Spatially modulated phases in AdS/CFT

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Based on work with J. P. Gauntlett

Outline

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1. Motivation / Introduction
2. A diagnostic for BH perturbative instabilities
3. Holographic stripes
4. Helical superconductors
   - Helical superconducting black holes
5. Final comments

Motivation

- The AdS/CFT correspondence is a powerful tool to study strongly coupled (conformal) quantum field theories
- Interest in application to strongly coupled Condensed Matter Theory systems
  - One focus: systems with strongly coupled "quantum critical points" - phase transition at zero temperature
  - Another focus: thermally driven symmetry breaking phase transitions e.g. superconductivity / superfluidity [Gubser; Hartnoll, Herzog, Horowitz]

$\Rightarrow$ What about spatially modulated order? e.g. spin density waves, charge density waves, stripe phase of underdoped cup rate superconductors, FFLO [Nakamura, Ooguri, Park] [AD, Gauntlett] [Bergman, Jokela, Lifschytz, Lippert]

Normal/broken phase transitions

Normal phase

- Critical points exhibiting full relativistic conformal invariance could be described by AdS geometries in string or M-theory
- The boundary field theory at finite temperate is described by black hole (black brane) solutions asymptoting to AdS
- Finite chemical potential would correspond to a charged black hole (black brane) with the charge carried by a bulk gauge field

Phase transition

- Certain fields can condense due to an instability, at a critical temperature T_c, spontaneously breaking a space-time (density waves, nematic phases) and/or internal symmetry (superfluidity)
- Emergence of new black hole branch

Bottom-up approach

- Study phase transitions at a minimal setting
- The AdS Reissner-Nordström black hole is the canonical example of charged black hole in Einstein-Maxwell theory to play the role of a normal phase
- Couple additional fields which become unstable below a critical black hole temperature T_c

Advantages

- Discover new mechanisms for instabilities/condensation
- Uncovers universal behaviour close to T_c

Disadvantages

- Dual field theory existence not guaranteed
- Low temperature behavior model dependent

Top-down approach

- Consider $AdS_d \times M$ solution of string/M-theory
- Compactify on M to generate an infinite tower of KK modes in d dimensions
- For SUSY compactifications in $d = 4, 5$ there is a consistent truncation with at least a SUGRA multiplet $(g_{\mu\nu}, A_\mu) \rightarrow$ electric RN black hole [Gauntlett, Varela]
- Also possible to retain additional fields (multiplets) in the consistent truncation e.g. $AdS_4 \times SE^7$, $AdS_5 \times SE^5$, $AdS_5 \times H^2 \times S^4$, $AdS_4 \times H^3 \times S^4$

Advantages

- Guaranteed to have a field theory dual
- Couplings, scalar potentials are not arbitrary

Disadvantages

- Hard
- Branches/instabilities outside consistent truncation [AD, Gauntlett]

Plan

- Electric *AdS* RN black hole (normal phase) solution in string/M-theory
- Consider perturbative coupling of minimal $\mathcal{N} = 2$ SUGRA in $D = 4, 5$ to additional fields (multiplets)
- Study stability of normal phase against perturbations
- Embed the mechanism in known string/M-theory reductions

$\rightsquigarrow$ SUGRA couplings can in general break translational invariance!

$\rightsquigarrow$ Rich structure of competing orders in string/M-theory

Outline

Electric AdS_4 RN black hole

The Einstein-Maxwell in $d = 4$ is

$$\mathcal{L}_{EM} = \frac{1}{2} R * 1 + 6 * 1 - \frac{1}{2} F \wedge * F$$

The electrically charged AdS RN black hole is

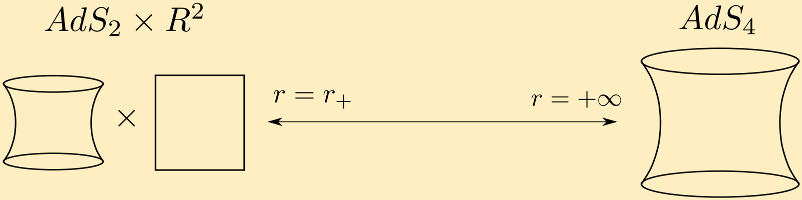
$$ds_4^2 = -f\, dt^2 + \frac{dr^2}{f} + r^2 \left(dx_1^2 + dx_2^2 \right)$$
$$A = \mu \left(1 - \frac{r_+}{r} \right) dt$$
$$f = 2r^2 - \left(2r_+^2 + \frac{\mu^2}{2} \right) \frac{r_+}{r} + \mu^2 \frac{r_+^2}{2r^2}$$

