SUPERCONFORMAL INDEX

A WINDOW ON DUALITY

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Realistic quantum field theories are difficult to get to grips with outside perturbation theory except in some cases using lattice methods and intensive computer simulations.

The fundamental insight of Wilson was that QFT's have a scale μ, related to the cut off, and the QFT 'flows' in the space of theories, with given symmetries and field content, as μ is varied.

The flow has fixed points which are often 'universal' independent of the particular starting theory.

The goal in a non perturbative understanding of QFT is to determine possible fixed points and the qualitative structure of the RG flows between them.

In perturbation theory flow equations for the couplings are typically

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g^i = \beta^i(g)$$

where

$$t = \ln \mu \to \infty \quad \text{is an UV fixed point} \qquad t \to -\infty \quad \text{is an IR fixed point}$$

IR fixed points control low energy physics.

Massive fields decouple, only massless fields remain.

At an IR fixed point scale invariance $\Rightarrow$ conformal invariance (usually)

Conformal field theories are relevant to the fixed points of RG flow: minimal data are the scaling dimensions, spins and other quantum numbers of the operators.

Conventional wisdom < 1994 non free conformal field theories in four dimensions are rare.

After 1994 conformal field theories in four dimensions appear to be very common. Their presence depends on non trivial interactions between gauge fields and fermions and chiral symmetries play a crucial role. Until recently hard to simulate on a lattice.

For QCD like theories, $SU(N)$ gauge group N_f flavours

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right) + \frac{g^5}{(16\pi^2)^2}\left(-\frac{34}{3}N^2 + \frac{10}{3}NN_f + \frac{N^2-1}{N}N_f\right)$$

For the right choice of N_f the first term is negative, the second is positive

$$\beta(g_*) = 0 \quad \text{is an IR fixed point}$$

For g_* small approximation is credible but of limited validity.

Supersymmetric theories are easier to analyse. Additional constraints due to R-symmetry and analyticity of the superpotential. There are arguments for fixed points away from the perturbative regime. In suitable limits these match on to the perturbative fixed points.

In 1994 Seiberg analysed $\mathcal{N} = 1$ supersymmetric gauge theories which are the direct SUSY extensions of QCD.

$SU(N)$ gauge group

Flavour symmetry

$SU(N_f) \quad \times \quad SU(N_f) \quad \times \quad U(1)_B \quad \times \quad U(1)_R$

vector & axial symmetries   baryon charge R-charge

Each 2-component fermion $\rightarrow$ chiral scalar superfield

A theory with vector gauge fields and no gauge anomalies requires two 'quark' chiral superfields $Q, \tilde{Q}$ belonging to the fundamental representation and its conjugate

Gauge fields are contained in a chiral spinor superfield λ_α belonging to the adjoint rep

Field	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
Q	f	f	1	$1/N$	r_Q
$\tilde{Q}$	$\bar{f}$	1	$\bar{f}$	$-1/N$	r_Q
λ_α	adj.	1	1	0	1

f = fundamental representation dimension, dimension N for $SU(N)$

$\bar{f}$ is conjugate

adj. is the adjoint, dimension $N^2 - 1$

In addition there are conjugate anti-chiral fields

R-symmetry is anomaly free, unique in this theory

r_Q is the R-charge for Q, $r_\lambda = 1$ from conformal symmetry and gauge invariance

For chiral scalar fields at a superconformal fixed point scale dimension $\Delta = \frac{3}{2}r$

From unitarity $\Delta \geq 1$ or $r \geq \frac{2}{3}$

$r = \frac{2}{3}$ is a free theory

But there may be R-charges $< \frac{2}{3}$ in a confining theory

In the $\mathcal{N} = 1$ theory with confinement physical gauge singlet states are mesons $Q\tilde{Q}$ and baryons Q^N, $\tilde{Q}^N$

$Q\tilde{Q}$ has R-charge $2r_Q$, $Q^N, \tilde{Q}^N$ R-charge Nr_Q

Require $2r_Q \geq \frac{2}{3}$

For this theory there is an exact β-function formula

$$\beta_g(g) = -\frac{g^3}{16\pi^2}\frac{3N - N_f + 2N_f\gamma_Q(g)}{1 - \frac{g^2}{8\pi^2}N}, \qquad \gamma_Q(g) = -\frac{g^2}{16\pi^2}\frac{N^2-1}{N} + \dots$$

$\gamma_Q(g)$ is the anomalous dimension for $Q, \tilde{Q}$ chiral fields.

