Holographic V-QCD at Finite Temperature

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Outline

1. Review of (holographic) V-QCD models

   [MJ, Kiritsis arXiv:1112.1261]

2. Turning on finite temperature

   [Alho, MJ, Kajantie, Kiritsis, Tuominen arXiv:121x.xxxx]

Motivation

QCD: $SU(N_c)$ gauge theory with N_f quark flavors (fundamental)

- Often useful: "quenched" or "probe" approximation, $N_f \ll N_c$

However, many features cannot be captured in this approximation:

- Phase diagram of QCD at zero temperature, baryon density, and quark mass, varying $x = N_f/N_c$
- The QCD thermodynamics as a function of x
- Phase diagram as a function of baryon density

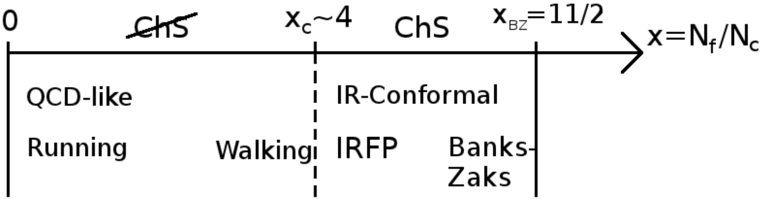
$\Rightarrow$ Veneziano limit: large N_f, N_c with $x = N_f/N_c$ fixed

Holographic V-QCD

Holographic bottom-up models (V-QCD) that describe the QCD phase diagram in the Veneziano limit

[MJ, Kiritsis arXiv:1112.1261]

- Conformal window for $x_c < x < x_{BZ}$, ChSB for $0 < x < x_c$
- Critical value $x_c \sim 4$ arising from dynamics
- Walking backgrounds for x slightly below x_c



The fusion

1. IhQCD: model for glue by using dilaton gravity
   [Gursoy, Kiritsis, Nitti; Gubser, Nellore]
2. Adding flavor and chiral symmetry breaking via tachyon brane actions
   [Klebanov,Maldacena; Bigazzi,Casero,Cotrone,Iatrakis,Kiritsis,Paredes]

Consider 1 + 2 with full backreaction ⇒ V-QCD models

Defining V-QCD

Degrees of freedom:

- $\tau \leftrightarrow \bar{q}q \quad ; \qquad \lambda \leftrightarrow \mathrm{Tr} F^2$
- λ is identified as the 't Hooft coupling $g^2 N_c$

$$\mathcal{S}_{\mathrm{V-QCD}} = N_c^2 M^3 \int d^5x \sqrt{g} \left[R - \frac{4}{3} \frac{(\partial\lambda)^2}{\lambda^2} + V_g(\lambda) \right] - N_f N_c M^3 \int d^5x \, V_f(\lambda,\tau) \sqrt{-\det(g_{ab} + \kappa(\lambda)\partial_a\tau\partial_b\tau)}$$

$$V_f(\lambda,\tau) = V_{f0}(\lambda)\exp(-a(\lambda)\tau^2) \; ; \qquad ds^2 = e^{2A}(dr^2 + \eta_{\mu\nu}x^\mu x^\nu)$$

The simplest and most reasonable potential choices do the job!

Matching to QCD

- In the UV ($\lambda \to 0$):
  - UV expansions of the various potentials can be matched with the perturbative QCD beta function and the anomalous dimension of the quark mass/chiral condensate
  - After this, a single undetermined parameter in the UV: W_0
- In the IR, the tachyon action $\propto e^{-a(\lambda)\tau^2}$ must become small
  - $V_g(\lambda)$ chosen as for Yang-Mills, so that a "good" IR singularity exists
  - $V_{f0}(\lambda)$, $a(\lambda)$, and $\kappa(\lambda)$ chosen to produce tachyon divergence: several possibilities
  - Extra constraints from the asymptotics of the meson spectra

Background analysis: zero temperature

Analysis of the backgrounds (r-dependent solutions of EoMs) at zero temperature

- Expect two kinds of solutions, with
  1. Nontrivial tachyon profile
  2. Identically vanishing tachyon
- Fully backreacted system $\Rightarrow$ rich dynamics, complicated numerical analysis . . .
  However, main features can be understood without going to details

Effective potential

For solutions with $\tau = \tau_* = \mathrm{const}$

$$\mathcal{S} = M^3 N_c^2 \int d^5x \sqrt{g} \left[R - \frac{4}{3} \frac{(\partial\lambda)^2}{\lambda^2} + V_g(\lambda) - xV_f(\lambda, \tau_*) \right]$$

IhQCD with an effective potential

$$V_{\mathrm{eff}}(\lambda) = V_g(\lambda) - xV_f(\lambda, \tau_*) = V_g(\lambda) - xV_{f0}(\lambda) \exp(-a(\lambda)\tau_*^2)$$

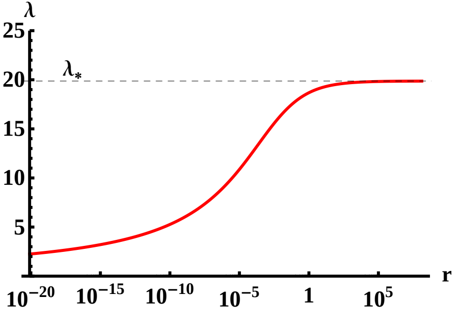
Minimizing for τ_* we obtain $\tau_* = 0$ and $\tau_* = \infty$

- $\tau_* = 0$: $V_{\mathrm{eff}}(\lambda) = V_g(\lambda) - xV_{f0}(\lambda)$;
  fixed point with $V'_{\mathrm{eff}}(\lambda_*) = 0$
- $\tau_* \to \infty$: $V_{\mathrm{eff}}(\lambda) = V_g(\lambda)$ (like YM, no fixed points)