- There is an outer horizon located at $r = r_+$
- Temperature is $T = (12\, r_+^2 - \mu^2)/(8\pi r_+)$, entropy is $s = 2\pi r_+^2$

$\Rightarrow$ Finite entropy at $T = 0$

The extremal limit

- At $T = 0$ the near horizon (IR) limit is $AdS_2 \times \mathbb{R}^2 \to 1$ dim. or the chiral sector of $1 + 1$ dim. CFT



- The $\mathbb{R}^2$ Fourier modes of bulk fields yield a continuum of dual operators $\mathcal{O}_{\vec{k}}$ in the IR CFT
- The modes $\vec{k} \neq 0$ break translations in the UV CFT
- Check unitarity (BF) bound of the IR CFT for all $\vec{k}$
- If for high T RN bh is stable and the IR CFT is unstable, there must be a T_c for the onset of the instability

A prototype instability

The $AdS_2 \times \mathbb{R}^2$ limit is (after rescaling)

$$ds^2 = -12r^2\, dt^2 + \frac{dr^2}{12r^2} + dx_1^2 + dx_2^2, \quad A = 2\sqrt{3}\, rdt$$

Add minimally coupled complex scalar

$$\mathcal{L} = \mathcal{L}_{EM} - \frac{1}{2}\left|D_\mu \psi\right|^2 - \frac{1}{2} m^2 \left|\psi\right|^2$$
$$D_\mu \psi = (\partial_\mu - \imath q\, A_\mu)\psi$$

For modes $\psi = \phi\, e^{\imath \vec{k}\vec{x}}$ the equation of motion gives

$$D^\mu D_\mu^* \phi - \vec{k}^2 \phi - m^2 \phi = 0 \Rightarrow$$
$$m_{eff}^2 = -q^2 + m^2 + \vec{k}^2$$

Violates AdS_2 BF bound if $m_{eff}^2 < -3$ but lightest mode always at $\vec{k} = 0$

Outline

The model

Consider the $d = 4$ theory of gravity $g_{\mu\nu}$ coupled to a gauge field A_μ and a pseudo scalar φ

$$\mathcal{L} = \frac{1}{2} R * 1 - \frac{1}{2} * d\varphi \wedge d\varphi - V(\varphi) * 1 - \frac{1}{2}\tau(\varphi) F \wedge *F - \frac{1}{2}\vartheta(\varphi) F \wedge F$$

- For perturbative considerations, we are interested in the first few terms

$$V = -6 + \frac{1}{2} m_s^2 \varphi^2 + \cdots , \quad \tau = 1 - \frac{n}{12} \varphi^2 + \cdots , \quad \vartheta = \frac{c_1}{2\sqrt{3}} \varphi + \cdots$$

- Purely electric/magnetic RN black hole still solution

$$\begin{aligned} d * d\varphi + V' * 1 + \frac{1}{2}\tau' F \wedge *F + \frac{1}{2}\vartheta' F \wedge F =& 0 \\ d(\tau * F + \vartheta F) =& 0 \end{aligned}$$

The AdS_2 perturbation

Einstein's equations read

$$R_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \tau \left(\frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} - F_{\mu\rho} F_\nu{}^\rho \right) + g_{\mu\nu} V$$

In general, perturbations of the gauge field will mix with metric perturbations

$$\begin{aligned} \delta A_{x_2} =& a(t,r)\sin(kx_1) \\ \delta\varphi =& w(t,r)\cos(kx_1) \\ \delta g_{tx_2} =& 2\sqrt{3}\, r h_{tx_2}(t,r)\sin(kx_1) \\ \delta g_{x_1x_2} =& h_{x_1x_2}(t,r)\cos(kx_1) \end{aligned}$$

The function $h_{x_1x_2}$ can be eliminated from the equations of motion to yield a second order system

The instability

The second order system in matrix form is

$$\Box_{AdS_2}\mathbf{v} - M^2\mathbf{v} = 0, \quad \mathbf{v} = (\phi_{x_1x_2}, a, w)$$

$$M^2 = \begin{pmatrix} k^2 & \frac{1}{\sqrt{3}}k & 0 \\ 24\sqrt{3}k & 24 + k^2 & -c_1 k \\ 0 & -c_1 k & k^2 + \tilde{m}_s^2 \end{pmatrix}, \quad \tilde{m}_s^2 = m_s^2 + n$$