There is then an IR fixed point for

$$\gamma_Q(g_*) = -\frac{3N - N_f}{2N_f} \quad \text{for} \quad N_f \lesssim 3N$$

The scale dimension of $Q, \tilde{Q}$

$$\Delta_Q = 1 + \gamma_Q(g_*) = \frac{3}{2}\frac{N_f - N}{N_f} = \frac{3}{2}r_Q$$

Expect confinement for $N_f < 3N$, $r_Q < \frac{2}{3}$

$r_Q > \frac{1}{3}$ for $N_f > \frac{3}{2}N$, $\frac{3}{2}N \leq N_f \leq 3N$ is the conformal window

There is also a dual magnetic theory with the same flavour symmetries

This theory has the same IR fixed point

Electric and Magnetic theories are in the same universality class

Duality: strong coupling electric $\leftrightarrow$ weak coupling magnetic

Dual theory is a $SU(\tilde{N})$ gauge theory, $\tilde{N} = N_f - N$

Chiral scalar fields $q, \tilde{q}$

Gauge chiral fermion field $\tilde{\lambda}_\alpha$

Additional gauge singlet chiral scalar M

Field	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
q	f	$\bar{f}$	1	$1/\tilde{N}$	r_q
$\tilde{q}$	$\bar{f}$	1	f	$-1/\tilde{N}$	r_q
$\tilde{\lambda}_\alpha$	adj.	1	1	0	1
M	1	f	$\bar{f}$	0	r_M

$q, \tilde{q}, M$ become free when $r_q = r_M = \frac{2}{3}$

Asymptotic freedom requires $N_f > \frac{3}{2}N$

baryons $Q^N \sim \tilde{q}^{\tilde{N}}$, $\tilde{Q}^N \sim q^{\tilde{N}}$, $M \sim Q\tilde{Q}$ or $Nr_Q = \tilde{N}r_q$, $r_M = 2r_Q$.

There is a superpotential $\tilde{q}qM$ with R-charge 2

This ensures $q\tilde{q} \sim 0$, require $2r_q + r_M = 2$.

For the magnetic theory there are two couplings

Gauge coupling g, also y where $W = y\,\tilde{q}Mq$

$$\beta_g(g,y) = -\frac{g^3}{16\pi^2}\,\frac{3\tilde{N} - N_f + 2N_f\gamma_q(g,y)}{1 - \frac{g^2}{8\pi^2}\tilde{N}}$$

$$\beta_y(g,y) = y\big(\gamma_M(g,y) + 2\gamma_q(g,y)\big)$$

At a fixed point $g = g_*, y = y_*$

$$-\gamma_{M*} = 2\gamma_{q*} = -\frac{3\tilde{N} - N_f}{N_f}$$

Hence

$$\Delta_q = 1 + \gamma_{q*} = \frac{3}{2}\,\frac{N_f - \tilde{N}}{N_f} = \frac{3}{2}r_q$$

$$\Delta_M = 1 + \gamma_{M*} = 3\,\frac{\tilde{N}}{N_f} = \frac{3}{2}r_M$$

Duality has many consistency checks

Most significant is the 't Hooft anomaly matching condition

Coefficient of triangle anomaly for fermion flavour currents has no dynamical corrections