Backgrounds at zero quark mass

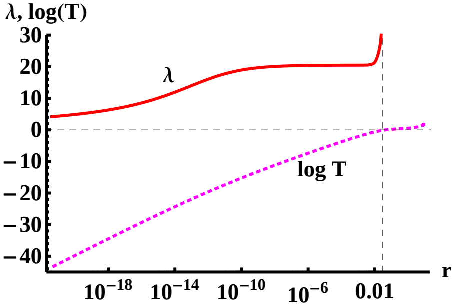
Sketch of behavior in the conformal window ($x > x_c$):

- Tachyon vanishes (no ChSB)
- IHQCD with $V_g \to V_{\rm eff}$ $\Rightarrow$ IR fixed point
- Dilaton flows between UV/IR fixed points



Right below the conformal window ($x < x_c$; $|x - x_c| \ll 1$)

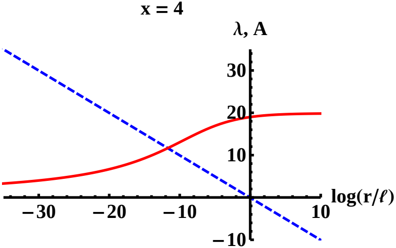
- Dilaton flows very close to the IR fixed point
- "Small" nonzero tachyon induces an IR singularity

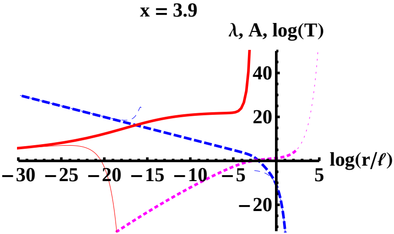


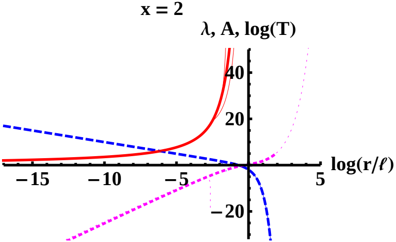
Result: "walking"

Numerical solutions for backgrounds

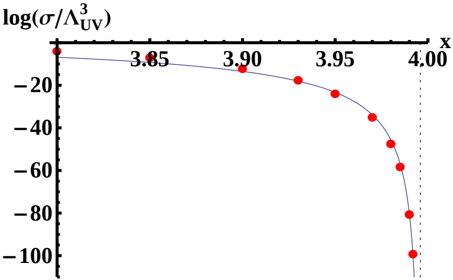
Color code:
λ, A, τ ($= T$ here)
UV: $r = 0$
IR: $r = \infty$
$A \sim \log \mu \sim -\log r$







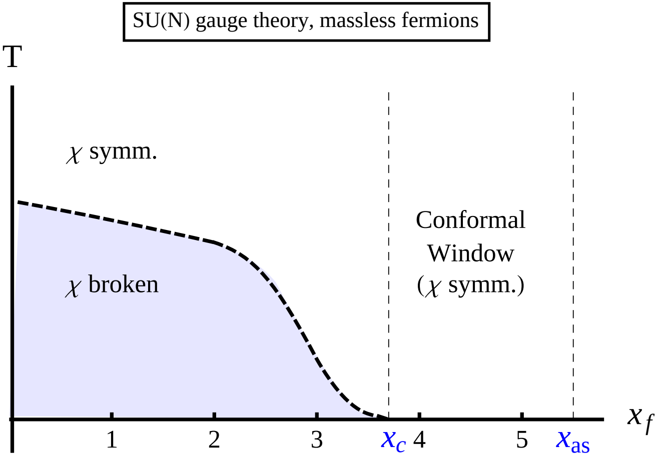
Important features



1. Miransky scaling as $x \to x_c$ from below
   - The ratio of the IR and UV scales behaves as expected
   - E.g. $\langle \bar{q}q \rangle \propto \sigma \sim \exp(-\kappa/\sqrt{x_c - x})$, with calculable κ
2. Metastable/unstable Efimov vacua observed for $x < x_c$
3. Turning on the quark mass modifies the dynamics in a natural way

Finite temperature

Expected phase diagram



Finite temperature – definitions

Lagrangian unchanged

$$\mathcal{S}_{\mathrm{V-QCD}} = N_c^2 M^3 \int d^5x \sqrt{g} \left[R - \frac{4}{3} \frac{(\partial\lambda)^2}{\lambda^2} + V_g(\lambda) \right]$$
$$- N_f N_c M^3 \int d^5x \, V_f(\lambda,\tau) \sqrt{-\det(g_{ab} + \kappa(\lambda)\partial_a\tau\partial_b\tau)}$$

A more general metric, A and f solved from EoMs

$$ds^2 = e^{2A} \left(\frac{dr^2}{f(r)} - f(r)dt^2 + d\mathbf{x}^2 \right) \;; \qquad f(r) = 1 + \mathcal{O}(r^4)$$

Black hole thermodynamics:

$$f(r) = 4\pi T (r_h - r) + \mathcal{O}\left((r-r_h)^2\right) \;; \qquad s = 4\pi M^3 N_c^2 e^{3A(r_h)}$$

Potential choices

IR asymptotics of the potentials (a, κ, V_{f0})

- Potentials I lead to $\tau \sim e^{Cr}$ as $r \to \infty$
- Potentials II lead to $\tau \sim \sqrt{r}$ as $r \to \infty$

Behavior of the IR fixed point

- For potentials I & II, the fixed point exist for all values of x
- For potentials I_* & II_*, the fixed point exist only above some x_*

UV parameter W_0

- Use fixed values $0, 12/11, 24/11$
- Or choose such that the Stefan-Boltzmann law for the thermodynamic functions is automatically obtained

Solutions & Parameters

Parameters:

- Black hole (BH) solutions most easily parametrized in terms of the horizon values τ_h, λ_h of the fields
- Can be mapped to physical parameters m_q (common quark mass) and T (temperature)

