From $\mathcal{N} = 4$ SYM to $\mathcal{N} = 2$ SuperConformal QCD Spin Chains and Holography

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arXiv:0912.4918 and arXiv:1006.0015 with Abhijit Gadde and Leonardo Rastelli
arXiv:1105.3972 with Pedro Liendo and Leonardo Rastelli
arXiv:1105.3487 with Christoph Sieg
discussions with Matthias Staudacher
work in progress with Pedro Liendo, Leonardo Rastelli and Christoph Sieg

Motivation: $\mathcal{N} = 4$ universality class

How general is the AdS/CFT?

- Only for a sparse set of 4-dim gauge theories we have *quantitative* evidence of an "exact" duality.
    - $D3$ branes in **critical string theory**.
    - Adjoint and bifundamental matter.
      (Fundamental flavors in the **probe** approximation)
    - Susy can be broken but there are always remnants of the transverse dimensions.
- Search for the gravity dual of **more "realistic"** theories:
    - Theories with **"genuinely" less susy**.
    - **Unquenched flavor**.

Previous work in this direction

1.
- $\exists$ constructions with **non-zero beta function** (for example for pure $\mathcal{N}=1$ SYM (Klebanov-Strassler, Maldacena-Nunez)) count as a physical "existence proofs" of the string duals.
- **But** FT low energy dynamics $\leftrightarrow$ **large curvature** on the string side.

2.
- $\mathcal{N}=1$ super QCD in the Seiberg **conformal** window has been argued to be dual to $6d$ non-critical backgrounds of the form $AdS_5 \times S^1$. (Klebanov-Maldacena, Fotopoulos-Niarchos-Prezas, Murthy-Troost,...)
- **However** SCFTs in the Seiberg window are strongly coupled, **isolated fixed points**.

The next simplest case: $\mathcal{N} = 2$ SuperConformal QCD

- $\mathcal{N} = 2$ SCQCD has an **exactly marginal coupling**, which can take **arbitrary values**!
- There is a weakly coupled Lagrangian description for $\lambda \to 0$.
- **Bottom-up** approach (perturbative analysis of the Dilatation operator, Wilson loops, Instantons, Scattering Amplitudes . . .).
- Use the **spin chain as a tool** to discover the **closed string**. (The same integrable structures arise in both sides . . .)

Outline

1. The theory
   - $\mathcal{N} = 2$ SCQCD
   - The interpolating theory
2. Spin Chains
3. Search for the Gravity dual

$\mathcal{N} = 2$ SuperConformal QCD (SCQCD)

$U(1) \times SU(2)_R$

$\mathcal{N} = 2$ vector multiplet **adjoint** in $SU(N_c)$:

$$\begin{matrix} & A_\mu & \\ \lambda^1_\alpha & & \lambda^2_\alpha \\ & \phi & \end{matrix} \;, \quad \lambda^{\mathcal{I}} = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} , \quad \mathcal{I} = 1, 2$$

$\mathcal{N} = 2$ hyper multiplet **fundamental** in $SU(N_c)$ and $U(N_f)$:

$$\begin{matrix} & \psi_{\alpha\, i} & \\ q_i & & (\tilde{q})^*_i \\ & \left(\tilde{\psi}_\alpha\right)^\dagger_i & \end{matrix} \;, \quad Q^{\mathcal{I}} = \begin{pmatrix} q \\ \tilde{q}^* \end{pmatrix} , \quad i = 1, \ldots N_f$$

$\beta = \frac{g^3_{YM}}{16\pi^2} (N_f - 2N_c) = 0$ $N_f = 2N_c$ **exactly marginal coupling**!

It should have an AdS dual description with $\lambda \equiv g^2_{YM} N_c \longrightarrow \left(\frac{R_{AdS}}{\ell_{st}}\right)^4$.

Veneziano expansion

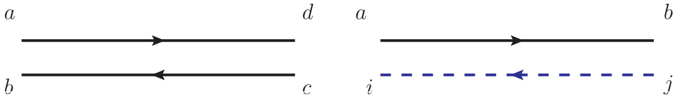
The theory admits a *Veneziano* expansion:

$$N_c \to \infty \quad \text{and} \quad N_f \to \infty$$

$$\text{with} \quad \frac{N_f}{N_c} \quad \text{and} \quad \lambda = g_{YM}^2 N_c \quad \text{kept fixed.}$$

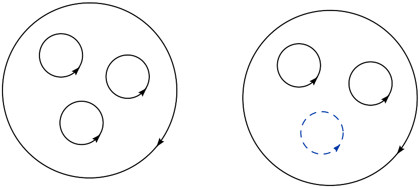
(Veneziano 1976)

Generalized double line notation:



Consequences of the Veneziano expansion

The two diagrams are of the same order $N_c \sim N_f$



Operators will mix:

$$\mathcal{O} \sim \mathrm{Tr}\left(\phi^{\ell}\bar{\phi}\right) + \mathrm{Tr}\left(\phi^{\ell-1}q_i\bar{q}^i\right)$$

