplitudes remain model-dependent).

Let us consider now the constraints imposed by scale and conformal invariance to the *n*-point correlators. Rotation and translation invariance require correlators of the operators at points $\vec{x}_1$ and $\vec{x}_2$ to depend on $|\vec{x}_1 - \vec{x}_2|$. As is well known, the correlator of two operators is completely determined by their scale dimensions whereas the functional form of 3-pt correlator is also determined by their dimensions.

$$\langle \sigma^I(\vec{x}_1)\, \sigma^J(\vec{x}_2) \rangle = \frac{c_{IJ}}{|\vec{x}_1 - \vec{x}_2|^{\Delta_I + \Delta_J}}, \tag{59}$$

$$\langle \sigma^I(\vec{x}_1) \sigma^J(\vec{x}_2) \sigma^K(\vec{x}_3) \rangle = \frac{c_{IJK}}{|\vec{x}_1 - \vec{x}_2|^{w_K} |\vec{x}_2 - \vec{x}_3|^{w_I} |\vec{x}_3 - \vec{x}_1|^{w_J}},$$

where $(w_I + w_J + w_K) = \Delta_I + \Delta_J + \Delta_K = 3\Delta$.

In momentum space

$$\langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \rangle' = c_{IJ}\, k_1^{\Delta_I + \Delta_J - 3}, \tag{60}$$

$$\begin{aligned} \langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \sigma^K_{\vec{k}_3} \rangle' &= c_{IJK} 2^{7-3\Delta} \pi^{\frac{5}{2}} \frac{\Gamma(3 - \frac{3\Delta}{2})\Gamma(\frac{3-\Delta_K}{2})}{\Gamma(\frac{\Delta_I}{2})\Gamma(\frac{\Delta_J}{2})} \\ &\times \; k_1^{3\Delta - 6} \int_0^1 \mathrm{d}u \frac{(1-u)^{\frac{1}{2}-\frac{\Delta_I}{2}} u^{\frac{1}{2}-\frac{\Delta_J}{2}}}{[(1-u)X + uY]^{3-\frac{3\Delta}{2}}} \\ &\times \; {}_2F_1\left(3 - \frac{3\Delta}{2}, \frac{\Delta_K}{2}, \frac{3}{2}, \mathcal{Z}\right) + \text{cyclic}, \end{aligned} \tag{61}$$

where

$$X = \frac{k_2^2}{k_1^2}, \qquad Y = \frac{k_3^2}{k_1^2}, \qquad \mathcal{Z} = 1 - \frac{u(1-u)}{(1-u)X + uY}. \tag{62}$$

This form of the 3-pt fuction is very general and one should consider various limits of it. The so-called squeezed limit $k_1 \ll k_2 \sim k_3$ of the 3-pt function is particularly interesting from the observationally point of view because it is associated to the simplest model of NG, the so-called local one in which the total initial adiabatic curvature is a local function of its gaussian counterpart $\zeta_{\rm g}$, *e.g.*

$$\zeta = \zeta_{\rm g} + \frac{3 f_{\rm NL}^{local}}{5}(\zeta_{\rm g}^2 - \langle \zeta_{\rm g}^2 \rangle) + \cdots$$

The local model leads to pronounced effects of NG on the clustering of DM halos and to strongly scale-dependent bias.

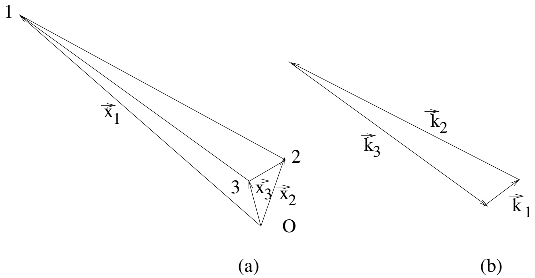


Figure: (a) Squeezed three-point configuration with two points (b) Local shape in k-space with $k_1 \ll k_2 \sim k_3$.