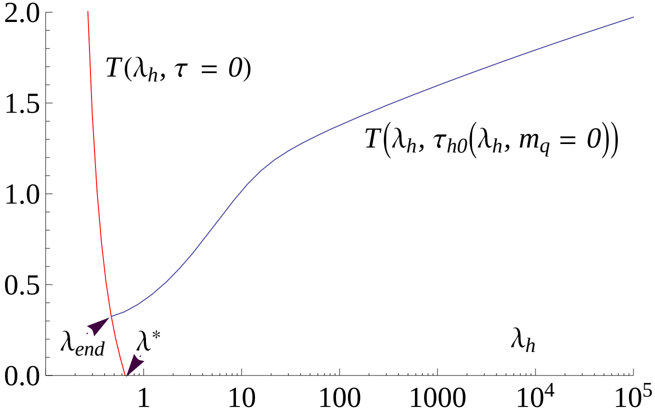
Types of solutions:

1. BH with vanishing tachyon – chirally symmetric (λ_h)
2. BH with nontrivial tachyon – chirally broken (λ_h, τ_h)
   - We require below $m_q = 0$, which fixes τ_h

- In addition, at low temperatures, the thermal gas solutions ($f \equiv 1$) – no black hole $\simeq$ zero T solutions

Black hole branches

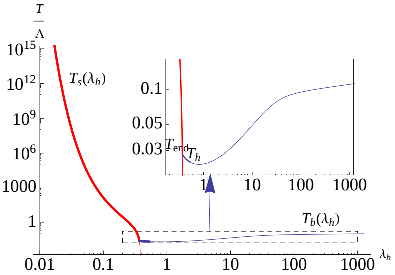
Example: PotII at $x = 3$, $W_0 = 12/11$



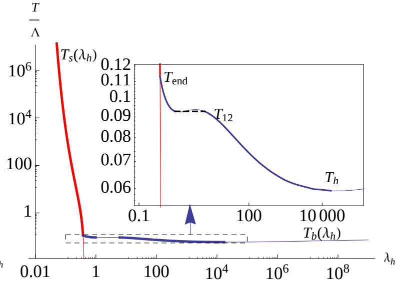
Simple phase structure: 1st order transition at $T = T_h$ from thermal gas to (chirally symmetric) BH

More complicated cases:

PotII at $x = 3$, W_0 SB



PotI at $x = 3.5$, $W_0 = 12/11$

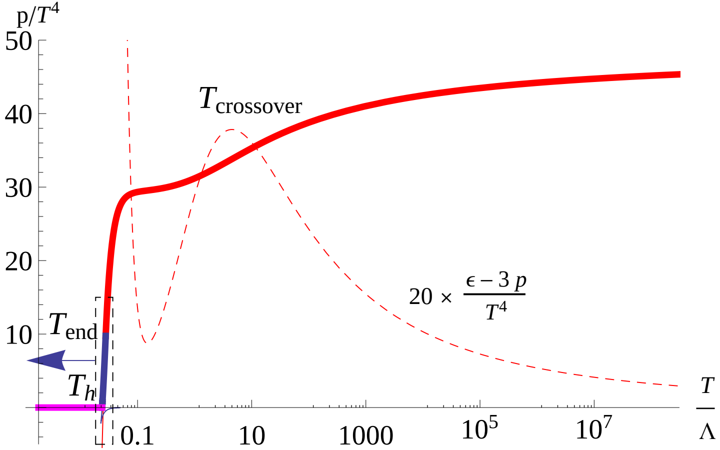


- Left: chiral symmetry restored at 2nd order transition with $T = T_{\text{end}} > T_h$
- Right: Additional first order transition between BH phases with broken chiral symmetry

Also other cases ...