Closed string states $\longrightarrow$ **"generalized single-trace"** operators

$$\mathrm{Tr}\left(\phi^{k_1}\mathcal{M}^{\ell_1}\phi^{k_2}\ldots\phi^{k_n}\mathcal{M}^{\ell_n}\right)\,, \quad \mathcal{M}^a{}_b \equiv \sum_{i=1}^{N_f} q^a{}_i\,\bar{q}^i{}_b\,, \quad a, b = 1, \ldots, N_c$$

One parameter family of $\mathcal{N} = 2$ SCFT

$\mathcal{N} = 2$ SCQCD is part of an **one parameter family** $\mathcal{N} = 2$ SCFT with a product gauge group $SU(N_c) \times SU(N_{\check{c}})$ and $N_{\check{c}} = N_c$, **two exactly marginal couplings** g and $\check{g}$. (define $\kappa = \check{g}/g$)

- For $\check{g} \to 0$ one recovers $\mathcal{N} = 2$ SCQCD *plus* a decoupled free vector multiplet.
- For $\check{g} = g$ one finds the well-known $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. (Kachru-Silverstein, Lawrence-Nekrasov-Vafa,. . .)

This theory has a global $SU(2)_L$ symmetry $\forall g, \check{g}$ values.

For $\check{g} = 0$ **symmetry enhancement**: $SU(N_{\check{c}}) \times SU(2)_L \to U(N_f = 2N_c)$.

The interpolating orbifold theory

$\mathcal{N}=2$ vector multiplet adjoint in $SU(N_c)$: $\left(\phi,\ \lambda^{\mathcal{I}}_{\alpha},\ \mathcal{F}_{\alpha\beta}\right)^{a}{}_{b}$

$\mathcal{N}=2$ vector multiplet adjoint in $SU(N_{\check{c}})$: $\left(\check{\phi},\ \check{\lambda}^{\mathcal{I}}_{\alpha},\ \check{\mathcal{F}}_{\alpha\beta}\right)^{\check{a}}{}_{\check{b}}$

$\mathcal{N}=2$ hypermultiplet bifundamental in $SU(N_c)$ and $SU(N_{\check{c}})$:

$$\left(Q^{\mathcal{I}},\ \psi_{\alpha},\ \bar{\tilde{\psi}}_{\alpha}\right)^{\hat{\mathcal{I}}\ a}{}_{\check{a}} \qquad \hat{\mathcal{I}}=1,2 \quad SU(2)_L$$

For $\check{g}=0$ **symmetry enhancement**: $(\hat{\mathcal{I}}\,,\ \check{a})\rightarrow i$
$SU(2)_L\times SU(N_{\check{c}})\rightarrow U(N_f=2N_c)$.

Spin chains

Spin chains – Integrability vs “Solvability”

$\mathcal{N} = 4$ SYM

- The $\mathcal{N} = 4$ spin chain is **integrable** (Minahan, Zarembo, . . .).
- $\exists$ the $\mathcal{N} = 4$ Scattering Matrix $\forall\lambda$ (Staudacher, Beisert,. . .).
- The complete determination of the exact operator spectrum of the planar $\mathcal{N} = 4$ SYM theory is within reach (Gromov, Kazakov, Vieira,. . .).

Beyond $\mathcal{N} = 4$

- Even if the spin chain is not integrable it is a very useful tool.
  - Calculate the **2-body** scattering matrix from both sides and compare.
- **“Solvability”** has to do with the size of the supergroup.
  - Scattering matrix for $SU(2|2)$ without further assumption.
- One-loop **pure QCD spin chain** has an **integrable subsector** self-dual field strengths with an arbitrary number of derivatives. (Beisert, Ferretti, Heise, Zarembo)

Constructing the Spin Chain

$\mathcal{N} = 4$ SYM

Each site hosts a "letter" that belongs to the single **ultrashort singleton** multiplet:

$$V_F = \mathcal{D}^n \left(X,\ Y,\ Z,\ \bar{X},\ \bar{Y},\ \bar{Z},\ \lambda^A_\alpha,\ \bar{\lambda}^{\dot{\alpha}}_A,\ \mathcal{F}_{\alpha\beta},\ \bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}} \right)$$

The **state space** at each lattice site is ∞-dim $\mathcal{V}_\ell = V_F$.

The total space is just $\otimes^L_\ell \mathcal{V}_\ell$.

