# The Hilbert Series of SQCD

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# Motivation: Hilbert series vs. Brane decay

1. Hilbert series of $\mathcal{N} = 1$ supersymmetric QCD

[Chen,Mekareeya arXiv:1104.2045]

$$g(t, \tilde{t}) = \frac{1}{N_c!} \int \prod_{a=1}^{N_c} \frac{d\tau_a}{2\pi} |\Delta(z)|^2 \prod_{a=1}^{N_c} \frac{1}{\left(1 - t z_a^{-1}\right)^{N_f} \left(1 - \tilde{t} z_a\right)^{N_f}}$$

2. (A contribution to the) emission amplitude for a closed string from a decaying brane (half $S$-brane)

$$Z(w) = \frac{1}{N!} \int \prod_{a=1}^{N} \frac{d\tau_a}{2\pi} |\Delta(z)|^2 \prod_{a=1}^{N} |1 - w z_a|^{2i\omega}$$

- Same integrals (for $t = \tilde{t}$)!
- Our work: known results from brane decay applied to the Hilbert series (+some new results)

[Jokela,MJ,Keski-Vakkuri, arXiv:1112.5454]

# Outline

- Introduction to Hilbert series (mostly stolen from Amihay Hanany's talks)
- Defining SQCD Hilbert series
- Hilbert series and Schur polynomials
- Hilbert series of SQCD in the Veneziano limit: the log-gas approach

# Introduction to Hilbert Series

Main idea:

- Hilbert series = generating function for numbers of gauge invariant BPS operators in $\mathcal{N} = 1$ supersymmetric gauge theory
- For a theory having $n$ $U(1)$ symmetries

$$H(t_1, \ldots, t_n) = \sum_{k_1,\ldots,k_n} c_{k_1,\ldots,k_n} t_1^{k_1} \cdots t_n^{k_n}$$

  - $c_{k_1,\ldots,k_n}$: number of operators having charges $k_1, \ldots, k_n$ under the symmetries
  - Variables $t_i$ termed "chemical potentials" or "fugacities"
- Admits a generalization to non-Abelian symmetries (definition however complicated...)
- Usually one restricts to one Abelian fugacity $t \to$ operators counted by their dimension

# Introduction to Hilbert Series – Example

Example: SQCD with $SU(2)$ gauge group and $N_f = 1$

$$H(t) = g(t) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + \cdots$$

- One single-trace operator $Q_1 Q_2$
- All operators of various degrees $\mathbf{1}$, $Q_1 Q_2$, $(Q_1 Q_2)^2$, $(Q_1 Q_2)^3$, ...
- "Freely generated" moduli space, dimension one

# Introduction to Hilbert Series – Generic Features

$$H(t) = \frac{Q(t)}{(1-t)^k} = \frac{P(t)}{(1-t)^{\dim(\mathcal{M})}}$$

- $Q(t)$, $P(t)$ polynomials
- $k$ is dimension of "embedding space"
  (For SQCD, mesonic+baryonic operators)
- $\dim(\mathcal{M})$, dimension of (classical) moduli space, equals the degree of the pole at $t = 1$
- $P(t = 1)$ is "degree of $\mathcal{M}$" (AdS/CFT $\rightarrow$ volume of the dual Sasaki-Einstein manifold)

# Introduction to Hilbert Series – More Examples

More SQCD examples $(N_f, N_c)$

[Gray,He,Hanany,Mekareeya,Jejjala, arXiv:0803.4257]

