Matrix Models, Duality & Spontaneous SUSY Breaking in 3D QFT

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based on

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1108.4963 with **Takeshi Morita** and ongoing work...

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Important progress has been recently achieved in the study of d=3 N=2 supersymmetric conformal field theories (SCFTs).

The central object of discussion has been the free energy of the Euclidean CFT on the three-sphere

$$F = -\log |Z_{S_3}|$$

The sphere partition function can be computed **exactly** in terms of a **matrix integral** expression using **localization** *(Kapustin et al. '09, Jafferis '10, Hama et al. '10).*

The exact information provided by F is useful for a variety of purposes.

Practical implications of *F*

- **F-maximization.** d=3 N=2 SCFTs have a conserved U(1) R-symmetry that sits in the same supermultiplet as the stress-energy tensor and controls the scaling dimension of chiral operators. This symmetry is **not** protected against quantum corrections. *F* is a functional of trial U(1) R-charges. The exact R-symmetry is the one that extremizes (maximizes?) *F (Jafferis '10)*.

  The exact spectrum of chiral operators provides information about potential supersymmetric RG flows.

- **Non-trivial checks of 3d dualities.** 3d theories are known to exhibit non-perturbative dualities (analogs of 4d S-duality and Seiberg duality). Matching *F* in dual pairs gives new non-trivial checks.

- **A 3d c-theorem.** *F* has been proposed to decrease along RG flows in 3d - ***F*-theorem** *(Myers et al. '10, '11, Jafferis et al. '11)*.
  [in ABJM F from weak to strong coupling: field theory counting of $N^{3/2}$ dof]

- **3D-4D connections.** The 3D sphere PF is closely related to 4D superconformal indices *(Dolan et al. '11, Gadde, Yan '11, Imamura '11)*.

- **Spontaneous SUSY breaking.** A new non-perturbative criterion for spontaneous supersymmetry breaking (SSB) in three-dimensional QFT *(Morita, VN '11)*.

  SSB occurs if and only if (Q-deformed Z_{S^3})=0.

We will discuss all these aspects in a specific theory:

U(N) N=2 Chern-Simons theory at level k coupled to a single adjoint chiral superfield X (with or w/o superpotential interactions).

Why is this theory interesting??

• W/o superpotential, this theory is believed to be exactly superconformal for any values N, k *(Gaiotto, Yin '07)*. *[Call this: **A-theory**]*

• The R-symmetry receives strong quantum corrections and the U(1) R-charge asymptotes to zero at strong coupling *(VN '09)*.
With increasing coupling more and more fields hit the unitarity bound and decouple as free fields: *the (interacting part of the) chiral ring is truncated from below.*

• With superpotential interactions of the form W=Tr X^{n+1} there is a range of parameters (*for any value of n*) where the superpotential is relevant and drives the theory to a new IR fixed point. There is also a range of parameters where the SUSY vacuum is lifted (SSB). *[Call deformed theory:* ***A_{n+1}-theory****]*

E.g. in planar t' Hooft limit $N, k \to \infty\ , \quad \lambda = \frac{N}{|k|} = \text{fixed}$

Tr X^{n+1} is relevant for any n for $\lambda^*_{n+1} < \lambda < n$

SSB occurs for $N > n|k|$, i.e. $\lambda > n$

• The $U(N)_k^{(n+1)}$ theory exhibits a 3d Seiberg duality *(VN '08)*

$$U(N)_k^{(n+1)} \ \leftrightarrow \ U(nk-N)_{-k}^{(n+1)}$$

Having a minimal matter content this is one of the simplest illustrations of 3d Seiberg duality and thus a useful playground for attempts to understand the general underpinnings of such dualities in field theory.

F-maximization

Independent non-perturbative information to be reproduced by F-max.

The SUSY breaking pattern and 3D duality in ***A_{n+1}-theory*** (most easily deduced from a brane construction *(VN '08)*) imply the non-perturbative inequalities *(VN '09)*

lower bound: deforming op. interacting before SSB

$$\left[\frac{n-3}{4}\right] \leq \lambda^*_{n+1} < \frac{n}{2}$$

upper bound: deforming op. relevant before SSB

$$\frac{1}{2(\lambda+1)} \leq R(\lambda) < \frac{2}{\lambda+1} \ , \quad \lambda = 1, 2, \dots$$

$$R(\lambda) < \frac{2}{2\lambda+1} \ , \quad \lambda = \frac{1}{2}, 1, \frac{3}{2}, \dots$$

from 3D Seiberg duality

Note: No holographic description of this theory in supergravity is expected. Cannot appeal to AdS/CFT for any information about this theory. A new type of non-perturbative test of F-extremization *(VN '11)*.

It is convenient to study these theories in the large-N 't Hooft limit.