$\mathcal{N} = 2$ SCQCD

Has **three** distinct irreducible superconformal representations:

- $\mathcal{V}$ the vector multiplet with shortening condition $\Delta = -r$ $\left(\bar{\mathcal{Q}}_{\mathcal{I}\,\dot{\alpha}} |h.w.\rangle = 0 \right)$
- $\bar{\mathcal{V}}$ the conjugate vector multiplet with $\Delta = r$
- $\mathcal{H}$ the hypermultiplet (real representation) with $\Delta = 2R$

The Spin Chain of $\mathcal{N} = 2$ SCQCD

The **state space** at each lattice site is ∞-dim, spanned by

$$\mathcal{V}_\ell = \{\mathcal{V}\,, \quad \bar{\mathcal{V}}\,, \quad \mathcal{H}\,, \quad \bar{\mathcal{H}}\}$$

$$\mathcal{V} = \mathcal{D}^n \left(\phi\,,\, \lambda^{\mathcal{I}}_\alpha\,,\, \mathcal{F}_{\alpha\beta}\right)^a{}_b \quad , \quad \mathcal{H} = \mathcal{D}^n \left(Q^{\mathcal{I}}\,,\, \psi\,,\, \bar{\tilde{\psi}}\right)^a{}_i$$

The index structure imposes **restrictions** on the total space $\otimes_\ell^L \mathcal{V}_\ell$:

$$\cdots\, \phi\, \phi\, Q_i\, \bar{Q}^i\, \phi\, \phi\, \cdots \to \cdots\, \phi\, \phi\, \mathcal{M}\, \phi\, \phi\, \cdots$$

We may use instead the **color-adjoint objects**:

$$\phi \qquad \bar{\phi} \qquad \mathcal{M}_1 \qquad \mathcal{M}_3$$

where the **flavor contracted mesons** $\mathcal{M}$ are viewed as "**dimers**" occupying two sites of the chain.

Elementary excitations

$\mathcal{N} = 4$ SYM

- Choice of vacuum $\mathrm{tr} Z^\ell$ with $\Delta - J = 0$.
- $8 + 8$ **elementary excitations** with $\Delta - J = 1$:

  $\lambda_\alpha^A, X, \bar{X}, Y, \bar{Y}, \mathcal{D}_{\alpha\dot{\alpha}}$ with $A = 1, \ldots, 4$ the $SU(4)$ index.
- $\Delta - J \geq 2$: $\bar{Z}, \mathcal{F}_{\alpha\beta}, \ldots$ are **composite states**.

Elementary excitations

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$\mathcal{N} = 2$ SCQCD

- Choice of vacuum $\mathrm{tr}\phi^\ell$ with $\Delta - J = 0$.
- $4 + 4$ **elementary excitations** with $\Delta - J = 1$:

  $\lambda_\alpha^{\mathcal{I}}$ and $\mathcal{D}_{\alpha\dot{\alpha}}$ with $\mathcal{I} = 1, 2$ the $SU(2)_R$ index.
- $\Delta - J \geq 2$: $\mathcal{T}$, $\widetilde{\mathcal{T}} = \phi\bar{\phi} \pm \mathcal{M}_{\mathbf{1}}$, $\mathcal{M}_{\mathbf{3}}$, $\mathcal{F}_{\alpha\beta}$ $\dots$ are **composite states**.

Elementary excitations: integrability at 1-loop

- $\lambda^{\mathcal{I}}_{\alpha}$ in the sea of ϕ's is **identical** to the $\mathcal{N} = 4$ paradigm:

$$E_\lambda(p) = 8\sin^2\left(\frac{p}{2}\right)$$

Sector	Scattering Matrix
$\mathbf{3}_\alpha \otimes \mathbf{3}_R$	-1
$\mathbf{1}_\alpha \otimes \mathbf{3}_R$	$\mathcal{S}_{XXX}$
$\mathbf{1}_\alpha \otimes \mathbf{1}_R$	-1

($\mathbf{3}_\alpha \otimes \mathbf{3}_R$ is the $SU(1|1)$ subsector – closed to all loops)

- The **non-compact** $SU(1,1)$ **bosonic sector** (closed to all loops): ϕ's and one kind of **derivative** $\mathcal{D}_{+\dot{+}}$ is also **identical** to $\mathcal{N} = 4$!

Scalar excitations (1-loop)

$$H_{\ell,\ell+1} = \begin{array}{c} \phi\phi \\ Q\bar{Q} \\ \bar{Q}Q \\ \phi Q \end{array} \overset{\begin{array}{cccc} \phi\phi & Q\bar{Q} & \bar{Q}Q & \phi Q \end{array}}{\begin{pmatrix} 2\mathbb{I} + \mathbb{K} - 2\mathbb{P} & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 2\,(2\mathbb{I} - \mathbb{K}) & 0 & 0 \\ 0 & 0 & 2\mathbb{K} & 0 \\ 0 & 0 & 0 & 2\mathbb{I} \end{pmatrix}}$$

Diagonalize the **"scalar"- impurities** in the "sea" of ϕ's:

$$\mathcal{T} = \bar{\phi}\phi - \mathcal{M}_{\mathbf{1}} \quad \text{with} \quad E = 4\sin^2\left(\frac{p}{2}\right)$$

$$\widetilde{\mathcal{T}} = \bar{\phi}\phi + \mathcal{M}_{\mathbf{1}} \quad \text{with} \quad E = 8$$

$$\mathcal{M}_{\mathbf{3}} \quad \text{with} \quad E = 8$$

- **"single"- impurities** of the 1-loop scalar sector of SCQCD are really **composite** (dimeric - they occupy two sites) and $\Delta - J = 2$.