| $(N_f, N_c)$ | $\mathcal{M}$ | Hilbert Series $H(\mathcal{M}; t)$ |
|---|---|---|
| $(2,2)$ | $(5,4\vert 5[2:2:2:2:2:2]\vert 4^1)$ | $\frac{1+t^2}{(1-t^2)^5}$ |
| $(3,2)$ | $(9,896\vert 14[2^{15}]\vert 4^{15})$ | $\frac{1+6t^2+6t^4+t^6}{(1-t^2)^9}$ |
| $(4,2)$ | $(13,4325376\vert 27[2^{28}]\vert 4^{70})$ | $\frac{1+15t^2+50t^4+50t^6+15t^8+t^{10}}{(1-t^2)^{13}}$ |
| $(5,2)$ | $(17,383862702080\vert 44[2^{45}]\vert 4^{210})$ | $\frac{1+28t^2+196t^4+490t^6+490t^8+196t^{10}+28t^{12}+t^{14}}{(1-t^2)^{17}}$ |
| $(3,3)$ | $(10,6\vert 10[2^9:3^2]\vert 6^1)$ | $\frac{1+t^3}{(1-t^2)^9(1-t^3)}$ |
| $(4,3)$ | $(16,88128\vert 23[2^{16}:3^8]5^8 6^{16} 7^{12})$ | $\frac{1+4t^2+4t^3+10t^4+8t^5+14t^6+8t^7+10t^8+4t^9+4t^{10}+t^{12}}{(1-t^2)^{12}(1-t^3)^4}$ |
| $(4,4)$ | $(17,8\vert 17[2^{16}:4^2]\vert 8^1)$ | $\frac{1+t^4}{(1-t^2)^{16}(1-t^4)}$ |

Palindromic property of $P(t) \Rightarrow$ moduli space is a Calabi-Yau

$$H(t) = \frac{Q(t)}{(1-t)^k}$$

Three families of moduli spaces:

1. Freely generated ($N_c > N_f$): $Q(t) = 1$
2. Complete intersection ($N_c = N_f$): $Q(t) = 1 - t^d$
3. The rest ($N_c < N_f$)

# Introduction to Hilbert Series – Plethystics

Plethystic exponential $PE$ and logarithm $PL = PE^{-1}$

$$PE[f(t)] \equiv \exp\left(\sum_{k=1}^{\infty} \frac{f(t^k)}{k}\right)$$

Plethystic logarithm of $H$:

- Generators of the moduli space – first positive terms
- Relations of the moduli space – first negative terms

For SQCD, taking the plethystic logarithm of $H^{(N_f,N_c)}(t)$:

$PL[H^{(1,2)}(t)] = t^2$
$PL[H^{(2,2)}(t)] = 6t^2 - t^4$
$PL[H^{(2,3)}(t)] = 4t^2$
$PL[H^{(3,2)}(t)] = 15t^2 - 15t^4 + 35t^6 - \cdots$
$PL[H^{(3,3)}(t)] = 9t^2 + 2t^3 - t^6$

# The SQCD Hilbert Series – U(N)

For $U(N_c)$ gauge group (simpler), refined Hilbert series

$$g_{N_f,U(N_c)}(t_i,\tilde{t}_i) = \frac{1}{N_c!}\prod_{a=1}^{N_c}\int_0^{2\pi}\frac{d\tau_a}{2\pi}|\Delta(z)|^2\prod_{a=1}^{N_c}\prod_{i=1}^{N_f}\frac{1}{(1-t_i z_a^{-1})(1-\tilde{t}_i z_a)}$$

- $t_i$, $\tilde{t}_i$ and $z_a = e^{i\tau_a}$ are the flavor, "antiflavor" and color fugacities, respectively
- $\Delta(z)$ is the Vandermonde determinant

Understanding the expression:

- $\prod_{a=1}^{N_c}\prod_{i=1}^{N_f}\frac{1}{1-t_i z_a^{-1}}$ generates all operators involving $Q_i^a$
- $\prod_{a=1}^{N_c}\prod_{i=1}^{N_f}\frac{1}{1-\tilde{t}_i z_a}$ generates all operators involving $\tilde{Q}_i^a$
- $\frac{1}{N_c!}\prod_{a=1}^{N_c}\int_0^{2\pi}\frac{d\tau_a}{2\pi}|\Delta(z)|^2$ picks up gauge invariant terms

# The SQCD Hilbert Series – SU(N)

For $SU(N_c)$, add a constraint for the phases $\tau_a$

$$g_{N_f,SU(N_c)}(t_i,\tilde{t}_i) = \frac{1}{N_c!}\prod_{a=1}^{N_c}\int_0^{2\pi}\frac{d\tau_a}{2\pi}|\Delta(z)|^2\sum_{k=-\infty}^{\infty}\delta\left(\sum_a \tau_a - 2\pi k\right) \times \prod_{a=1}^{N_c}\prod_{i=1}^{N_f}\frac{1}{(1-t_i z_a^{-1})(1-\tilde{t}_i z_a)}$$