Thermodynamics

Example: PotII, $x = 3$, W_0 SB

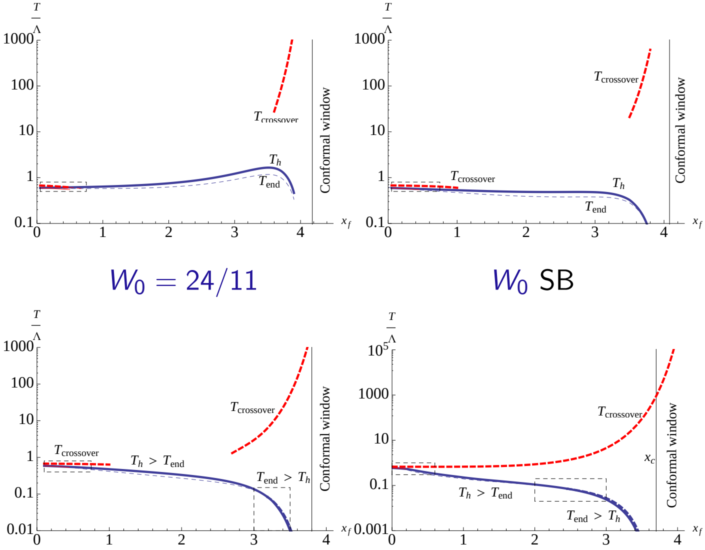


Notice the crossover at high temperature

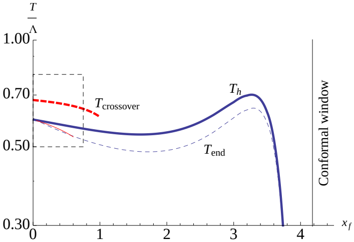
Phase diagrams on the (x, T)-plane

PotII $W_0 = 0$ | $W_0 = 12/11$

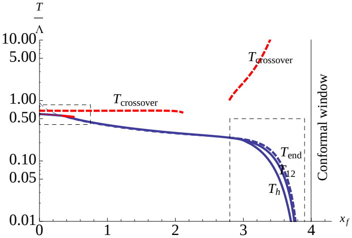
$W_0 = 24/11$ | W_0 SB



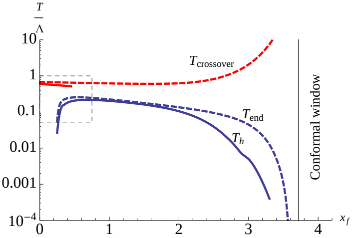
PotI $W_0 = 0$



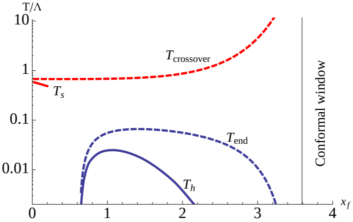
$W_0 = 12/11$



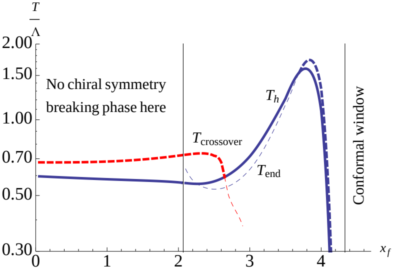
$W_0 = 24/11$



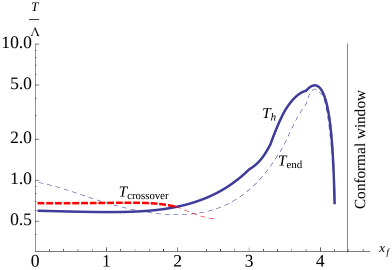
W_0 SB



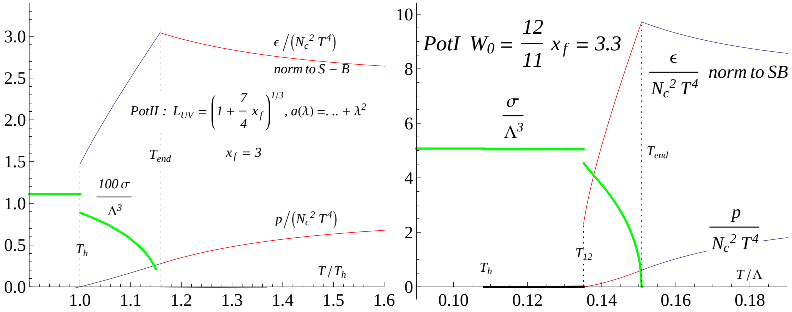
PotI_* W_0 SB



PotII_* W_0 SB



$\langle \bar{q}q \rangle (T)$, pressure and energy density in some cases having several transitions



Conclusion

- A class of holographic bottom-up models (V-QCD) was obtained by a fusion of IhQCD with tachyonic brane action
- A large class of V-QCD models have phase diagrams which meet expectations from QCD both at zero and at finite temperature
- Also in progress: fluctuation analysis (with Arean, Iatrakis, Kiritsis)

Extra slides

Extra slides . . .

A step back: Glue – 5D dilaton gravity

For YM, “improved holographic QCD” (IhQCD): well-tested string-inspired bottom-up model

[Gursoy, Kiritsis, Nitti arXiv:0707.1324, 0707.1349]
[Gubser, Nellore arXiv:0804.0434]

$$\mathcal{S}_{\rm g} = M^3 N_c^2 \int d^5x \sqrt{g} \left[R - \frac{4}{3}(\partial\phi)^2 + V_g(\phi) \right]$$

with Poincaré invariant metric

$$ds^2 = e^{2A}(dr^2 + \eta_{\mu\nu}x^\mu x^\nu)$$

- Potential $V_g \leftrightarrow$ QCD β-function
  - $A \to \log\mu$ energy scale
  - $e^\phi \to \lambda$ ’t Hooft coupling $g^2 N_c$

$$V_g = \frac{12}{\ell^2}(1 + c_1\lambda + \cdots), \quad \lambda \to 0, \qquad V_g \sim \lambda^{4/3}\sqrt{\log\lambda}, \quad \lambda \to \infty$$

Agrees well with pure YM, both a zero and finite temperature

[Gursoy, Kiritsis, Mazzanti, Nitti; Panero; . . .]

A step back: Adding flavor

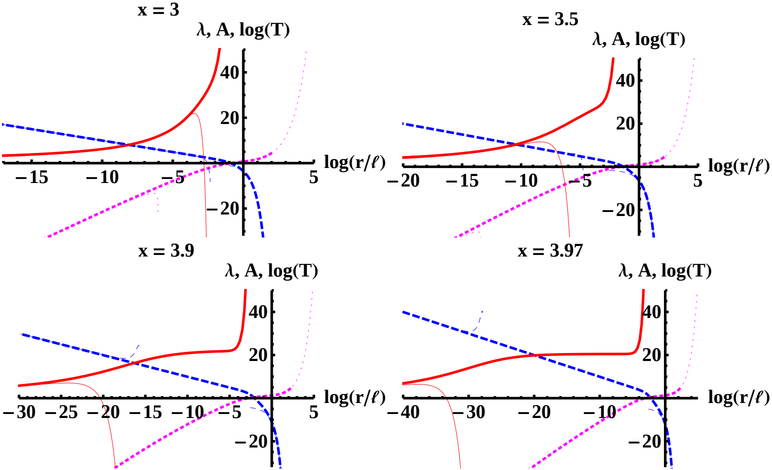
- Fundamental quarks $\rightarrow$ probe $D4 - \bar{D}4$ branes in 5D
- For the vacuum structure only the tachyon is relevant
- A tachyon action motivated by the Sen action
  - Confining asymptotics of the geometry trigger ChSB
  - Gell-Mann-Oakes-Renner relation
  - Linear Regge trajectories for mesons
  - A very good fit of the light meson masses

[Klebanov,Maldacena]

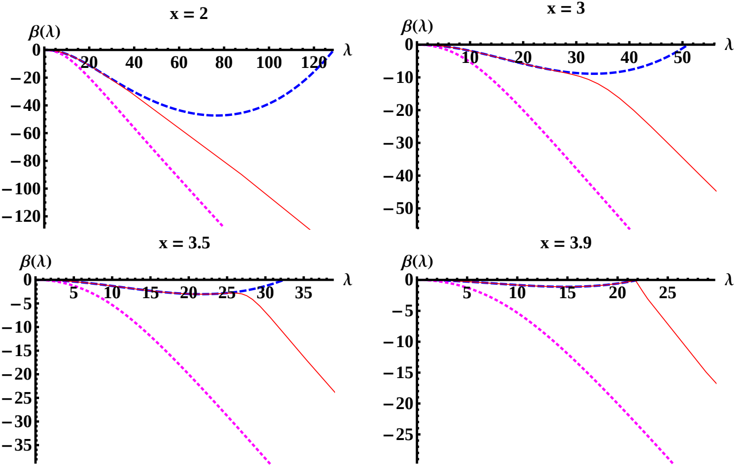
[Bigazzi,Casero,Cotrone,Iatrakis,Kiritsis,Paredes hep-th/0505140,0702155; arXiv:1003.2377,1010.1364]