Composite States with $\Delta - J = 2$

$\mathcal{N} = 4$ SYM

$$
\begin{aligned}
& \cdots Z\, Z\, \bar{Z}\, Z\, Z \cdots \\
\xhookrightarrow{H_1} \quad & \cdots Z\, Z \left(X \bar{X} + Y \bar{Y} \right) Z\, Z \cdots \\
\xhookrightarrow{H_1} \quad & \cdots Z\, Z\, X\, Z\, \bar{X}\, Z\, Z \cdots
\end{aligned}
$$

Interpretation: 2-body problem, $Z\bar{Z}$ is a **composite state** of $X\bar{X}$ and $Y\bar{Y}$.

Composite States with $\Delta - J = 2$

$\mathcal{N} = 4$ SYM

$$\cdots Z\, Z\, \bar{Z}\, Z\, Z \cdots$$
$$\xhookrightarrow{H_1} \cdots Z\, Z \left(X\bar{X} + Y\bar{Y}\right) Z\, Z \cdots$$
$$\xhookrightarrow{H_1} \cdots Z\, Z\, X\, Z\, \bar{X}\, Z\, Z \cdots$$

Interpretation: 2-body problem, $Z\bar{Z}$ is a **composite state** of $X\bar{X}$ and $Y\bar{Y}$.

$\mathcal{N} = 2$ SCQCD

$$\cdots \phi\, \phi\, \mathcal{T}\, \phi\, \phi \cdots$$
$$\xhookrightarrow{H_{\ell>1}} \cdots \phi\, \phi\, \lambda\, \lambda\, \phi\, \phi \cdots$$
$$\xhookrightarrow{H_1} \cdots \phi\, \phi\, \lambda\, \phi\, \lambda\, \phi\, \phi \cdots$$

Interpretation: $\mathcal{T}$, $\widetilde{\mathcal{T}}$ and $\mathcal{M}_3$ are **composite states** of two λs.

"Regularizing" $\mathcal{N} = 2$ SCQCD

- Consider the interpolating orbifold theory (SCQCD $\kappa \to 0$)

  $\kappa = \frac{\check{g}}{g}$ should be thought of as a **regulator**.

- We **regularize by inserting** $\check{\phi}$**s between the** Q**s** giving the dimeric impurities the possibility to split:

$$\cdots \; \phi \, \phi \, Q \, \check{\phi} \, \check{\phi} \, \cdots \, \check{\phi} \, \check{\phi} \, \bar{Q} \, \phi \, \phi \, \cdots$$

- Now the Qs can move independently

$$\Delta - J = 1$$

  and can be interpreted as **elementary excitations**!

- Back to $8 + 8$ **elementary excitations** with $\Delta - J = 1$.

Scalar Hamiltonian of the interpolating theory

$$H_{k,k+1} =$$

$$\begin{array}{c|cccccccc}
 & \phi\phi & Q\bar{Q} & \check{\phi}\check{\phi} & \bar{Q}Q & \phi Q & Q\check{\phi} & \check{\phi}\bar{Q} & \bar{Q}\phi \\
\hline
\phi\phi & (2+\mathbb{K}-2\mathbb{P}) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q\bar{Q} & 1 & (2-\mathbb{K})\hat{\mathbb{K}}+2\kappa^2\mathbb{K} & 0 & 0 & 0 & 0 & 0 & 0 \\
\check{\phi}\check{\phi} & 0 & 0 & \kappa^2(2+\mathbb{K}-2\mathbb{P}) & \kappa^2 & 0 & 0 & 0 & 0 \\
\bar{Q}Q & 0 & 0 & \kappa^2 & \kappa^2(2-\mathbb{K})\hat{\mathbb{K}}+2\mathbb{K} & 0 & 0 & 0 & 0 \\
\phi Q & 0 & 0 & 0 & 0 & 2 & -2\kappa & 0 & 0 \\
Q\check{\phi} & 0 & 0 & 0 & 0 & -2\kappa & 2\kappa^2 & 0 & 0 \\
\check{\phi}\bar{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 2\kappa^2 & -2\kappa \\
\bar{Q}\phi & 0 & 0 & 0 & 0 & 0 & 0 & -2\kappa & 2
\end{array}$$

where $\kappa \equiv \frac{g}{\check{g}}$

- Recall that $Q^{\mathcal{I}\hat{\mathcal{I}}}$ with $\mathcal{I} \in SU(2)_R$ and $\hat{\mathcal{I}} \in SU(2)_L$.

Scalar impurities in the interpolating theory (1-loop)

- We have two vacua $\mathrm{tr}\phi^\ell$ and $\mathrm{tr}\check{\phi}^\ell$.

Single Q in the sea of ϕ's: $\ldots\ \phi\,\phi\,\phi\,Q\,\check{\phi}\,\check{\phi}\,\check{\phi}\ \ldots$ (open chain)

The dispersion relation:

$$g^2 E(p) = 2(g-\check{g})^2 + 8\,g\,\check{g}\sin^2\left(\frac{p}{2}\right)$$

- the **gap is zero only** when $g=\check{g}$ ($\mathcal{N}=4$ **orbifold** (Beisert-Roiban))
- for $\check{g}\to 0$: $E(p)=2$ the Q's **freeze** – can not move any more.