Some notation (important to recall!):

- For $U(N_f)_L$ fugacities (separation into $SU(N_f)_L \times U(1)_Q$)

$$(t_1, t_2, \ldots, t_{N_f}) \equiv \left(x_1, \frac{x_2}{x_1}, \ldots, \frac{1}{x_{N_f-1}}\right) t \equiv (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{N_f})t$$

- For $U(N_f)_R$ fugacities

$$(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{N_f}) \equiv \left(\frac{1}{y_1}, \frac{y_1}{y_2}, \ldots, y_{N_f-1}\right)\tilde{t} \equiv (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{N_f})\tilde{t}$$

# The SQCD Hilbert Series – Unrefining

(Unrefined, standard) Hilbert series: set all $x_i = 1 = y_j$ (or $t = t_1 = \cdots t_n$), e.g.

$$g_{N_f,U(N_c)}(t,\tilde{t}) = \frac{1}{N_c!}\prod_{a=1}^{N_c}\int_0^{2\pi}\frac{d\tau_a}{2\pi}|\Delta(z)|^2\prod_{a=1}^{N_c}(1-tz_a^{-1})^{-N_f}(1-\tilde{t}z_a)^{-N_f}$$

Often in addition set $t = \tilde{t}$ (the most interesting case) $\Rightarrow$ real integrand

# The SQCD Hilbert Series – As Matrix Integral

$$g_{N_f,U(N_c)}(t,\tilde{t}) = \frac{1}{N_c!}\prod_{a=1}^{N_c}\int_0^{2\pi}\frac{d\tau_a}{2\pi}|\Delta(z)|^2\prod_{a=1}^{N_c}(1-tz_a^{-1})^{-N_f}(1-\tilde{t}z_a)^{-N_f}$$
$$= \int d\mu_{U(N_c)}\det(\mathbf{1}-tU^\dagger)^{-N_f}\det(\mathbf{1}-\tilde{t}U)^{-N_f}$$
$$= \left\langle \det(\mathbf{1}-tU^\dagger)^{-N_f}\det(\mathbf{1}-\tilde{t}U)^{-N_f}\right\rangle_{\mathrm{CUE}}$$

- An expectation value in the circular unitary ensemble
- $d\mu_{U(N_c)}$ is the Haar measure

Integrals can be evaluated $\Rightarrow$ Toeplitz determinant

$$g_{N_f,U(N_c)}(t,\tilde{t}) = \det T[f] \equiv \det(\hat{f}_{i-j})_{i,j=1,\ldots,N_c} \quad \text{with}$$

$$\hat{f}_n(t,\tilde{t}) = (-1)^n\binom{-N_f}{|n|}\,{}_2F_1(N_f+|n|,N_f,|n|+1;t\tilde{t})\times\begin{cases}\tilde{t}^n & n\geq 0\\ t^{-n} & n<0\end{cases}$$

[Chen,Mekareeya; Jokela,MJ,Keski-Vakkuri]

However result cumbersome for $N_c \gtrsim 3$

# Hilbert Series & Schur Polynomials

Schur polynomials: symmetric polynomials of $n$ variables

$$s_\lambda(z_1, z_2, \ldots, z_n) = \frac{\det\left(z_j^{\lambda_{n-i+1}+i-1}\right)_{i,j=1,\ldots,n}}{\det\left(z_j^{i-1}\right)_{i,j=1,\ldots,n}}$$

$$= \frac{1}{\Delta(z)} \begin{vmatrix} z_1^{\lambda_n} & z_2^{\lambda_n} & \cdots & z_n^{\lambda_n} \\ z_1^{\lambda_{n-1}+1} & z_2^{\lambda_{n-1}+1} & \cdots & z_n^{\lambda_{n-1}+1} \\ \vdots & & & \vdots \\ z_1^{\lambda_1+n-1} & z_2^{\lambda_1+n-1} & \cdots & z_n^{\lambda_1+n-1} \end{vmatrix}$$

- $\lambda = (\lambda_1, \ldots, \lambda_n)$ partition of $|\lambda| = \sum_i \lambda_i$ or a Young diagram
- Example: $\lambda = (2, 1, 1) =$ 