Applying the squeezed limit to the general expression we find

$$\langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \sigma^K_{\vec{k}_3} \rangle' \sim \gamma_s \frac{c_{IJK}}{k_1^{3-2w} k_2^{3-w}} + \text{cyclic} \qquad (k_1 \ll k_2 \sim k_3). \quad (63)$$

The OPE is a very powerful tool to analyze the squeezed limit of the three-point correlator and the collapsed and squeezed limit of the four-point correlator. Let us consider two generic operators $\sigma^I(\vec{x})$ and $\sigma^J(\vec{y})$ at the points $\vec{x}$ and $\vec{y}$ on a $\tau = \mathrm{constant}$ hypersurface of de Sitter spacetime. Then, we expect that the product of local operators at distances small compared to the characteristic length of the system should look like a local operator short-distance expansion of the form

$$\sigma^I(\vec{x})\sigma^J(\vec{y}) \overset{\vec{x}\to\vec{y}}{\sim} \sum_n C_n(\vec{x}-\vec{y}; g)\mathcal{O}_n(\vec{y}), \tag{64}$$

where $C_n(\vec{x}-\vec{y})$ are c-number functions (in fact distributions), $\mathcal{O}_n$ local operators and g is the coupling.

Let us consider the OPE

$$\sigma^I(\vec{x})\sigma^J(\vec{y}) \overset{\vec{x}\to\vec{y}}{\sim} \sum_K C_K^{IJ}(\vec{x}-\vec{y};g)\sigma^K(\vec{y}) + \cdots \tag{65}$$

The n- and $(n+1)$-point functions are given by

$$g_{n+1}^{I_1\cdots I_{n+1}}(x_1,\ldots,x_{n+1};\mu,g) = \langle\sigma^{I_1}(x_1)\cdots\sigma^{I_{n+1}}(x_{n+1})\rangle', \tag{66}$$

$$g_n^{I_1\cdots I_n}(x_1,\ldots,x_n;\mu,g) = \langle\sigma^{I_1}(x_1)\cdots\sigma^{I_n}(x_n)\rangle', \tag{67}$$

where μ a mass scale.

These correlators satisfy the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \sum_I \gamma_I\right) g_i^I = 0, \qquad (i = n, n+1), \tag{68}$$

where β is the usual β-function and γ_I the anomalous dimension of σ^I. Using the OPE expansion one finds immediately that

$$g_{n+1}^{I_1 \ldots I_{n+1}} = \sum_K C_K^{I_n I_{n+1}} g_n^{I_1 \ldots I_{n-1} K}. \tag{69}$$

Then, the coefficients of the OPE expansion are also satisfy the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_I + \gamma_J - \gamma_K\right) C_K^{IJ}(x, y; \mu, g) = 0. \tag{70}$$

In particular, for a a conformal field theory for which $\beta = 0$, dimensional arguments and the fact that renormalized operators can be chosen such that they do not depend μ lead to

$$C_K^{IJ}(x, y; g) = \frac{C^{IJ}(g)}{x^{2w_I + 2w_J - 2w_K}}, \tag{71}$$

where $w_{I,J,K}$ are the dimensions of the fields σ^I, σ^J and σ^K, respectively. Therefore we can write the OPE

$$\sigma^I(\vec{x})\sigma^J(\vec{y}) \overset{\vec{x}\to\vec{y}}{\sim} \sum_K \frac{C^{IJ}(g)}{|\vec{x} - \vec{y}|^{2w_I + 2w_J - 2w_K}} \sigma^K(\vec{y}) + \cdots \tag{72}$$

If we wish to consider the three-point correlator in the squeezed limit, the configuration in real space is such that two points, say $\vec{x}_1$ and $\vec{x}_2$ are very close and the third one very far. Let us therefore consider the OPE expansion for the two fields σ^I and σ^J in the (12) channel at the coincident point

$$\sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2) = \left(\frac{C_0^{IJ}}{x_{12}^{2w}} + \frac{C^{IJ}{}_M}{x_{12}^{w}} \sigma^M(\vec{x}_2) + \cdots \right). \qquad (73)$$

Here $w \simeq m^2/3H^2 \ll 1$, where m is the mass of the fields, is the conformal weight of the fields involved (remember that the weight of the fields σ^I and σ^I must be the same due to the special conformal symmetry). The three-point correlator in the squeezed limit can be evaluated by employing the OPE as

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\rangle = \Big\langle \left(\frac{C_0^{IJ}}{x_{12}^{2w}} + \frac{C_A^{IJ}}{x_{12}^{w}}\sigma^A(\vec{x}_2) + \cdots \right) \sigma^K(\vec{x}_3) \Big\rangle . (74)$$