2-body scattering: The Scattering matrix factorizes

Using the S-matrix of the **XXZ** chain, with $\Delta_{XXZ} = \kappa$

$$\mathcal{S}(p_1, p_2, \kappa) \equiv -\frac{1 - 2\kappa e^{ip_1} + e^{i(p_1+p_2)}}{1 - 2\kappa e^{ip_2} + e^{i(p_1+p_2)}}$$

The scattering matrix of two Qs:

$$S(p_1, p_2; \kappa) = \frac{S_L(p_1, p_2; \kappa) S_R(p_1, p_2; \kappa)}{S_{3\otimes 3}(p_1, p_2; \kappa)}$$

$SU(2)_L$	$S_L(p_1, p_2; \kappa)$	$SU(2)_R$	$S_R(p_1, p_2; \kappa)$
1	$\mathcal{S}(p_1, p_2; \kappa - \frac{1}{\kappa})$	**1**	-1
3	$\mathcal{S}(p_1, p_2; \kappa)$	**3**	$\mathcal{S}(p_1, p_2; \kappa)$

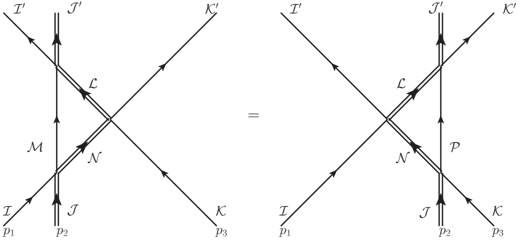
2-body scattering: Bound states

Besides the continuum of states with real momenta p_1 and p_2, there can be **bound** and **“anti-bound”** states for special complex values of the momenta.

$$\mathcal{T} \qquad\qquad E = 4\sin^2\left(\frac{P}{2}\right)$$

$$\widetilde{\mathcal{T}} \text{ and } \mathcal{M}_3 \qquad E = \frac{4\kappa^2}{(1-\kappa^2)}\left(\frac{2}{\kappa^2} - 1 - \sin^2\left(\frac{P}{2}\right)\right)$$

where $P = p_1 + p_2$ is the total momentum of $Q(p_1)$ and $\bar{Q}(p_2)$.



Necessary but not sufficient condition for integrability.

- $g = \check{g}$: **satisfied** (orbifold is known to be integrable (Beisert-Roiban))
- generic $g \neq \check{g}$: **not satisfied** (it is not integrable)
- $\check{g} = 0$: **satisfied** (is $\mathcal{N} = 2$ SCQCD integrable?)

The Complete-one loop Hamiltonian

$\mathcal{N} = 4$ SYM

All "letters" belong to the single **ultra-short singleton** multiplet.

$$V_F \otimes V_F = \sum_{j=0}^{\infty} \mathcal{V}_j$$

Identify a simple closed subsector of the theory, such that each of the $\mathcal{V}_j$ modules contains a representative within the subsector.

- $\mathcal{D}^n Z$ "letters" $\rightarrow SU(1,1)$ subsector (compute Feynman diagrams)
- $\mathcal{D}^n \lambda$ "letters" $\rightarrow SU(1,1) \times U(1|1)$ subsector (algebraic evaluation)

where $\mathcal{D} \equiv \mathcal{D}_{+\dot{+}}$ and $\lambda \equiv \lambda_{+}^{1}$

$$H_{\ell,\ell+1} = \sum_{j=0}^{\infty} 2h(j)\mathcal{P}_{\ell,\ell+1}^{(j)} \qquad \text{with} \quad h(j) \equiv \sum_{k=1}^{j} \frac{1}{k}$$

(Beisert)

The Complete one-loop Hamiltonian

$\mathcal{N} = 2$ there are three singleton representations

$$\begin{aligned}
\mathcal{H} \otimes \mathcal{H} &= \hat{\mathcal{B}}_1 \oplus \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})} \\
\mathcal{H} \otimes \mathcal{V} &= \bigoplus_{q=-1}^{\infty} \hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q}{2})} = \mathcal{V} \otimes \mathcal{H} \\
\mathcal{H} \otimes \bar{\mathcal{V}} &= \bigoplus_{q=-1}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q+1}{2})} = \bar{\mathcal{V}} \otimes \mathcal{H} \\
\mathcal{V} \otimes \mathcal{V} &= \bar{\mathcal{E}}_{2(0,0)} \oplus \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q+1}{2},\frac{q-1}{2})} \\
\bar{\mathcal{V}} \otimes \bar{\mathcal{V}} &= \mathcal{E}_{2(0,0)} \oplus \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q-1}{2},\frac{q+1}{2})} \\
\mathcal{V} \otimes \bar{\mathcal{V}} &= \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{0(\frac{q}{2},\frac{q}{2})} = \bar{\mathcal{V}} \otimes \mathcal{V}
\end{aligned}$$

All modules contain a representative in $D^n_{++\dot{+}}\left(\lambda^1_+\,,\,\bar{\lambda}^1_{\dot{+}}\,,\,Q^1\,,\,\bar{Q}^1\right)$

- whose Hamiltonian we compute either by Feynman diagrams or by a **purely algebraic** approach.