Backgrounds in the walking region

Backgrounds with zero quark mass, $x < x_c \simeq 3.9959$ (λ, A, τ)



Beta functions along the RG flow (evaluated on the background),
zero tachyon, YM $x_c \simeq 3.9959$



Holographic beta functions

Generalization of the holographic RG flow of IhQCD

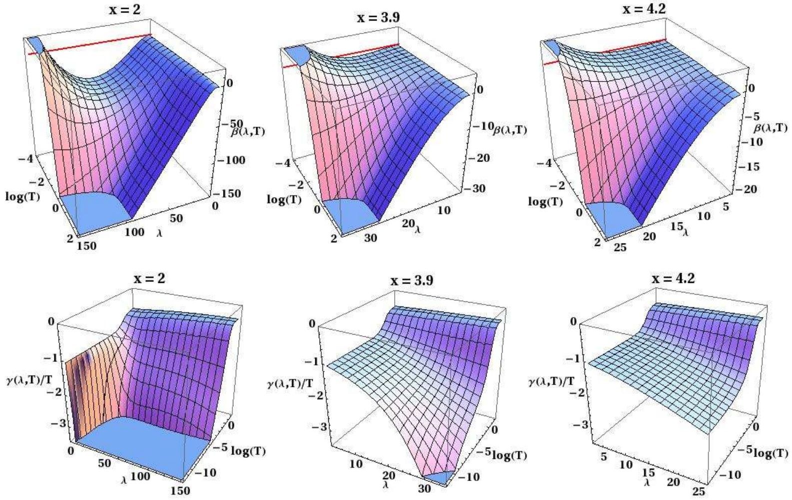
$$\beta(\lambda,\tau) \equiv \frac{d\lambda}{dA}\ ; \qquad \gamma(\lambda,\tau) \equiv \frac{d\tau}{dA}$$

linked to

$$\frac{dg_{\rm QCD}}{d\log\mu}\ ; \qquad \frac{dm}{d\log\mu}$$

The full equations of motion boil down to two first order partial non-linear differential equations for β and γ

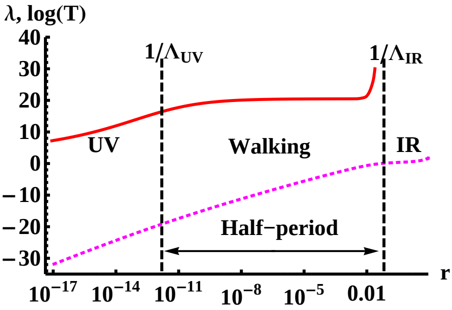
“Good” solutions numerically (unique)



Miransky/BKT scaling

As $x \to x_c$ from below: walking, dominant solution

- BF-bound for the tachyon violated at the IRFP
- x_c fixed by the BF bound:
  $\Delta = 2$ & $\gamma_* = 1$
  at the edge of the conformal window



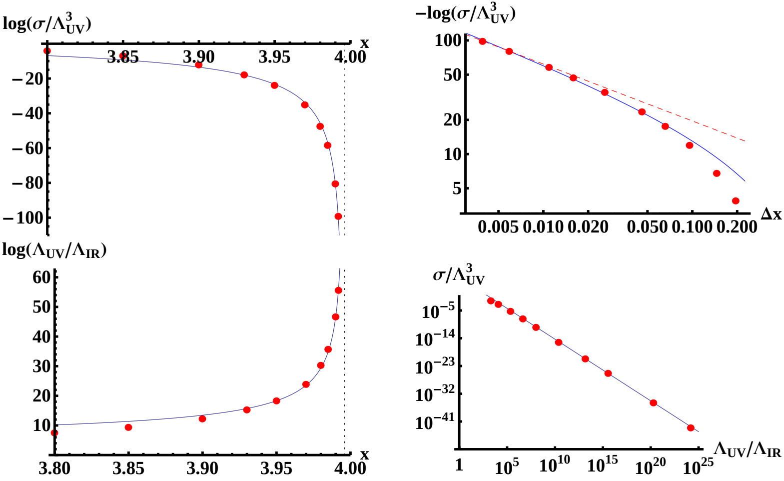
- $T(r) \sim r^2 \sin(\kappa\sqrt{x_c - x} \log r + \phi)$ in the walking region
- "0.5 oscillations" $\Rightarrow$ Miransky/BKT scaling, amount of walking $\Lambda_{\rm UV}/\Lambda_{\rm IR} \sim \exp(\pi/(\kappa\sqrt{x_c - x}))$

As $x \to x_c$

with known κ

$$\langle \bar{q}q \rangle \sim \sigma \sim \exp(-2\pi/(\kappa\sqrt{x_c - x}))$$

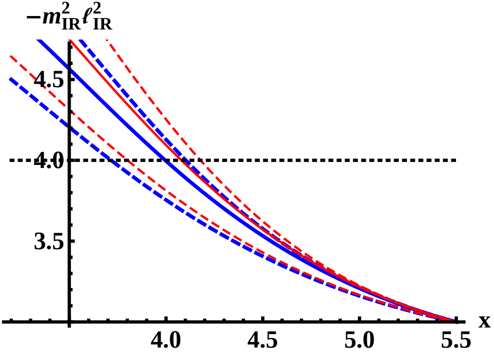
$$\Lambda_{\mathrm{UV}}/\Lambda_{\mathrm{IR}} \sim \exp(\pi/(\kappa\sqrt{x_c - x}))$$



Prediction for x_c

Dependence on the UV parameter W_0 and (reasonable) "IR choices" for the potentials