$s$$(z_1, z_2, z_3, z_4) = z_1^2 z_2 z_3 + z_1 z_2^2 z_3 + z_1 z_2 z_3^2 + z_1^2 z_2 z_4$

$+z_1 z_2^2 z_4 + z_1^2 z_3 z_4 + 3 z_1 z_2 z_3 z_4 + z_2^2 z_3 z_4 + z_1 z_3^2 z_4 + z_2 z_3^2 z_4 + z_1 z_2 z_4^2$

$+z_1 z_3 z_4^2 + z_2 z_3 z_4^2$

# Hilbert Series & Schur Polynomials – Properties

The Schur polynomials have special properties

- Orthogonality ($z_i = e^{i\tau_i}$)

$$\frac{1}{n!}\prod_{i=1}^{n}\int_0^{2\pi}\frac{d\tau_i}{2\pi}|\Delta(z)|^2 s_\lambda(z_1,\ldots,z_n)s_\kappa(\bar{z}_1,\ldots,\bar{z}_n) = \delta_{\lambda,\kappa}$$

- The Cauchy identity

$$\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-z_i w_j} = \sum_\lambda s_\lambda(z)s_\lambda(w)$$

- $s_\lambda(\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_{N_f}) = s_\lambda\left(x_1,\frac{x_2}{x_1},\ldots,\frac{1}{x_{N_f-1}}\right)$ are characters of $SU(N_f)$

# Hilbert Series & Schur Polynomials – Results

Using the properties one easily proves an earlier conjecture for the refined Hilbert series of $U(N_c)$ SQCD:

[Constable, Larsen, hep-th/0305177]

$$g_{N_f,U(N_c)}(t,\tilde{t},x,y) = \sum_{\lambda:\ell(\lambda)\leq \min(N_f,N_c)} (t\tilde{t})^{|\lambda|} s_\lambda(\tilde{x}) s_\lambda(\tilde{y})$$

- $\ell(\lambda)$ is the width of the Young diagram $\lambda$

For $SU(N_c)$ some extra algebra is required, giving

$$g_{N_f,SU(N_c)}(t,\tilde{t},x,y) = \sum_{k=-\infty}^{\infty} I_k \quad \text{with}$$

$$I_k = \begin{cases} \sum_{\lambda,\kappa} t^{|\lambda|} s_\lambda(\tilde{x}) \tilde{t}^{|\kappa|} s_\kappa(\tilde{y}) \delta_{\lambda,\kappa+k} \ ; & k \geq 0 \ , \\ \sum_{\lambda,\kappa} t^{|\lambda|} s_\lambda(\tilde{x}) \tilde{t}^{|\kappa|} s_\kappa(\tilde{y}) \delta_{\lambda+|k|,\kappa} \ ; & k < 0 \ . \end{cases}$$

# Hilbert Series & Schur Polynomials – U(N) Results

In some cases one can use Cauchy identity to sum the series:

- For $N_f \leq N_c$: (freely generated)

$$g_{N_f, U(N_c)}(t, \tilde{t}, x, y) = \prod_{i=1}^{N_f} \prod_{j=1}^{N_f} \frac{1}{1 - t\tilde{t}\tilde{x}_i \tilde{y}_j}$$

- For $N_f = N_c + 1$: (complete intersection)

$$g_{N_c+1, U(N_c)}(t, \tilde{t}, x, y) = \left(1 - (t\tilde{t})^{N_c+1}\right) \prod_{i,j=1}^{N_c+1} \frac{1}{1 - t\tilde{t}\tilde{x}_i \tilde{y}_j}$$

- $N_f = N_c + 2$ also calculable, a big mess
- These are new results!