The Complete one-loop Hamiltonian

$\underline{\mathcal{V} \times \mathcal{V}}$:

$$H_{12} = 0 \times \mathcal{P}_{\bar{\mathcal{E}}} + \sum_{q=0}^{\infty} 4\, h(q+1) \mathcal{P}_{(\frac{q+1}{2}, \frac{q-1}{2})}$$

the same as in $\mathcal{N} = 4$.

$\underline{\mathcal{V} \times \mathcal{H} \leftrightarrow \mathcal{H} \times \mathcal{V}}$:

$$H_{12} = 2 \sum_{q=-1}^{\infty} \begin{pmatrix} h(q+2) + h(q+1) & -\frac{\kappa}{q+2} \\ -\frac{\kappa}{q+2} & \kappa^2 (h(q+2) + h(q+1)) \end{pmatrix} \mathcal{P}_{(\frac{q+1}{2}, \frac{q}{2})}$$

A quick check: for ϕQ and $Q \check{\phi}$: $\quad q = -1 \quad \longrightarrow \quad H_{12} = \begin{pmatrix} 2 & -2\kappa \\ -2\kappa & 2\kappa^2 \end{pmatrix}$

$\underline{\mathcal{V} \times \bar{\mathcal{V}} \leftrightarrow \bar{\mathcal{V}} \times \mathcal{V} \leftrightarrow \mathcal{H} \times \mathcal{H}}$: $\quad 3 \times 3$ matrix.

$h(k)$ are the harmonic numbers $h(k) = \sum_{j=1}^{k} \frac{1}{j}$ and $h(0) \equiv 0$.

We need to go to higher loops

We have:

- **Dispersion relations** and **Scattering matrices** at 2-loops.
  For subsectors that include **only color adjoint fields** from one color group (e.g. $SU(1|1)$ or $SU(1,1)$) they are the same as in $\mathcal{N}=4$.

- **Dispersion relation** of Q to 3-loops and the Hamiltonian of a very special scalar sector.

To do:

- $SU(1|1)$ and $SU(1,1)$ to 3-loops.

- At 2-loops three bosons can turn into two fermions:

$$\phi Q \bar{Q} \to \lambda^{\alpha} \lambda_{\alpha}$$

- Explicitly get $\mathcal{T}$ or $\widetilde{\mathcal{T}}$ and $\mathcal{M}_3$ as **composite states of two** λ**s**.

Symmetries in the excitation picture

$\mathcal{N} = 4$ SYM Choise of the BMN vacuum $\mathrm{tr}Z^\ell$ breaks

$$PSU(2,2|4) \to PSU(2|2)_R \times PSU(2|2)_L \times \mathbb{R}$$

where the common central factor $\mathbb{R}$ corresponds to the Hamiltonian.

$\mathcal{N} = 2$ orbifold Further breaks:

$$PSU(2|2)_L \to SU(2)_L \times SU(2)_\alpha$$

The preserved symmetry in the excitations picture is:

$$SU(2|2)_R \times SU(2)_L \times SU(2)_\alpha$$

$\mathcal{N} = 2$ SCQCD

$SU(2)_L$ singlet: $SU(2|2)_R \times SU(2)_\alpha$

Beisert's all loop Scattering Matrix

$\mathcal{N} = 4$ SYM

	$SU(2)_{\dot\alpha}$	$SU(2)_R$	$SU(2)_\alpha$	$SU(2)_L$
$SU(2)_{\dot\alpha}$	$\mathcal{L}^{\dot\alpha}{}_{\dot\beta}$	$\mathcal{Q}^{\dot\alpha}{}_{\mathcal{J}}$	$\mathcal{D}^{\dagger\dot\alpha}{}_{\beta}$	$\lambda^{\dagger\dot\alpha}{}_{\hat{\mathcal{J}}}$
$SU(2)_R$	$\mathcal{S}^{\mathcal{I}}{}_{\dot\beta}$	$\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}$	$\lambda^{\dagger\mathcal{I}}{}_{\beta}$	$\mathcal{X}^{\dagger\mathcal{I}}{}_{\hat{\mathcal{J}}}$
$SU(2)_\alpha$	$\mathcal{D}^{\alpha}{}_{\dot\beta}$	$\lambda^{\alpha}{}_{\mathcal{J}}$	$\mathcal{L}^{\alpha}{}_{\beta}$	$\mathcal{Q}^{\alpha}{}_{\hat{\mathcal{J}}}$
$SU(2)_L$	$\lambda^{\hat{\mathcal{I}}}{}_{\dot\beta}$	$\mathcal{X}^{\hat{\mathcal{I}}}{}_{\mathcal{J}}$	$\mathcal{S}^{\hat{\mathcal{I}}}{}_{\beta}$	$\mathcal{R}^{\hat{\mathcal{I}}}{}_{\hat{\mathcal{J}}}$