Should match between theory and its dual

Non trivial tests from matching of anomalies for

$SU(N_f)^3$, $SU(N_f)^2U(1)_B$, $SU(N_f)^2U(1)_R$, $U(1)_B^2U(1)_R$, $U(1)_R$, $U(1)_R^3$,

$$\begin{aligned}
SU(N_f)^3: &\quad N = -\tilde{N} + N_f \\
SU(N_f)^2U(1)_R: &\quad N(r_Q-1) = \tilde{N}(r_q-1) + N_f(r_M-1) \\
U(1)_B^2U(1)_R: &\quad \frac{N_f}{N}(r_Q-1) = \frac{N_f}{\tilde{N}}(r_q-1) \\
U(1)_R: &\quad 2NN_f(r_Q-1) + N^2 - 1 = 2\tilde{N}N_f(r_q-1) + \tilde{N}^2 - 1 + N_f^2(r_M-1) \\
U(1)_R^3: &\quad 2NN_f(r_Q-1)^3 + N^2 - 1 = 2\tilde{N}N_f(r_q-1)^3 + \tilde{N}^2 - 1 + N_f^2(r_M-1)^3
\end{aligned}$$

Describe here further very non trivial tests based on a supersymmetric index

Assume a SUSY charge $\mathcal{Q}$ and conjugate $\mathcal{Q}^+$ such that

$$\{\mathcal{Q}, \mathcal{Q}^+\} = 2\mathcal{H}\,, \qquad \mathcal{Q}^2 = 0\,, \qquad [\mathcal{H}, \mathcal{Q}] = 0$$

$\mathcal{Q}, \mathcal{Q}^+$ change the fermion number F by ± 1

Index I counts states $|\psi\rangle$, $\mathcal{Q}|\psi\rangle = 0$, $|\psi\rangle \neq \mathcal{Q}|\phi\rangle$

For such states $\mathcal{H}|\psi\rangle = 0$, and also $\mathcal{Q}^+|\psi\rangle = 0$,

otherwise $\mathcal{H}|\psi\rangle = E|\psi\rangle \Rightarrow |\psi\rangle = \mathcal{Q}\mathcal{Q}^+|\psi\rangle/E$

Must have $E \geq 0$, if $E > 0$ then $\{|\psi\rangle,\ \mathcal{Q}^+|\psi\rangle\}$ is a multiplet with 2-fold degeneracy

For the two states F differs by 1

Define

$$
\begin{aligned}
I &= \mathrm{tr}_{\mathcal{H}=0}\big((-1)^F\big) \\
&= \mathrm{tr}\big((-1)^F\, e^{-\beta\mathcal{H}}\big) \\
&= \mathrm{tr}\big((-1)^F e^{-\beta\mathcal{H}+X\mathcal{Q}+\mathcal{Q}^+X^+\}}\big) \qquad X \text{ arbitrary}
\end{aligned}
$$

If there are hermitian operators $\mathcal{R}$ commuting with $\mathcal{H}$, $\mathcal{Q}$, $\mathcal{Q}^+$ define

$$
I(t) = \mathrm{tr}_{\mathcal{H}=0}\big((-1)^F t^{\mathcal{R}}\big)
$$

I can be expressed as a functional integral, D Sen in 1987, Romelsburger in 2005

$$
I = \int_{\text{periodic fields on } S^3\times S^1} \mathrm{d}[\text{fields}] \quad e^{-S}
$$

S is a general vector, scalar $\mathcal{N}=1$ supersymmetric action on $S^3 \times S^1$, all fields periodic on S^1 up to a twisting to incorporate $t^{\mathcal{R}}$. $\delta_{\mathcal{Q}} S = 0$ where $\delta_{\mathcal{Q}}{}^2 = 0$. Can modify $S \to S + \delta_{\mathcal{Q}} X$. Use to *localise* the action so evaluation becomes a Gaussian integral. I is a topological invariant under deformations of S

Application to $\mathcal{N}=1$ superconformal theories, in 4-dimensions spinor indices $\alpha, \dot{\alpha}=1,2$

The $\mathcal{N}=1$ superconformal algebra involves supercharges $Q_\alpha, \bar{Q}_{\dot{\alpha}}$, momenta $P_{\alpha\dot{\alpha}}$, two $SU(2)$ like angular momentum operators J_3, $J_\pm$, $\bar{J}_3$, $\bar{J}_\pm$, dilation operator H and the R-charge R plus some conformal partners $S^\alpha, \bar{S}^{\dot{\alpha}}$ and $K^{\dot{\alpha}\alpha}$