- The mass spectrum is found after diagonalizing to give three AdS_2 masses as functions of k
- The lowest mass matrix eigenvalue m^2_{min} occurs at non-zero k for sufficiently large c_1
- It can violate the AdS_2 BF bound even with a stable $k = 0$ sector

M-theory embedding

Consider skew-whiffed $AdS_4 \times SE^7$ and dimensionally reduce to obtain $\mathcal{N} = 2$ minimal gauged SUGRA coupled to 1 vector mult.

Black hole static modes

To determine T_c we need to construct a static, normalizable mode
The AdS_2 analysis suggests the perturbation

$$\delta g_{tx_2} = \lambda \left[r\,(r - r_+)\; h(r)\,\sin(kx_1) \right]$$
$$\delta A_{x_2} = \lambda \left[a(r)\,\sin(kx_1) \right]$$
$$\delta\varphi = \lambda \left[w(r)\,\sin(kx_1) \right], \qquad \lambda << 1$$

- The equations of motion lead to a system of three second order ODEs for h, a, and w
- Demanding regular perturbation near the black hole horizon leads to the expansion

$$h = h_+ + \mathcal{O}(r - r_+), \quad a = a_+ + \mathcal{O}(r - r_+), \quad w = w_+ + \mathcal{O}(r - r_+)$$

- With the system being linear we can always choose one of the constants of integration to be equal to one

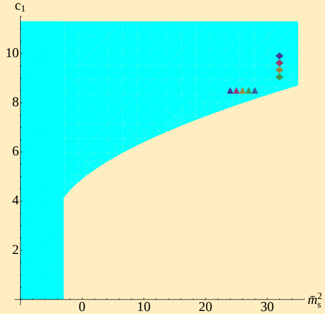
Asymptotic boundary conditions

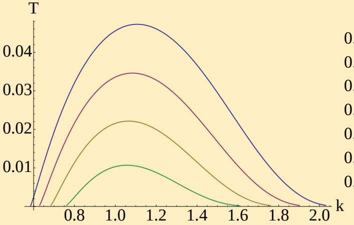
- In general two constants of integration at infinity

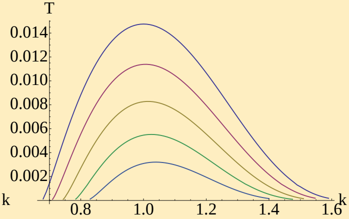
$$h = h_0 + \cdots + h_3 \frac{1}{r^3} + \cdots$$
$$a = a_0 + \cdots + a_1 \frac{1}{r} + \cdots$$
$$w = w_1 \frac{1}{r} + \cdots + w_2 \frac{1}{r^2} + \cdots$$

- The constant h_0 would correspond to a boost "chemical potential" and a_0 to a current chemical potential. For spontaneous symmetry breaking we set $a_0 = h_0 = 0$
- We assumed that $m_s^2 = -4$. We choose $\Delta\left(\mathcal{O}_\phi\right) = 2$ and set $w_1 = 0$
- For a fixed wavenumber k we have a total of 6 free variables 2 (horizon)+ 3 (infinity)+ T $\Rightarrow$ One solution (Discreet more precisely!)

Fix AdS_4 mass $m_s^2 = -4$







Questions for perturbation theory

- Higher order perturbative analysis shows that modulated black hole branches exist
- Can be used to study thermodynamics close to T_c

$\rightsquigarrow$ In general they are continuous transitions (second order)

Would still like to ask

- Transport properties of modulated phases?
- What is the low temperature behaviour? Modulation persists at low temps?
- If yes, new emergent IR with modulation?