Resulting variation of the edge of conformal window $x_c = 3.7 \ldots 4.2$

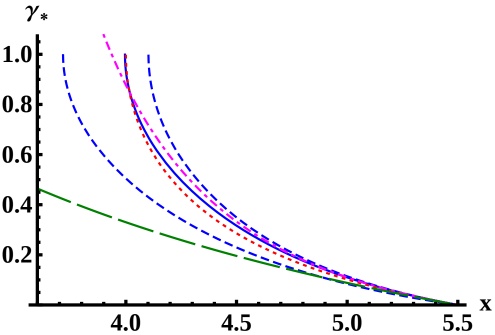


γ_* in the conformal window

Comparison to other guesses

V-QCD (dashed: variation due to W_0)
Dyson-Schwinger
2-loop PQCD
All-orders β

[Pica, Sannino arXiv:1011.3832]

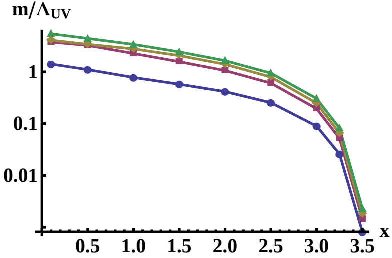


Mass spectra

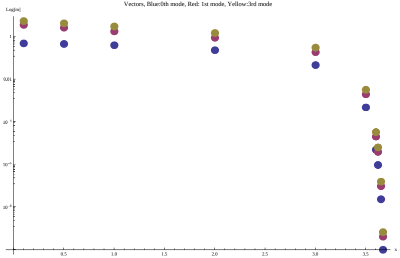
Full fluctuation analysis

- Miransky scaling
- Ratios depend mildly on x

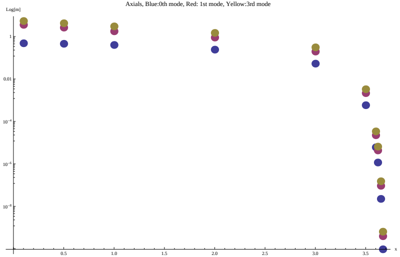
Scalars



Vectors



Axial vectors



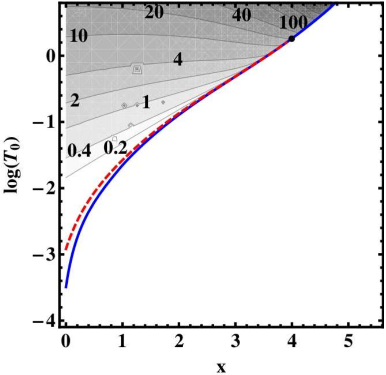
Parameters

Understanding the solutions for generic quark masses requires discussing parameters

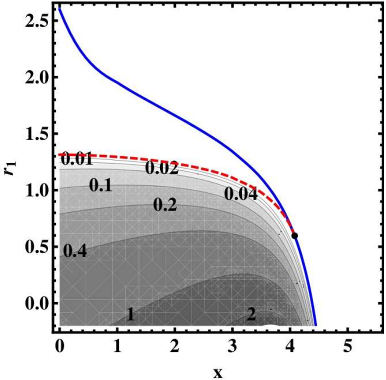
- YM or QCD with massless quarks: no parameters
- QCD with flavor-independent mass m: a single (dimensionless) parameter $m/\Lambda_{\rm QCD}$
- In this model, after rescalings, this parameter can be mapped to a parameter (τ_0 or r_1) that controls the diverging tachyon in the IR
- x has become continuous in the Veneziano limit

Map of all solutions

All "good" solutions ($\tau \neq 0$) obtained varying x and τ_0 or r_1
Contouring: quark mass (zero mass is the red contour)

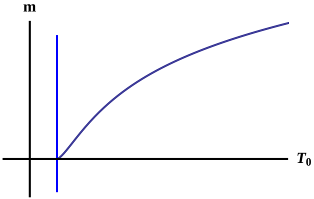


"Potentials I" $\leftrightarrow$ T_0



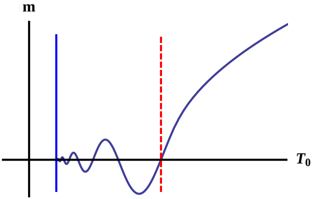
"Potentials II" $\leftrightarrow$ r_1

Mass dependence and Efimov vacua



Conformal window ($x > x_c$)

- For $m = 0$, unique solution with $\tau \equiv 0$
- For $m > 0$, unique "standard" solution with $\tau \neq 0$

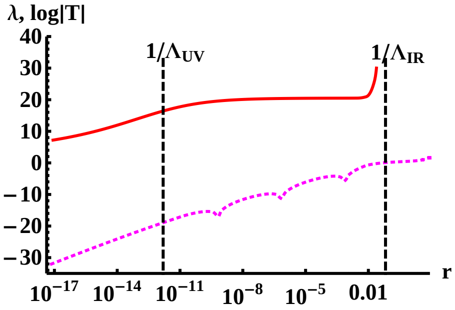


Low $0 < x < x_c$: Efimov vacua

- Unstable solution with $\tau \equiv 0$ and $m = 0$
- "Standard" stable solution, with $\tau \neq 0$, for all $m \geq 0$
- Tower of unstable Efimov vacua (small $|m|$)

Efimov solutions

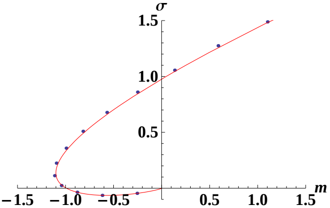
- Tachyon oscillates over the walking regime
- $\Lambda_{UV}/\Lambda_{IR}$ increased wrt. "standard" solution