Label states by $\Delta, r, j, \bar{j}$ eigenvalues of of $H, R, J_3, \bar{J}_3$, Δ is the scale dimension, $\Delta=0$ is the vacuum

Choose $\mathcal{Q}=\bar{Q}_1$, $\mathcal{Q}^+=\bar{S}^1$ then superconformal algebra gives

$$\{\mathcal{Q},\mathcal{Q}^+\}=2\mathcal{H} \qquad \text{for} \qquad \mathcal{H}=H-2\bar{J}_3-\tfrac{3}{2}R$$

This requires $\Delta \geq 2\bar{j}+\frac{3}{2}r$

If $\mathcal{Q}|\psi\rangle=\mathcal{Q}^+|\psi\rangle=0$ and hence $\Delta=2\bar{j}+\frac{3}{2}r$ the representations of the full algebra are short or BPS.

A representation with $\Delta>2\bar{j}+\frac{3}{2}r$ is long,

as $\Delta\rightarrow 2\bar{j}+\frac{3}{2}r$ it may be decomposed into two short representations

In the superconformal algebra $\mathcal{R} = R + 2\bar{J}_3$ and J_3 commute with $\mathcal{H}$. The index is then defined by

$$I(t,x) = \mathrm{tr}_{\mathcal{H}=0}\big((-1)^F t^{\mathcal{R}} x^{2J_3}\big)$$

Chiral free fermion superfield $\varphi \to \psi_\alpha$ then

$\varphi(0)|0\rangle$ has $\Delta = 1$, $r = \frac{2}{3}$, $j = \bar{j} = 0 \qquad \Rightarrow \qquad \mathcal{H}\,\varphi(0)|0\rangle = 0$

For anti-chiral conjugate $\bar{\varphi} \to \bar{\psi}_{\dot{\alpha}}$ then

$\bar{\psi}_2(0)|0\rangle$ has $\Delta = \frac{3}{2}$, $r = -\frac{2}{3} + 1$, $j = 0$, $\bar{j} = \frac{1}{2} \qquad \Rightarrow \qquad \mathcal{H}\,\bar{\psi}_2(0)|0\rangle = 0$

momentum operators P_{12}, P_{22} have $\Delta = 1$, $j = \pm\frac{1}{2}$, $\bar{j} = \frac{1}{2}$, $r = 0$ and so commute with $\mathcal{H}$

For free chiral scalar fields single particle states with $\mathcal{H} = 0$ are $\big\{P_{12}^n P_{22}^m \varphi(0)|0\rangle, P_{12}^n P_{22}^m \bar{\psi}_2(0)|0\rangle\big\}$

For these states

$$i_S(t,x) = \mathrm{tr}_{\mathcal{H}=0}\big((-1)^F t^{\mathcal{R}} x^{2J_3}\big) = \frac{t^{\frac{2}{3}} - t^{\frac{4}{3}}}{(1-tx)(1-tx^{-1})}$$

For the gauge multiplet $\lambda_\alpha \to f_{\alpha\beta}$ and for the conjugate $\bar{\lambda}_{\dot{\alpha}} \to \bar{f}_{\dot{\alpha}\dot{\beta}}$

$\lambda_\alpha(0)|0\rangle$ has $\Delta = \frac{3}{2}$, $r = 1$, $j = \pm\frac{1}{2}$, $\bar{j} = 0 \qquad \Rightarrow \qquad \mathcal{H}\,\lambda_\alpha(0)|0\rangle = 0$

$\bar{f}_{22}(0)|0\rangle$ has $\Delta = 2$, $r = -1+1$, $j = 0$, $\bar{j} = 1 \qquad \Rightarrow \qquad \mathcal{H}\,\bar{f}_{22}(0)|0\rangle = 0$

Since $P_{22}\lambda_1(0)|0\rangle = P_{12}\lambda_2(0)|0\rangle$ a basis of states with $\mathcal{H} = 0$ is then