$\rightsquigarrow$ Easier to answer in some 5D models

Outline

Helical superconductors

- Spatially modulated phases exist in holography
  Superconducting phases exist in holography
  Combination of the two?
- FFLO phase (1964) - a variation of BCS:
  Construct a situation in which the Fermi momentum of spin up quasi-particles is not equal to that of spin down. If they form Cooper pairs they will have net momentum leading to a spatially modulated superconducting order parameter
- Perhaps seen in some heavy fermions (eg $CeCOIn_5$) and some organic superconductors

Holographic superfluids

- Ingredients: theory of gravity with gauge field which is dual to a conformal field theory with global $U(1)$ symmetry. In addition we need charged fields in the bulk which provide the order parameter for the superconductivity in the dual CFT
- s-wave superconductors have $l = 0$ order parameter $\rightarrow$ use charged bulk scalar fields. [Gubser][Hartnoll, Herzog, Horowitz]
- p-wave superconductors have $l = 1$ order parameter
  Seen in eg He_3, heavy fermions, organics, Sr_2RuO_4
  In $D = 4, 5$ use $SU(2)$ gauge fields. Take the background to be charged with respect to $U(1) \subset SU(2)$ [Gubser] and then spontaneously break the $U(1)$
  In $D = 5$ use a charged first-order two-form
  [Aprile,Franco,Rodriguez,Russo]

The charged 2-form model

Consider the $D = 5$ model

$$\mathcal{L} = (R + 12) * 1 - \frac{1}{2} * F \wedge F - \frac{1}{2} * C \wedge \bar{C} - \frac{\imath}{2m} C \wedge \bar{H}$$

$$F = dA, \quad H = dC + \imath \frac{q_e}{\sqrt{3}} A \wedge C$$

- It is a consistent truncation of $D = 5$, $\mathcal{N} = 8$ gauged SUGRA for $q_e = 1$, $m = 1$
- The electric RN black brane is a solution with $C = 0$
- To analyze stability of C, simply examine its equation of motion on $AdS_2 \times \mathbb{R}^3$/RN bh background

$$* H = \imath m C$$

and is dual to a charge q_e, $d = 4$ self-dual tensor operator with

$$\Delta(\mathcal{O}_C) = 4 + m$$

2-form perturbations

Consistency of the 2-form equation of motion requires to consider

$$\delta C = dr \wedge (u_1\, dx_1 - v_1\, dx_2) + \imath\, dt \wedge (u_2\, dx_1 - v_2\, dx_2) \\ + dx_3 \wedge (u_3\, dx_2 + v_3\, dx_1)$$

Two independent cases

- $u_i = d_i \cos(kx_3), \quad v_i = d_i \sin(kx_3)$, p-wave
- $u_i = d_i\, e^{\imath k x_3}, \quad v_i = \imath d_i\, e^{\imath k x_3}$, $p + \imath p$-wave

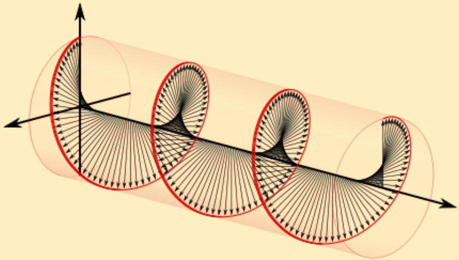
- Solve for d_1 and d_2 in terms of d_3
- The component d_3 satisfies a second order equation
- For the $AdS_2 \times \mathbb{R}^3$ background

$$\left(\mathcal{D}^2 - L^2\,(m^2 + k^2) + \frac{kq}{3\sqrt{2}m}\right) d_3 = 0, \quad \mathcal{D} = \nabla + \frac{\imath q}{\sqrt{3}} A$$

$$\rightsquigarrow \quad m^2_{eff} = L^2\,(k^2 + m^2) - \frac{kq}{3\sqrt{2}m} - \frac{q^2}{18}$$

The order parameter is

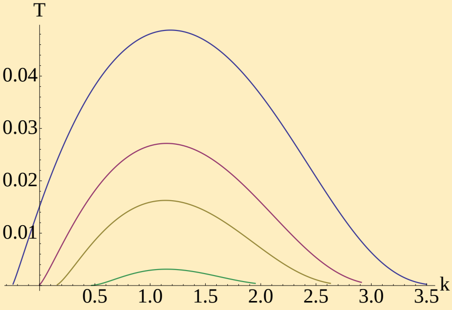
$$\delta C = \cdots - d_3 \left[\cos(kx_3)\, dx_2 + \sin(kx_3)\, dx_1\right] \wedge dx_3$$



A helical p_x-wave structure. Also possible to describe a helical $p_x + \imath p_y$-wave

Static normalizable modes

For fixed chemical potential and $m = 1$ construct static spontaneous symmetry breaking modes for $q = 2$, 1.8, 1.7, 1.5