Using again the orthogonality of the two-point functions we get

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\rangle = \frac{{C^{IJ}}_A}{x_{12}^{w}}\langle \sigma^A(\vec{x}_2)\sigma^K(\vec{x}_3)\rangle = \frac{C^{IJK}}{x_{12}^{w}x_{23}^{2w}} \quad (x_{12} \simeq 0). (75)$$

For an almost scale invariant spectrum $w \approx 0$, the Fourier transform of Eq. (75) is

$$\langle \sigma^I_{\vec{k}_1} \sigma^J_{\vec{k}_2} \sigma^K_{\vec{k}_3} \rangle' \sim C^{IJK} P_{\vec{k}_1} P_{\vec{k}_2} \left[1 + \mathcal{O}\left(\frac{k_1^2}{k_2^2} \right) \right], \quad (k_1 \ll k_2 \sim k_3). (76)$$

The non-universal contribution to the three-point correlator in the squeezed limit has therefore the same shape of the universal contribution. Its amplitude is model-dependent.

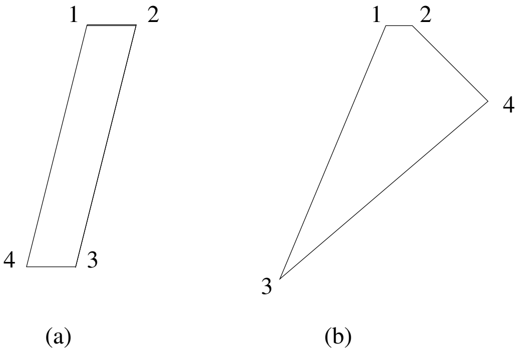


Figure: (a) Collapsed configuration projected on a plane in space where $x_{12} \approx x_{34} \approx 0$ with $x_{13} \gg x_{12}, x_{34}$. (b) Double squeezed configuration where $x_{34} \approx x_{13} \gg x_{24} \gg x_{12} \approx 0$.

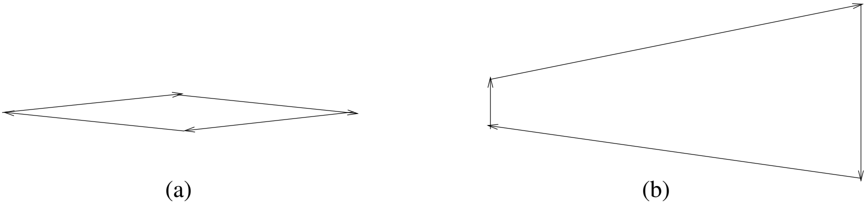


Figure: (a) Collapsed and (b) double squeezed shapes in momentum space.

If we wish to consider the four-point correlator in the collapsed limit, the configuration in real space is such that two pairs of points, say $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$, $\vec{x}_4$ are very far from each other. Let us therefore consider the OPE expansion (73) as well as the one for the other (34) channel at the coincident point

$$\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4) = \left(\frac{C_0^{KL}(w)}{x_{34}^{2w}} + \frac{C^{KL}{}_M(w)}{x_{34}^{w}} \sigma^M(\vec{x}_4) + \cdots \right). \qquad (77)$$

The four-point function in the collapsed limit

$$\langle \sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4)\rangle \qquad (x_{12} \simeq 0 \text{ and } x_{34} \simeq 0) \quad (78)$$

can be expressed as

$$\langle\sigma^I(\vec{x}_1)\sigma^J(\vec{x}_2)\sigma^K(\vec{x}_3)\sigma^L(\vec{x}_4)\rangle = \frac{C_0^{IJ}C_0^{KL}}{x_{12}^{2w}x_{34}^{2w}} + \frac{C^{IJ}{}_A C^{KL}{}_B}{x_{12}^{w}x_{34}^{w}}\langle\sigma^A(\vec{x}_2)\sigma^B(\vec{x}_4)\rangle + \cdots$$

whose Fourier transforms keeping the connected contribution gives

$$\langle\sigma^I_{\vec{k}_1}\sigma^J_{\vec{k}_2}\sigma^K_{\vec{k}_3}\sigma^L_{\vec{k}_4}\rangle' \sim C^{IJ}{}_A C^{KLA} P_{\vec{k}_{12}} P_{\vec{k}_2} P_{\vec{k}_4} + \text{ perm.}, \quad (\vec{k}_{12} \simeq \vec{0}).$$

The non-universal contribution to the four-point correlator in the collapsed limit has therefore the same shape of the universal contribution. Its amplitude is model-dependent.