- The **broken generators** (Goldstone excitations) and correspond to **gapless magnons**.
- The impurities transform in the fundamental of $SU(2|2)$

$$\Delta - |r| = 2\,\mathcal{C} = \sqrt{1 + h\,(g)\,\sin^2\left(\frac{p}{2}\right)}$$

- The two-body S-matrix is fixed by Beisert's centrally extended $SU(2|2) \times SU(2|2)$ symmetry. (Beisert)

$\mathcal{N}=2$ SCQCD

	$SU(2)_{\dot\alpha}$	$SU(2)_R$	$SU(2)_\alpha$	$SU(2)_L$
$SU(2)_{\dot\alpha}$	$\mathcal{L}^{\dot\alpha}{}_{\dot\beta}$	$\mathcal{Q}^{\dot\alpha}{}_{\mathcal{J}}$	$\mathcal{D}^{\dagger\dot\alpha}{}_{\beta}$	$\psi^{\dagger\dot\alpha}{}_{\hat{\mathcal{J}}}$
$SU(2)_R$	$\mathcal{S}^{\mathcal{I}}{}_{\dot\beta}$	$\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}$	$\lambda^{\dagger\mathcal{I}}{}_{\beta}$	$\bar{Q}^{\mathcal{I}}{}_{\hat{\mathcal{J}}}$
$SU(2)_\alpha$	$\mathcal{D}^{\alpha}{}_{\dot\beta}$	$\lambda^{\alpha}{}_{\mathcal{J}}$	$\mathcal{L}^{\alpha}{}_{\beta}$	
$SU(2)_L$	$\psi^{\hat{\mathcal{I}}}{}_{\dot\beta}$	$Q^{\hat{\mathcal{I}}}{}_{\mathcal{J}}$		$\mathcal{R}^{\hat{\mathcal{I}}}{}_{\hat{\mathcal{J}}}$

The broken generators → Goldstone excitations → Gapless magnons
Non-existing generators → non-Goldstone excitations → Gapped magnons

$$2\,\mathcal{C}_{\lambda,\mathcal{D}} = \sqrt{1+8\mathbf{g}^2\sin^2\left(\tfrac{p}{2}\right)} \qquad 2\,\mathcal{C}_{Q,\psi} = \sqrt{1+2(\mathbf{g}-\check{\mathbf{g}})^2+8\mathbf{g}\check{\mathbf{g}}\sin^2\left(\tfrac{p}{2}\right)}$$

$$\mathbf{g}^2 = h(g,\check{g}) = g^2+\cdots \qquad (\mathbf{g}-\check{\mathbf{g}})^2 = f_1(g,\check{g}) = (g-\check{g})^2+\cdots$$

$$\check{\mathbf{g}}^2 = \check{h}(g,\check{g}) = \check{g}^2+\cdots \qquad \mathbf{g}\check{\mathbf{g}} = f_2(g,\check{g}) = g\check{g}+\cdots$$

The S-matrix of highest weight states in $SU(2)_\alpha$ and $SU(2)_L$ is fixed by the centrally extended $SU(2|2)$. (Gadde, Rastelli)

3-loop calculation in the interpolating theory

$\mathcal{N} = 4$ SYM

No loop corrections!! (... Sieg ...)

$$h(g) = g^2$$

$\mathcal{N} = 2$ SCFT

We find maximum transandentality $\zeta(3)$ contributions at three loops (that disappears at the orbifold point $g = \check{g}$)

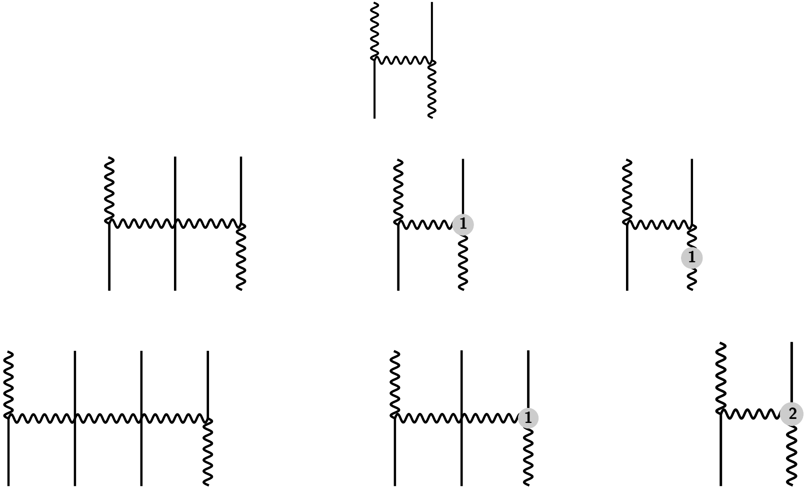
$$(\mathbf{g} - \check{\mathbf{g}})^2 = f_1(g, \check{g}) = (g - \check{g})^2 - \zeta(3)\,(g - \check{g})^2\left(g^4 + g\check{g}\left(g^2 + \check{g}^2\right) + \check{g}^4\right)$$

$$\mathbf{g}\check{\mathbf{g}} = f_2(g, \check{g}) = g\check{g} \;-\; \zeta(3)\,g\check{g}\,\left(g^2 - \check{g}^2\right)^2 + \cdots$$