$\left\{P_{12}^n P_{22}^m \lambda_1(0)|0\rangle, P_{22}^m \lambda_2(0)|0\rangle P_{12}^n P_{22}^m \bar{f}_{22}(0)|0\rangle\right\}$

The corresponding index becomes

$$i_V(t,x) = \mathrm{tr}_{\mathcal{H}=0}\big((-1)^F t^{\mathcal{R}} x^{2J_3}\big) = -\frac{tx}{1-tx} - \frac{tx^{-1}}{1-tx^{-1}}$$

Extension to additional flavour symmetries and gauge group

For flavour and gauge groups there are characters

$\chi_R(y) = \mathrm{tr}_R(y_1^{H_1} y_2^{H_2} \dots)$, H_i Cartan generators, for each flavour group representation R

$\chi_G(z)$ for each gauge group representation G

For $SU(N)$, $\chi_R(y)$ can be expressed as a symmetric homogeneous function of variables $y_1, \dots, y_N$, $\prod_i y_i = 1$

For the conjugate $\chi_{\bar{R}}(y) = \chi_R(y^{-1})$, $\chi_{\bar{G}}(z) = \chi_G(z^{-1})$

Let $p = tx$, $q = tx^{-1}$, then

$$i_S(p,q,y,z) = \frac{t^r \chi_R(y)\chi_G(z) - t^{2-r}\chi_R(y^{-1})\chi_G(z^{-1})}{(1-tx)(1-tx^{-1})} = \frac{\chi_R(u)\chi_G(z) - pq\,\chi_R(u^{-1})\chi_G(z^{-1})}{(1-p)(1-q)}$$

absorbing t in y so that $t^r\chi_R(y) = \chi_R(u)$, $t^{-r}\chi_R(y^{-1}) = \chi_R(u^{-1})$,

Also

$$i_V(p,q,z) = -\left(\frac{p}{1-p} + \frac{q}{1-q}\right)\chi_{\mathrm{adj.}}(z)$$

The total index requires summing over 'multi-particle states' and projecting on gauge singlet states, $i = i_V + i_S$,

$$I(p,q,y) = \int_{\text{gauge group}} \mathrm{d}\mu(z)\, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\, i\big(p^n, q^n, y^n, z^n\big) \right)$$

The group integration is what is left of the functional integral, the exponential is a one loop determinant, each n corresponds to 'n-particle states', (also referred to as the plethystic exponential)

The exponential sums can be evaluated using

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{y^n - p^n q^n / y^n}{(1-p^n)(1-q^n)} \right)$$
$$= \prod_{j,k\geq 0} \frac{1 - y^{-1} p^{j+1} q^{k+1}}{1 - y\, p^j\, q^k} \equiv \Gamma(y; p, q)$$

$$\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\Big)(z^n+z^{-n})\Big)$$
$$=\frac{1}{(1-z)(1-z^{-1})\,\Gamma(z;p,q)\Gamma(z^{-1};p,q)}$$
$$\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{p^n}{1-p^n}+\frac{q^n}{1-q^n}\Big)\Big)=(p;p)\,(q;q)\,,$$

$$(p;p)\equiv\prod_{j\geq 0}(1-p^{j+1})$$

Require $|p|,\ |q|<1$

The infinite product Γ plays an important role, it is an elliptic Gamma function

$$\Gamma(y;p,q)\,\Gamma(pq/y;p,q)=1$$

There are many other non trivial identities

Resumé on infinite products

$$(x;p) = \prod_{j=0}^{\infty}(1 - xp^j)\,, \qquad \theta(x;p) = (x;p)\,(px^{-1};p)$$

θ is related to Jacobi theta functions, $\Gamma(x, x^{-1}; p, q) = 1/\theta(x;p)\theta(x^{-1}, q)$

Crucial property of Γ 'q-difference equation'

$$\Gamma(qx;p,q) = \theta(x;p)\,\Gamma(x;p,q) \qquad \Gamma(px;p,q) = \theta(x;q)\,\Gamma(x;p,q)$$