We are instructed to maximize the free energy

$$F = -\log \left| \frac{1}{N!} \int \left(\prod_{j=1}^{N} e^{\frac{i\pi N}{\lambda} t_j^2} dt_j \right) \prod_{i<j}^{N} (2\sinh(\pi t_{ij}))^2 \prod_{i,j=1}^{N} e^{\ell(1-R+it_{ij})} \right|$$

- This function was computed (and extremized) in the large-N limit using the saddle point approximation. This entails solving the algebraic eqs

$$\mathcal{I}_i \equiv \frac{i}{\lambda} t_i + \frac{1}{N} \sum_{j\neq i} \left[\coth(\pi t_{ij}) - \frac{(1-R)\sinh(2\pi t_{ij}) + t_{ij}\sin(2\pi R)}{\cosh(2\pi t_{ij}) - \cos(2\pi R)} \right] = 0 \ , \quad i = 1, 2, \ldots, N$$

at a saddle point configuration

$$\mathcal{F}(\lambda, N) = -\log N! + \sum_{j=1}^{N} \frac{i\pi N}{\lambda} t_j^2 + \sum_{i<j} \log(4\sinh^2(\pi t_{ij})) + \sum_{i,j=1}^{N} \ell(1-R+it_{ij})$$

$$\ell(z) := -z\log\left(1-e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi i z}\right)\right) - \frac{i\pi}{12}$$

• In general, the t_i's that solve these equations are complex numbers.

• We solved these equations numerically. Practically, we introduced a ficticious time coordinate τ and solved the differential equations

$$a\frac{dt_i(\tau)}{d\tau} = \mathcal{I}_i$$

With suitably chosen coefficient *a* the solution converges very quickly to the equilibrium configuration we are looking for.

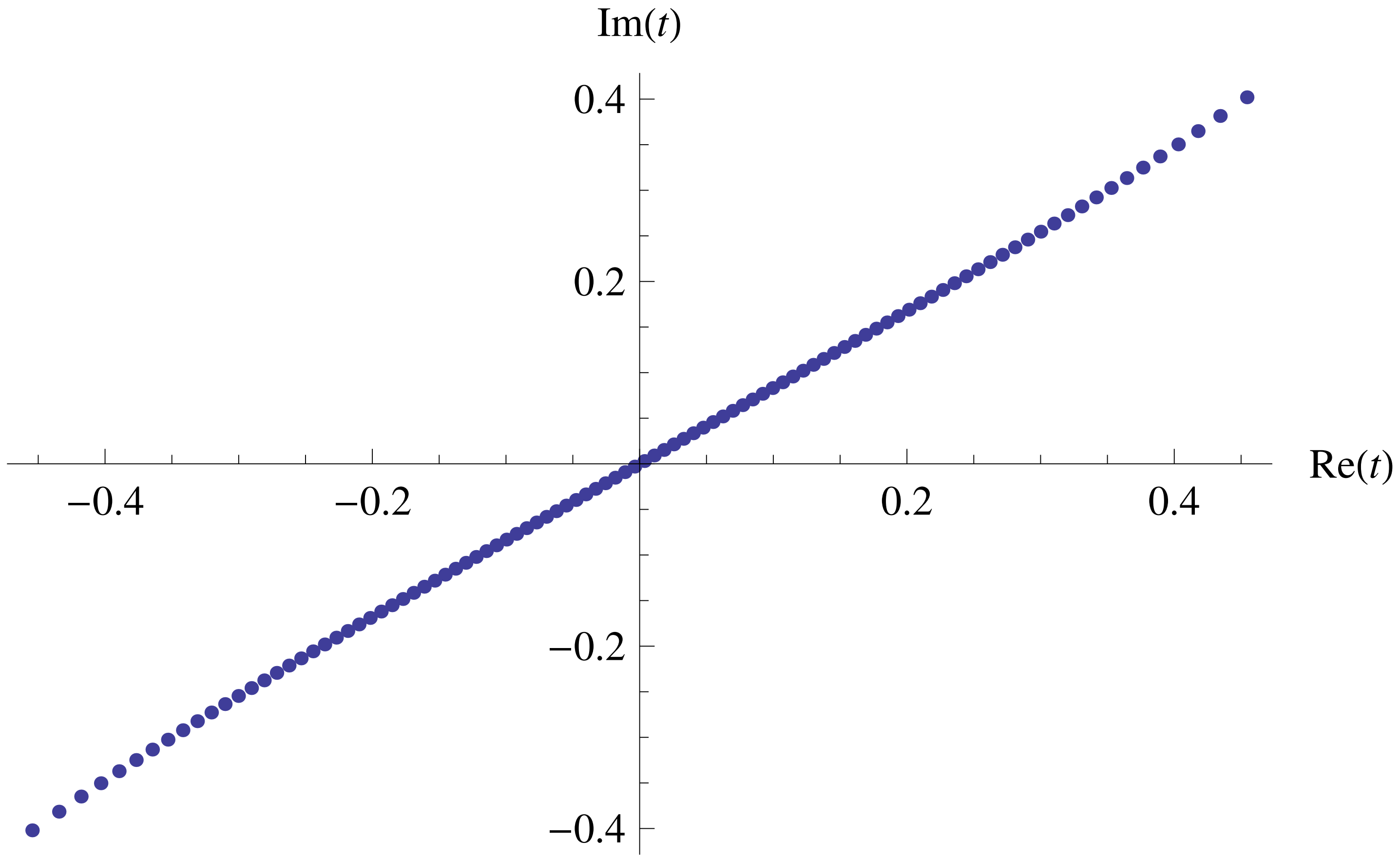
• The solution is sensitive to *a* and initial conditions (which results to different types of solutions: one-cut, multi-cut).