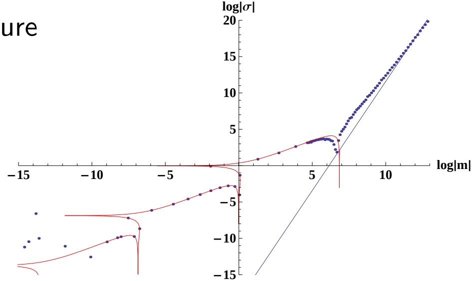


Efimov spiral

Ongoing work: $\sigma(m)$ dependence

- For $x < x_c$ spiral structure





- Dots: numerical data
- Continuous line: (semi-)analytic prediction

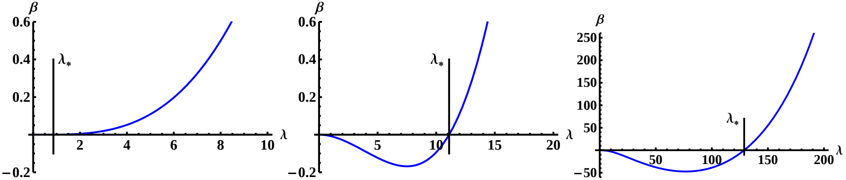
Allows to study the effect of double-trace deformations

Effective potential: zero tachyon

Start from Banks-Zaks region, $\tau_* = 0$, chiral symmetry conserved $(\tau \leftrightarrow \bar{q}q)$, $V_{\rm eff}(\lambda) = V_g(\lambda) - xV_{f0}(\lambda)$

- $V_{\rm eff}$ defines a β-function as in IhQCD – Fixed point guaranteed in the BZ region, moves to higher λ with decreasing x
- Fixed point λ_* runs to ∞ either at finite $x(< x_c)$ or as $x \to 0$

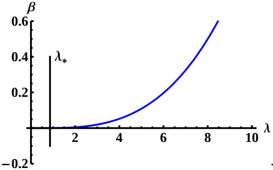
Banks-Zaks	Conformal Window	
$x \to 11/2$	$x > x_c$	$x < x_c$??

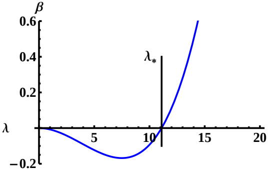


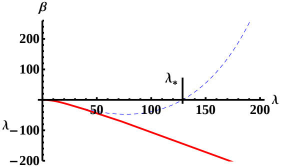
Effective potential: what actually happens

Banks-Zaks | Conformal Window

$x \to 11/2$ | $x > x_c$ | $x < x_c$







$\tau \equiv 0$ | $\tau \equiv 0$ | $\tau \neq 0$

- For $x < x_c$ vacuum has nonzero tachyon (checked by calculating free energies)
- The tachyon screens the fixed point
- In the deep IR τ diverges, $V_{\text{eff}} \to V_g \Rightarrow$ dynamics is YM-like

Where is x_c?

How is the edge of the conformal window stabilized?
Tachyon IR mass at $\lambda = \lambda_*$ $\leftrightarrow$ quark mass dimension

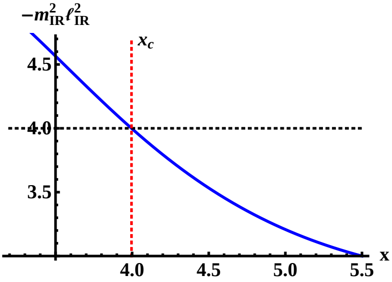
$$-m_{\rm IR}^2 \ell_{\rm IR}^2 = \Delta_{\rm IR}(4 - \Delta_{\rm IR}) = \frac{24 a(\lambda_*)}{\kappa(\lambda_*)(V_g(\lambda_*) - x V_0(\lambda_*))}$$

$$\gamma_* = \Delta_{\rm IR} - 1$$

Breitenlohner-Freedman (BF) bound (horizontal line)

$$-m_{\rm IR}^2 \ell_{\rm IR}^2 = 4 \;\Leftrightarrow\; \gamma_* = 1$$

defines x_c



Why $\gamma_* = 1$ at $x = x_c$?

No time to go into details ... the question boils down to the linearized tachyon solution at the fixed point

- For $\Delta_{\rm IR}(4 - \Delta_{\rm IR}) < 4 \qquad (x > x_c)$:

$$\tau(r) \sim m_q r^{\Delta_{\rm IR}} + \sigma r^{4-\Delta_{\rm IR}}$$

- For $\Delta_{\rm IR}(4 - \Delta_{\rm IR}) > 4 \qquad (x < x_c)$:

$$\tau(r) \sim C r^2 \sin\left[(\mathrm{Im}\Delta_{\rm IR}) \log r + \phi\right]$$

Rough analogy:
Tachyon EoM $\leftrightarrow$ Gap equation in Dyson-Schwinger approach
Similar observations have been made in other holographic frameworks

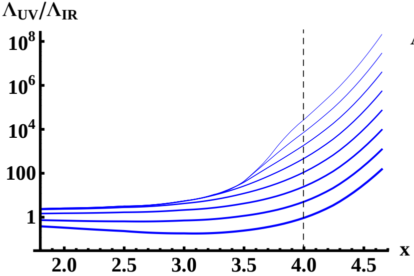
[Kutasov, Lin, Parnachev arXiv:1107.2324, 1201.4123]

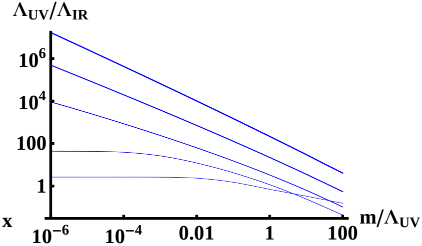
Mass dependence

For $m > 0$ the conformal transition disappears

The ratio of typical UV/IR scales $\Lambda_{\mathrm{UV}}/\Lambda_{\mathrm{IR}}$ varies in a natural way

$m/\Lambda_{\mathrm{UV}} = 10^{-6}, 10^{-5}, \ldots, 10 \qquad x = 2, 3.5, 3.9, 4.25, 4.5$





sQCD phases

The case of $\mathcal{N} = 1$ $SU(N_c)$ superQCD with N_f quark multiplets is known and provides an interesting (and more complex) example for the nonsupersymmetric case. From Seiberg we have learned that:

- $x = 0$ the theory has confinement, a mass gap and N_c distinct vacua associated with a spontaneous breaking of the leftover R symmetry Z_{N_c}.
- At $0 < x < 1$, the theory has a runaway ground state.
- At $x = 1$, the theory has a quantum moduli space with no singularity. This reflects confinement with ChSB.
- At $x = 1 + 1/N_c$, the moduli space is classical (and singular). The theory confines, but there is no ChSB.
- At $1 + 2/N_c < x < 3/2$ the theory is in the non-abelian magnetic IR-free phase, with the magnetic gauge group $SU(N_f - N_c)$ IR free.
- At $3/2 < x < 3$, the theory flows to a CFT in the IR. Near $x = 3$ this is the Banks-Zaks region where the original theory has an IR fixed point at weak coupling. Moving to lower values, the coupling of the IR $SU(N_c)$ gauge theory grows. However near $x = 3/2$ the dual magnetic $SU(N_f - N_c)$ is in its Banks-Zaks region, and provides a weakly coupled description of the IR fixed point theory.
- At $x > 3$, the theory is IR free.

Saturating the BF bound (sketch)

Why is the BF bound saturated at the phase transition (massless quarks)??

$$\Delta_{\mathrm{IR}}(4-\Delta_{\mathrm{IR}}) = \frac{24a(\lambda_*)}{\kappa(\lambda_*)(V_g(\lambda_*) - xV_0(\lambda_*))}$$

- For $\Delta_{\mathrm{IR}}(4-\Delta_{\mathrm{IR}}) < 4$:
  $\tau(r) \sim m_q r^{4-\Delta_{\mathrm{IR}}} + \sigma r^{\Delta_{\mathrm{IR}}}$
- For $\Delta_{\mathrm{IR}}(4-\Delta_{\mathrm{IR}}) > 4$:
  $\tau(r) \sim C r^2 \sin\left[(\mathrm{Im}\Delta_{\mathrm{IR}})\log r + \phi\right]$
- Saturating the BF bound, the tachyon solutions will engtangle $\longrightarrow$ required to satisfy boundary conditions
- Nodes in the solution appear trough UV $\longrightarrow$ massless solution

Saturating the BF bound (sketch)

Does the nontrivial (ChSB) massless tachyon solution exist?
Two possibilities:

- $x > x_c$: BF bound satisfied at the fixed point $\Rightarrow$ only trivial massless solution ($\tau \equiv 0$, ChS intact, fixed point hit)
- $x < x_c$: BF bound violated at the fixed point $\Rightarrow$ a nontrivial massless solution exist, which drives the system away from the fixed point

Conclusion: phase transition at $x = x_c$
As $x \to x_c$ from below, need to approach the fixed point to satisfy the boundary conditions $\Rightarrow$ nearly conformal, "walking" dynamics

Potentials I

$$
\begin{aligned}
V_g(\lambda) &= 12 + \frac{44}{9\pi^2}\lambda + \frac{4619}{3888\pi^4}\frac{\lambda^2}{(1+\lambda/(8\pi^2))^{2/3}}\sqrt{1+\log(1+\lambda/(8\pi^2))} \\
V_f(\lambda,\tau) &= V_{f0}(\lambda)e^{-a(\lambda)\tau^2} \\
V_{f0}(\lambda) &= \frac{12}{11} + \frac{4(33-2x)}{99\pi^2}\lambda + \frac{23473-2726x+92x^2}{42768\pi^4}\lambda^2 \\
a(\lambda) &= \frac{3}{22}(11-x) \\
\kappa(\lambda) &= \frac{1}{\left(1+\frac{115-16x}{288\pi^2}\lambda\right)^{4/3}}
\end{aligned}
$$

In this case the tachyon diverges exponentially:

$$
\tau(r) \sim \tau_0 \exp\left[\frac{81\ 3^{5/6}(115-16x)^{4/3}(11-x)}{812944\ 2^{1/6}}\frac{r}{R}\right]
$$

Potentials II

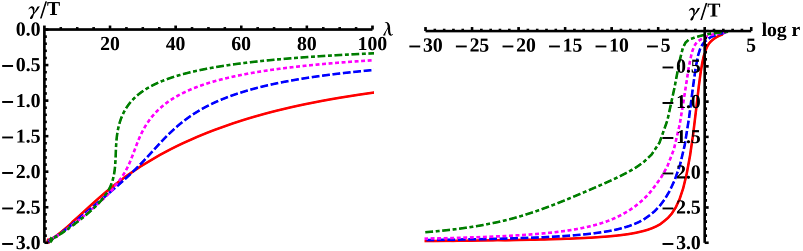
$$
\begin{aligned}
V_g(\lambda) &= 12 + \frac{44}{9\pi^2}\lambda + \frac{4619}{3888\pi^4}\frac{\lambda^2}{(1+\lambda/(8\pi^2))^{2/3}}\sqrt{1+\log(1+\lambda/(8\pi^2))} \\
V_f(\lambda,\tau) &= V_{f0}(\lambda)e^{-a(\lambda)\tau^2} \\
V_{f0}(\lambda) &= \frac{12}{11} + \frac{4(33-2x)}{99\pi^2}\lambda + \frac{23473-2726x+92x^2}{42768\pi^4}\lambda^2 \\
a(\lambda) &= \frac{3}{22}(11-x)\frac{1+\frac{115-16x}{216\pi^2}\lambda + \lambda^2/(8\pi^2)^2}{(1+\lambda/(8\pi^2))^{4/3}} \\
\kappa(\lambda) &= \frac{1}{(1+\lambda/(8\pi^2))^{4/3}}
\end{aligned}
$$

In this case the tachyon diverges as

$$
\tau(r) \sim \frac{27\ 2^{3/4}3^{1/4}}{\sqrt{4619}}\sqrt{\frac{r-r_1}{R}}
$$

Gamma functions

Massless backgrounds: gamma functions $\frac{\gamma}{\tau} = \frac{d \log \tau}{dA}$



$x = 2, 3, 3.5, 3.9$