- The $k = 0$ mode becomes quantum critical for charge $q = 1.8$ [Aprile,Franco,Rodriguez,Russo]
- The string/M-theory consistent truncation has $q = 1 \to$ low temps, numerical error

Helical superconducting black holes

- The linearised static mode takes the form

$$
\begin{aligned}
\delta C &= (\imath c_1(r)\, dt + c_2(r)\, dr) \wedge \omega_2 + c_3\, \omega_1 \wedge \omega_3 \\
\omega_1 &= dx_1 \\
\omega_2 &= \cos(kx_1)\, dx_2 - \sin(kx_1)\, dx_3 \\
\omega_3 &= \sin(kx_1)\, dx_2 + \cos(kx_1)\, dx_3
\end{aligned}
$$

- Key observation in constructing the back reacted geometries is that the unstable mode preserves the Bianchi VII_0 group

$$
\begin{aligned}
L^1 &= \partial_{x_1} + k\, (x_3 \partial_{x_2} - x_2 \partial_{x_3}) \\
L^2 &= \partial_{x_2} \\
L^3 &= \partial_{x_3} \\
\rightsquigarrow \quad & \mathcal{L}_{L^i} \omega_j = 0 \Rightarrow \mathcal{L}_{L^i} \delta C = 0
\end{aligned}
$$

- Back-reaction should respect this symmetry

Helical superconducting black holes

- Non-linear ansatz

$$ds^2 = -gf^2\,dt^2 + g^{-1}\,dr^2 + h^2\,\omega_1^2 + r^2\,\left(e^{2\alpha}\,\omega_2^2 + e^{-2\alpha}\,\omega_3^2\right)$$
$$C = (\imath\,c_1\,dt + c_2\,dr)\wedge\omega_2 + c_3\,\omega_1\wedge\omega_3$$
$$A = a\,dt$$

- Where g, f, h, α, c_i and a are only functions of r and we also used the Bianchi VII_0 invariant one-forms

$$\omega_1 = dx_1$$
$$\omega_2 = \cos(kx_1)\,dx_2 - \sin(kx_1)\,dx_3$$
$$\omega_3 = \sin(kx_1)\,dx_2 + \cos(kx_1)\,dx_3$$

- Plugging in the $5D$ equations of motion yields a consistent system of non-linear ODEs with k a parameter

Helical superconducting black holes

The RN black hole

At high temperatures we only find the RN black brane solution (normal phase)

$$f = 1, \quad h = r, \quad \alpha = 0, \quad c_i = 0$$
$$a = \mu \left(1 - \frac{r_+^2}{r^2}\right), g = r^2 - \frac{r_+^4}{r^2} + \frac{\mu^2}{3} \left(\frac{r_+^4}{r^4} - \frac{r_+^2}{r^2}\right)$$

with the $5D$ fields reading

$$ds^2 = -g\, dt^2 + g^{-1}\, dr^2 + r^2 \left(dx_1^2 + dx_2^2 + dx_3^2\right)$$
$$A = a\, dt, \quad C = 0$$

Helical superconducting black holes

Modulated black hole phase

- Starting at T_c and finite k we find new black hole branches
- Use shooting method to solve ODEs
- Demand regularity on the horizon $r = r_+$

$$g(r_+) = a(r_+) = 0$$

with all functions having an analytic expansion at r_+

- Demand expansion at infinity appropriate for spontaneous symmetry breaking

$$g = r^2 \left(1 - M\, r^{-4} + \cdots\right), \quad f = 1 - c_h\, r^{-4} + \cdots$$
$$h = r \left(1 + c_h\, r^{-4}\right), \quad \alpha = c_\alpha\, r^{-4} + \cdots$$
$$a = \mu + q\, r^{-2} + \cdots, \quad c_3 = c_v\, r^{-|m|} + \cdots$$