$f(x;p) \equiv f(x_1, \ldots, x_n; p)$ is p-elliptic if it is invariant under $x_i \to px_i$ for each i

For $n = 1$ since $\theta(px;p) = -\theta(x;p)/x$ can take

$$f(x;p) = \prod_{k=1}^{N} \frac{\theta(t_k x;p)}{\theta(w_k x;p)} \qquad \text{if} \qquad \prod_{k=1}^{N} t_k = \prod_{k=1}^{N} w_k$$

$F(x;p,q) \equiv F(x_1, \ldots x_n; p, q)$ is elliptic hypergeometric if it satisfies q-difference equations

$$F(x;p,q)|_{x_i \to qx_i} = h_i(x;p)\,F(x;p,q) \qquad \text{for} \qquad h_i(x;p)\ p\text{-elliptic} \qquad \text{for all } i$$

Can construct such F's using elliptic Gamma functions. Spiridonov conjecture: ratios of integrands for electric and magnetic indices for dual theories are elliptic hypergeometric

Apply index to Seiberg dual theories. R-charges and flavour and gauge group representations determine i_S, i_V which then determine I.

The equality $I_E = I_M$ is a non trivial test for duality

Consider the simplest non trivial case is $N = 2$, $N_f = 3$

Then $SU(3) \times SU(3) \times U(1)_B \to SU(6)$ since $SU(2)$ representations are self conjugate

$SU(6)$ characters depend on $y_1, y_2, y_3, y_4, y_5, y_6$ subject to $\prod_i y_i = 1$

Gauge group $SU(2)$ characters depend on a single variable z

$$\text{fundamental } \chi_2(z) = z + z^{-1}, \quad \text{adjoint } \chi_3(z) = z^2 + 1 + z^{-2}$$

In the electric theory $Q, \tilde{Q}$ belong to the 6-dimensional fundamental representation of $SU(6)$, $r = \frac{1}{3}$

The dual magnetic theory has gauge group $SU(1)$ which is trivial, implying a free theory

$q, \tilde{q}, M$ belong to the 15-dimensional antisymmetric representation of $SU(6)$, $r = \frac{2}{3}$

single particle indices

$$i_E(p,q,y,z) = -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\left(z^2+1+z^{-2}\right) + \frac{1}{(1-p)(1-q)}\left(\sum_i u_i - pq\sum_i u_i^{-1}\right)\left(z+z^{-1}\right),$$

$$u_i = (pq)^{\frac{1}{6}} y_i \qquad \prod_i u_i = pq$$

$$i_M(p,q,y) = \frac{1}{(1-p)(1-q)}\left(\chi_{15}(u) - pq\,\chi_{15}(u^{-1})\right)$$

$$\chi_{15}(u) = \sum_{1\le i<j\le 6} u_i u_j$$

multiparticle indices

$$
\begin{aligned}
&I_E(p,q,y)\\
&= \int_{SU(2)} \mathrm{d}\mu(z)\ \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_E\big(p^n, q^n, y^n, z^n\big)\right)\\
&= (p;p)\,(q;q)\,\frac{1}{4\pi i}\oint_{|z|=1} \frac{\mathrm{d}z}{z}\,\frac{\prod_{i=1}^{6}\Gamma(u_i z; p,q)\,\Gamma(u_i z^{-1}; p,q)}{\Gamma(z^2;p,q)\,\Gamma(z^{-2};p,q)}
\end{aligned}
$$

$$
\begin{aligned}
I_M(p,q,y) &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_M\big(p^n, q^n, y^n\big)\right)\\
&= \prod_{1\le i<j\le 6} \Gamma(u_i u_j; p,q)
\end{aligned}
$$

The contour encloses infinitely many poles

The identity of I_E and I_M is a very non trivial result found by Spiridonov in 2001

The proof is not an explicit evaluation by summing up residues of poles, but involves show that both sides satisfy the same recurrence relations and are equal in some special limiting case

When $q = 0$ and the integral depends only on p, as well as y, the result was only known in the 1980's

For Seiberg dual theories for gauge group $SU(N)$ flavour group $SU(N_f) \times SU(N_f) \times U(1)_B$