# Hilbert Series & Schur Polynomials – SU Results

- For $N_f < N_c$: (freely generated)

$$g_{N_f,SU(N_c)}(t,\tilde{t},x,y) = \prod_{i=1}^{N_f}\prod_{j=1}^{N_f}\frac{1}{1-t\tilde{t}\tilde{x}_i\tilde{y}_j}$$

- For $N_f = N_c$: (complete intersection)

$$g_{N_c,SU(N_c)}(t,\tilde{t},x,y) = \frac{1-(t\tilde{t})^{N_c}}{(1-t^{N_c})(1-\tilde{t}^{N_c})}\prod_{i=1}^{N_c}\prod_{j=1}^{N_c}\frac{1}{1-t\tilde{t}\tilde{x}_i\tilde{y}_j}$$

- $N_f = N_c + 1$ also calculable, a big mess

# Hilbert Series & Schur Polynomials – Results

Unrefining ($x_i \to 1$, $y_j \to 1$)

$$g_{N_f,U(N_c)}(t,\tilde{t}) = \sum_{\lambda:\ell(\lambda)\leq N_c} (t\tilde{t})^{|\lambda|} d_\lambda^2$$

$$g_{N_f,SU(N_c)}(t,\tilde{t}) = \sum_{k=0}^{\infty} t^{kN_c} \sum_{\kappa:\ell(\kappa)\leq N_c} (t\tilde{t})^{|\kappa|} d_{\kappa+k} d_\kappa + \cdots$$

- $d_\lambda$ is the dimension of the $SU(N_f)$ representation $\lambda$

A brute force calculation confirms:

$$g_{N_f,U(N_c)}(t,\tilde{t}) \sim \frac{1}{(1-t\tilde{t})^{2N_cN_f-N_c^2}} = \frac{1}{(1-t\tilde{t})^{\dim(\mathcal{M})}}$$

$$g_{N_f,SU(N_c)}(t,\tilde{t}=t) \sim \frac{1}{(1-t)^{2N_cN_f-N_c^2+1}} = \frac{1}{(1-t)^{\dim(\mathcal{M})}}$$

# Veneziano Limit

We take $t = \tilde{t}$ in the (unrefined) Hilbert series :

$$g_{N_f,U(N_c)}(t) = \frac{1}{N_c!}\prod_{a=1}^{N_c}\int_0^{2\pi}\frac{d\tau_a}{2\pi}|\Delta(z)|^2\prod_{a=1}^{N_c}|1-tz_a^{-1}|^{-2N_f}$$

In the Veneziano limit,

$$N_c, N_f \to \infty \ ; \qquad N_f/N_c \text{ fixed}$$

$$g_{N_f,U(N_c)}(t) \simeq g_{N_f,SU(N_c)}(t) \simeq e^{-\beta\mathcal{E}}$$

- Here $\beta = 2$
- $\mathcal{E}$ is the saddle-point value for the (continuum limit of) 2d log-gas on the unit circle

$$H = -\sum_{1\leq a<b\leq N_c}\log|e^{i\tau_a}-e^{i\tau_b}| + N_f\sum_{a=1}^{N_c}\log|e^{i\tau_a}-t|$$

with an "external charge" $N_f \sim N_c$ at the real axis at $t$

# Veneziano Limit – Log-gas Electrostatics

Minimize

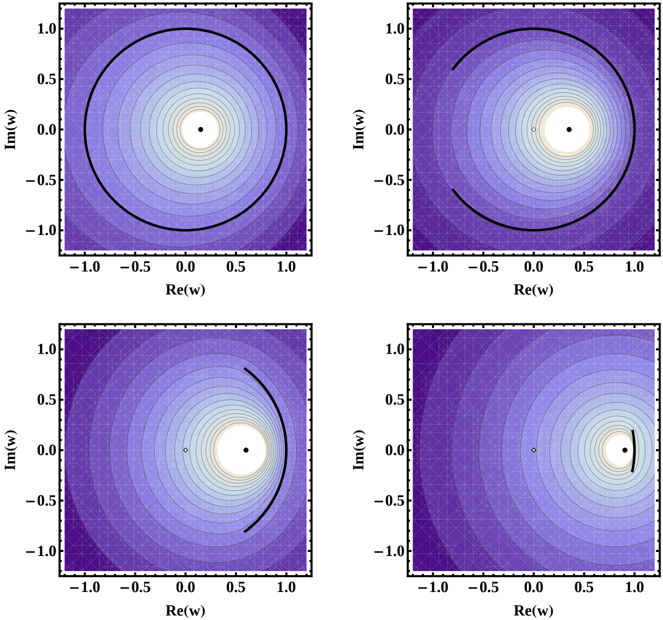
$$-\frac{1}{2}\int_0^{2\pi} d\tau_1 d\tau_2 \rho(\tau_1)\rho(\tau_2)\log\left|e^{i\tau_1}-e^{i\tau_2}\right| + N_f\int_0^{2\pi} d\tau\rho(\tau)\log\left|e^{i\tau}-t\right|$$