Helical superconducting black holes

- Integration constants at infinity associated with physical quantities of the dual field theory
- The vev of the superconducting order parameter is determined by both c_v and k

$$\langle \mathcal{O}\left(k\right)_C \rangle \propto c_v$$

- In addition we have charge density

$$\langle J^0 \rangle \propto q$$

Helical superconducting black holes

- Holographic renormalisation reveals the stress tensor

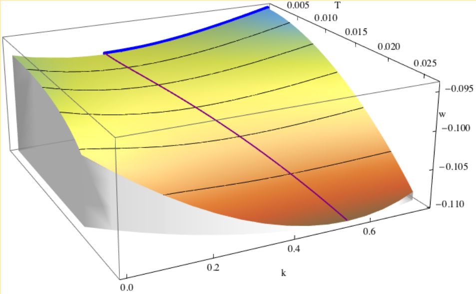
$$
\begin{aligned}
T_{tt} &= 3M + 8c_h \\
T_{x_1x_1} &= M + 8c_h \\
T_{x_2x_2} &= M + 8c_\alpha \cos\left(2kx_1\right) \\
T_{x_3x_3} &= M - 8c_\alpha \cos\left(2kx_1\right) \\
T_{x_2x_3} &= 8c_\alpha \sin\left(2kx_1\right)
\end{aligned}
$$

⇝ Spatially modulated pressure and shear in the $x_2 - x_3$ plane

- Free energy density $w\,\mathrm{vol}_3 \equiv T\left[I_{Tot}\right]_{OS}$ with $w = w_k\left(T,\mu\right) = -M$

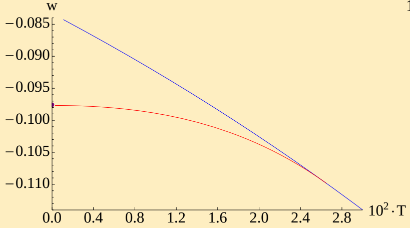
Helical superconducting black holes

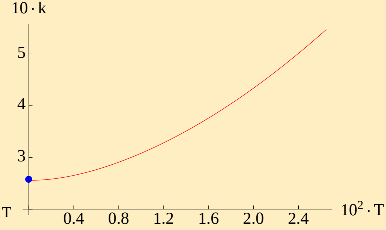
- A parameter count reveals that, for fixed μ, there exists a two parameter family of black holes labeled by T and k.
- For fixed T and μ minimise the free energy with respect to k

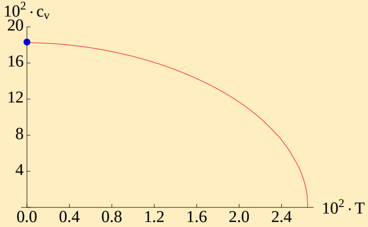


- Free energy minimised on the red curve
- All black holes have $c_v \neq 0$ and hence are helical superconductors
- All black holes have $w_{helical} < w_{RN}$

For the thermodynamically preferred $k(T)$







Helical superconducting black holes

Zero temperature limit

- As $T \to 0$ our black hole solutions approach smooth domain walls interpolating between AdS_5 in the UV and a new scaling solution in the IR

$$ds^2 = -r^{2z}\,dt^2 + r^{-2}\,dr^2 + h_0^2\,\omega_1^2 + r^2\,\left(e^{2\alpha_0}\,\omega_2^2 + e^{-2\alpha_0}\,\omega_3^2\right)$$
$$C = \left(\imath\, c_{1(0)}\, r^{z+1}\, dt + c_{2(0)}\, dr\right) \wedge \omega_2 + c_{3(0)}\, r\, \omega \wedge \omega_3$$
$$A = a_0 r^z\, dt$$

- z, h_0, α_0, $c_{i(0)}$, a_0 are constants

- Scaling $t \to \lambda^z t$, $x_{2,3} \to \lambda x_{2,3}$, $x_1 \to x_1$, $r \to \lambda^{-1} r$

Outline

Final comments

- Studied instabilities of electrically charged black holes leading to a wide range of spatially modulated black holes
- Constructed the first black holes dual to spatially modulated phases $\rightarrow$ helical p-wave superconducting order, novel scaling symmetry in the IR
- For the values of m and q_e we looked at the IR solutions has real scaling dimensions. For other values they can be complex. Stable?
- Interesting to calculate transport coefficients using linear response they
- We looked at p-wave order. There is also $p + \imath p$ wave order. Which one wins?

Final comments

- Other examples:
  - $D = 5$ Einstein-Maxwell Chern-Simons : similar story (to appear)
  - $D = 5$ with $SU(2) \times U(1)$: expect a similar story (in progress)
  - Example in $D = 4$ with axion and gauge-field would involve solving PDEs
  - Modulated instabilities of magnetic branes
- Rich set of examples with couplings natural in string/M-theory. Are they generic ground states of deformed CFTs?