Electric index

$$I_E(p,q,y,y',v)_{SU(N),r}$$

$$= (p;p)^{N-1}(q;q)^{N-1}\frac{1}{N!}\int\prod_{j=1}^{N-1}\frac{\mathrm{d}z_j}{2\pi i z_j}\frac{\prod_{1\le i\le N_f}\prod_{1\le j\le N}\Gamma\big(t^r v^{\frac{1}{N}}\,y_i z_j, t^r v^{-\frac{1}{N}}/(y'_i z_j);p,q\big)}{\prod_{1\le i<j\le N}\Gamma\big(z_i/z_j, z_j/z_i;p,q\big)}\Bigg|_{\prod_{j=1}^{N} z_j=1}$$

where $y=(y_1,\ldots,y_{N_f}),\ y'=(y'_1,\ldots,y'_{N_f})$ correspond to $SU(N_f)\times SU(N_f)$, v to $U(1)_B$ and

$$\Gamma(x_1,x_2;p,q)=\Gamma(x_1;p,q)\,\Gamma(x_2;p,q)$$

Magnetic index has an additional meson contribution

$$I_M(p,q,y,y',v)_{SU(\tilde N),\tilde r}=\prod_{1\le i,j\le N_f}\Gamma(t^{r_M}y_i/y'_j;p,q)\ I_E(p,q,y^{-1},y'^{-1},v)_{SU(\tilde N),\tilde r}$$

Require $y^{-1}=(y_1{}^{-1},\ldots,y_{N_f}{}^{-1}),\ y'^{-1}=(y'_1{}^{-1},\ldots,y'_{N_f}{}^{-1})$ and

$$\prod_{j=1}^{N_f} y_j=1 \qquad \prod_{j=1}^{N_f} y'_j=1 \qquad t^2=pq \qquad r=r_Q \quad \tilde r=r_q$$

Rains proved in 2003

$$I_E(p,q,y,y',v)_{SU(N),r} = I_M(p,q,y,y',v)_{SU(\tilde{N}),\tilde{r}}$$

so long as

$$N+\tilde{N}=N_f \qquad r+\tilde{r}=1 \qquad \tilde{N}\tilde{r}=Nr \qquad r_M=2r$$

The identities only follow for the precise values of the R-charges, and representation content, required by Seiberg duality. The index does not know about the conformal window

There are corresponding results for gauge groups $SO(N)$ and $Sp(2N)$, with flavour symmetry groups $SU(N_f)$ and $SU(2N_f)$, require different but similar integral identities proved by Rains

Conformal windows are respectively

$$\tfrac{3}{2}(N-2) \leq N_f \leq 3(N-2)\,, \quad \tfrac{3}{2}(N+1) \leq N_f \leq 3(N+1)$$

Many extensions to theories with various additional matter fields, leads to as yet unproved identities

Special case $N = 2$, $N_f = 4$, $SU(4) \times SU(4) \times U(1) \rightarrow SU(8)$

Electric theory

Field	$SU(2)$	$SU(8)$	$U(1)_R$
Q	f	f	$\frac{1}{2}$
λ_α	adj.	1	1

Dual magnetic theory, gauge group $SU(2)$

Field	$SU(2)$	$SU(8)$	$U(1)_R$
q	f	$\bar{f}$	$\frac{1}{2}$
$\tilde{\lambda}_\alpha$	adj.	1	1
M	1	A	1

A is the 28-dimensional antisymmetric tensor representation of $SU(8)$, $A \sim Q^2$

However in this case there are multiply dual theories

The formula for the index has a set of discrete symmetries which match these dualities

Conclusions

The index is a very non trivial test of Seiberg duality

Spiridonov has constructed the index for essentially all dual theories constructed in the 1990's and has found many new mathematical identities.

Conversely known identities can be a method of finding new dualities, especially for multiply dual theories

It may be applicable in cases where the usual techniques fail, e.g. in three dimensions where 't Hooft anomaly matching cannot be used

Duality may be significant in realistic theories, no handle on it at present without supersymmetry