ABJM

Four-loop contribution is proportional to $\zeta(2)$ and hence homogeneous maximal transcendental in three dim. (Minahan, Sax & Sieg)

$SU(1|1)$ – a good subsector for explicit computations



$$H_{\mathcal{N}=2} - H_{\mathcal{N}=4} \sim \lambda^3 \left(\mathbb{I} - \mathbb{P}\right)$$

A nice subsector

A **guess to be checked**:

a non-compact $SU(2,1|2)$ subsector:

$$\phi \qquad \lambda^{\mathcal{I}}_{+} \qquad \mathcal{D}_{+\dot{\alpha}}$$

closed to all loops

enjoys $SU(2|2)_R$ symmetry

(we know the **scattering matrix to all orders** in λ)

and **should be integrable**!

Guessing the **gravity dual**

of $\mathcal{N} = 2$ SCQCD

by computing its **Chiral Spectrum**.

Chiral spectrum of $\mathcal{N} = 2$ SCQCD

Start by matching **the chiral spectrum** of the gauge theory to the *KK* **spectrum** of the appropriate supergravity.

- $\mathcal{N} = 4$ SYM a half-BPS multiplet that is protected $\rightarrow$ one and only gravity multiplet of type *IIB* supergravity on $AdS_5 \times S^5$.

The **flavor singlet** protected states of $\mathcal{N} = 2$ SCQCD more complicated:

- Chiral Ring: $\mathrm{tr}\mathcal{M}_{\mathbf{3}}, \ \mathrm{tr}\,\phi^{\ell}$
- $\mathrm{tr}\left(T\,\phi^{\ell}\right)$
- “extra” multiplets that include **higher spin** states.

$T = \phi\bar{\phi} - \left(Q\bar{Q}\right)_{\mathbf{1}}$ is the highest weigh state in the $T_{\mu\nu}$ multiplet.

Chiral spectrum of $\mathcal{N} = 2$ SCQCD

Start by matching **the chiral spectrum** of the gauge theory to the *KK* **spectrum** of the appropriate supergravity.

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The **flavor singlet** protected states of $\mathcal{N} = 2$ SCQCD more complicated:

- Chiral Ring: $\mathrm{tr}\mathcal{M}_{\mathbf{3}}$, $\mathrm{tr}\,\phi^\ell$ (KK reduction of 6d tensor multiplet on $AdS_5 \times S^1$)
- $\mathrm{tr}\left(T\,\phi^\ell\right)$ (7d sugra (Samtleben & Weidner 2005) on $AdS_5 \times S^1 \times \mathcal{I}$)
- "extra" multiplets that include **higher spin** states.

$T = \phi\bar{\phi} - (Q\bar{Q})_{\mathbf{1}}$ is the highest weigh state in the $T_{\mu\nu}$ multiplet.

The string dual of the Interpolating SCFT

Type IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_2$

with

$$\beta \equiv \int_{S^2} B_{NS}$$

(Kachru-Silverstein, Lawrence-Nekrasov-Vafa,. . .)

$$\frac{1}{g_{YM}^2} + \frac{1}{\check{g}_{YM}^2} = \frac{1}{2\pi g_{st}} \qquad \text{and} \qquad \frac{\check{g}_{YM}^2}{g_{YM}^2} = \frac{\beta}{1-\beta}$$

(Nekrasov-Klebanov, Graña-Polchinski)

- $g_{YM} = \check{g}_{YM} \longleftrightarrow \beta = 1/2$ at the orbifold point (Aspinwall).
- To get $\mathcal{N} = 2$ SCQCD we must take $\beta \to 0$.
  But this limit is singular!

Conclusions

Bottom up:

- We already have a lot results concerning the Spin Chain description.
  - The complete picture is emerging!
- Wilson loops . . . (Passerini & Zarembo)
- Amplitudes . . .

Top Down:

- We need to start working on the gravity side . . .
- The interpolating SCFT is a good place to start.
  type IIB on $AdS_5 \times S^5/\mathbb{Z}_2$ (Kachru-Silverstein, Klebanov-Nekrasov,. . .)
  - S-matrix of the giant magnons
  - BMN oscillators
  - . . .

Conclusions: Take Home message

$\mathcal{N} = 2$ SCQCD is perhaps **the simplest** theory outside the $\mathcal{N} = 4$ universality class.

Continuously connected to the $\mathcal{N} = 4$ class by an interpolating $\mathcal{N} = 2$ SCFT.

It has a large $SU(2,2|2)$ to exploit!

May be the new **“solvable”** toy!