• Implemented this approach numerically for various values of N.
At N=100 the numerical result seems to approach the large-N asymptote within a few percent.

Lessons.

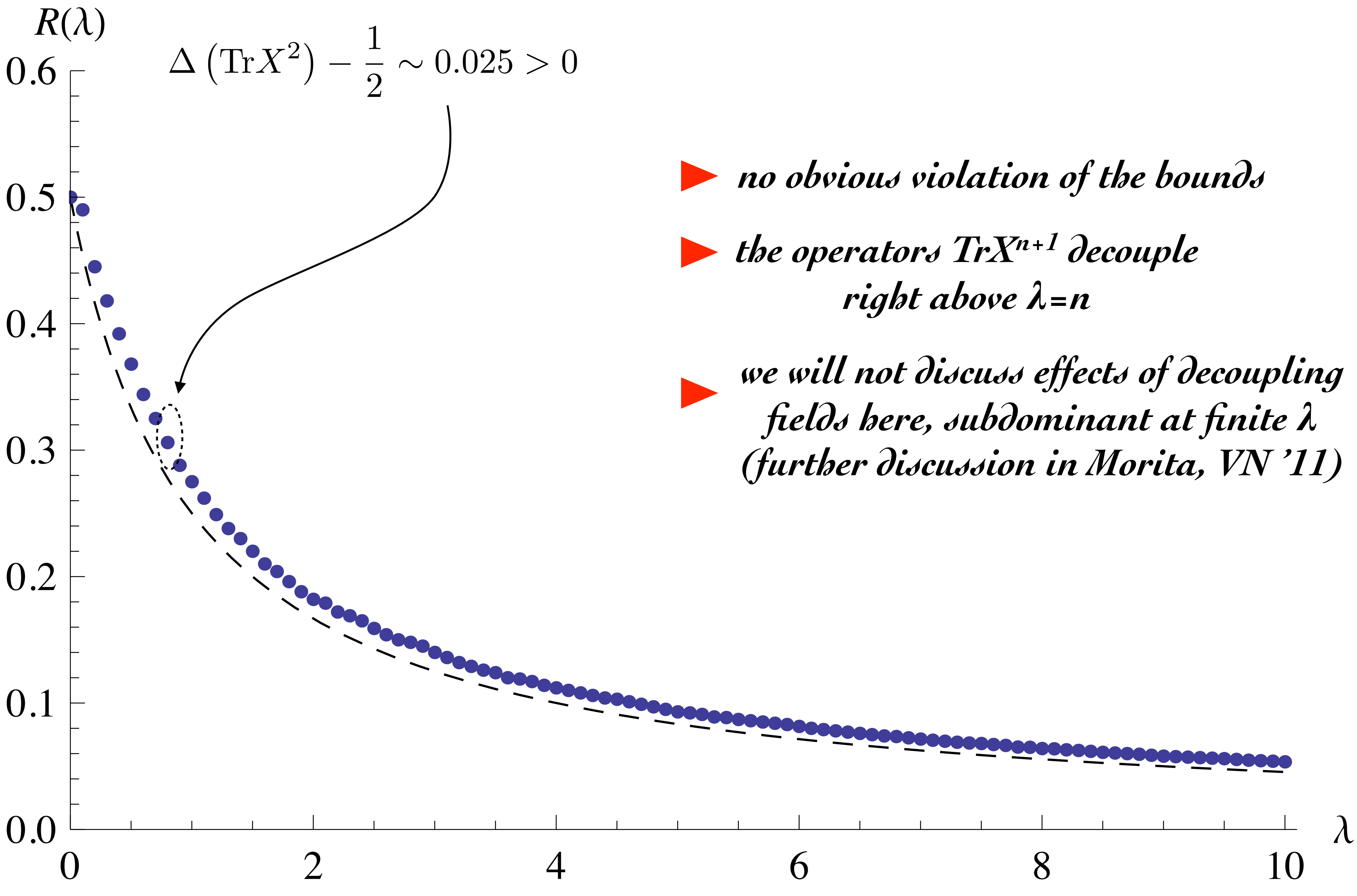
1) We compute in the planar limit soln's as a function of R, λ. When one-cut soln's exist they are observed to be dominant (minimal free energy).