with $\rho$ positive or zero $\Rightarrow$ two phases

- For $0 < t < t_c$: gapless phase
  - $\rho$ smooth over the unit circle
- For $t_c < t < 1$: one-gap phase (only if $N_f/N_c > 1$!)
  - A gap with vanishing $\rho$ opposite the external charge

$$t_c = \frac{N_c}{2N_f - N_c}$$

# Veneziano Limit – Log-gas Electrostatics



# Veneziano Limit – Results

- Small $t$, gapless phase $\sim$ freely generated

$$g_{N_f,N_c}(t) \simeq \frac{1}{(1-t^2)^{N_f^2}}$$

- Large $t$, one-gap phase

$$g_{N_f,N_c}(t) \simeq \left(\frac{\chi+1}{\delta(t)+1}\right)^{(2N_f-N_c)^2/2} \left(\frac{\chi-1}{\delta(t)-1}\right)^{N_c^2/2} \times \left(\frac{\delta(t)}{\chi}\right)^{N_f^2} \frac{1}{(1-t^2)^{N_f^2}}$$

$$\delta(t) = \frac{1+t}{1-t}\ ; \qquad \chi = \frac{N_f}{N_f-N_c}\ ; \qquad \delta(t_c) = \chi$$

# Veneziano Limit – Results

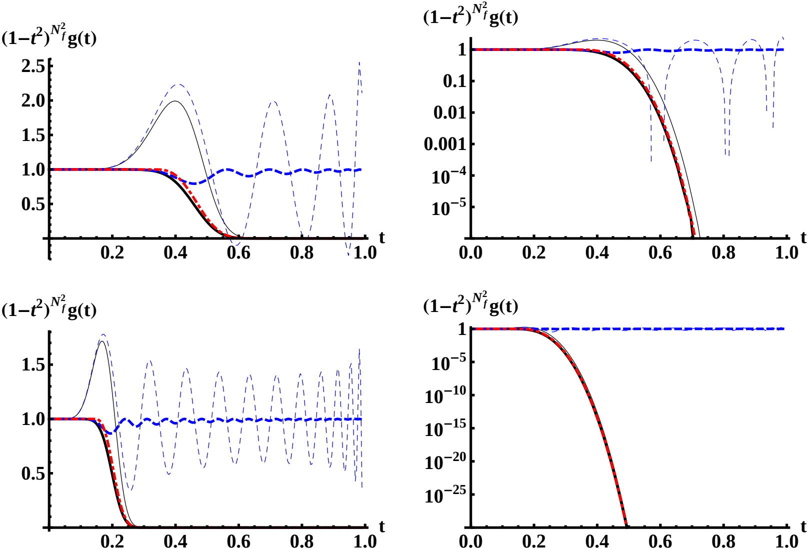
As $t \to 1$

$$g_{N_f,N_c}(t) \sim \left(\frac{\chi+1}{2}\right)^{(2N_f-N_c)^2/2}\left(\frac{\chi-1}{2}\right)^{N_c^2/2}\left(\frac{1}{\chi}\right)^{N_f^2}\frac{1}{(1-t)^{2N_fN_c-N_c^2}}$$

- Degree of singularity again agrees with the dimension of the moduli space
- Leading result for the residue could be obtained

# Veneziano Limit – Numerical Comparison

Exact $N_c = 8$ Log-gas GCBO **U(N)** SU(N)



# Conclusion

- The Hilbert series carries information of gauge invariant operators and the moduli space of $\mathcal{N} = 1$ supersymmetric gauge theories
- We calculated new result for the SQCD Hilbert series by using tools which were new to this framework
  - Schur polynomials could be used to calculate the refined Hilbert series (involving all fugacities) exactly in many cases
  - The Log-gas method provided an expression for the (unrefined) Hilbert series in the Veneziano limit for all values of $N_f/N_c$, and revealed a "phase transition" as $t$ was varied