The collapsed limit of the four-point correlator is particularly important because, together with the squeezed limit of the three-point correlator, it may lead to the so-called Suyama-Yamaguchi (SY) inequality . Consider a class of multi-field models which satisfy the following conditions:

a) scalar fields are responsible for generating curvature perturbations and

b) the fluctuations in scalar fields at the horizon crossing are scale invariant and **gaussian**.

By defining the nonlinear parameters $f_{\rm NL}$ and $\tau_{\rm NL}$ as

$$\begin{aligned} f_{\rm NL} &= \frac{5}{12}\frac{\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle'}{P^{\zeta}_{\vec{k}_1}P^{\zeta}_{\vec{k}_2}} \quad (k_1 \ll k_2 \sim k_3), \\ \tau_{\rm NL} &= \frac{1}{4}\frac{\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\zeta_{\vec{k}_4}\rangle'}{P^{\zeta}_{\vec{k}_1}P^{\zeta}_{\vec{k}_3}P^{\zeta}_{\vec{k}_{12}}} \quad (\vec{k}_{12} \simeq 0), \end{aligned} \tag{79}$$

and making use of the Cauchy-Schwarz inequality one can prove the SY inequality (at the tree-level)

$$\tau_{\rm NL} \geq (6 f_{\rm NL}/5)^2$$

where the equality holds in the case of a single scalar field.

But is the SY inequality valid for non-gaussian fluctuations as one might expect a contamination of the inequality if the light scalar fields are NG at horizon crossing.

By using OPEs of σ^I's and the Cauchy-Schwarz inequality, we got

$$\boxed{\tau_{\rm NL} \geq \left(\frac{6}{5} f_{\rm NL}\right)^2} \quad \text{(also for NG fields)}. \tag{80}$$

Therefore, the SY inequality is valid in all multifield models where the NG comes from light scalar fields other than the inflaton even when such light scalar fields are NG at horizon crossing. Loop corrections from the universal superhorizon NG part of the comoving curvature perturbation were shown not to change SY.

- Observationally, inflation has proven to be quite a robust paradigm, but we are still ignorant about many details: what mechanism is responsible for the cosmological perturbations?
- Even after Planck, there exists a huge class of possible inflationary models and we should go beyond linear theory.

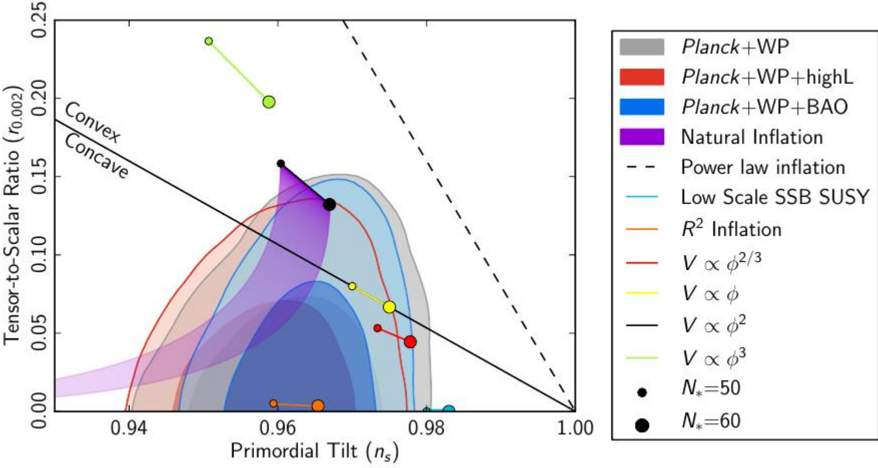


Figure: (a) Collapsed and (b) double squeezed shapes in momentum space.

- Non-Gaussianity is a powerful probe to ask what mechanism is responsible for the cosmological perturbations
- The symmetries of de Sitter are a powerful probe to characterize the shapes of non-Gaussianity and to tell us which are the relevant fields during inflation

THANK YOU