A typical distribution of the eigenvalues t_i in the complex plane in 1-cut sol's (this plot obtained for N=100, λ=1, R=0.225)



2) R(λ) after F-maximization using the 1-cut solutions.

>F-maximization passes the test of the non-perturbative inequalities<



- *no obvious violation of the bounds*
- *the operators TrX^{n+1} decouple right above λ=n*
- *we will not discuss effects of decoupling fields here, subdominant at finite λ (further discussion in Morita, VN '11)*

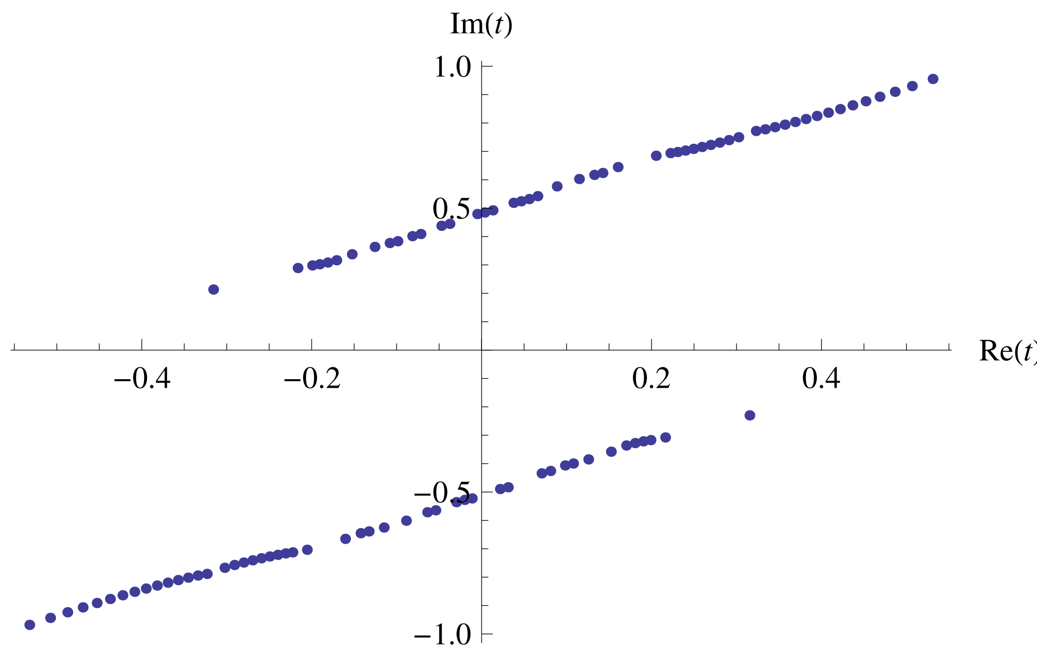
3) The one-cut soln's pass a number of independent checks.

i) they reproduce nicely the analytic perturbative soln *(Minwalla et al '11)*,

ii) reproduce quantitative predictions from 3d duality and *F*-theorem.

4) Multi-cut soln's crossing the imaginary axis through the points $\pm mi/2$, $\pm(R+m)i/2$ $(m=1,2,\ldots)$ exist for all range of parameters. They can be traced perturbatively at weak coupling and numerically at any coupling.

A typical 2-cut soln for N=100, λ=1, R=0.5.



5) In regimes where the 1-cut solution exist we observed that the function *F(λ,R)* (with fixed λ) has always a **single (smooth) maximum** but can also have smooth minima and points of divergence. The *R*-behavior changes as λ is varied.

6) There are regimes where the 1-cut soln's cease to exist and one is left with the multi-cut soln's.

This is a very interesting matrix model effect (previously observed in simpler matrix models by *Marino et al.* and related to discussions of the large-order behavior of matrix models in the 1/*N*-expansion). The exactly computable CS matrix model provides a clean illustration. This effect is closely related to zeros of the matrix model PF.

Our contribution: ***we associate such effects to SSB in quantum field theory***. (more about this later...)

3d Seiberg duality

The duality
$$U(N)_k^{(n+1)} \quad \Leftrightarrow \quad U(nk-N)_k^{(n+1)}$$

proposed by *VN '08* is supported by

- a D-brane argument,

- at *n=1* it reduces to level-rank duality in Chern-Simons theory,

- S^3 PF matching can be connected to SCI matching of 4d dualities via the 3d-4d connection (e.g. Intriligator duality for Sp(2*N*) SQCD theories *(Dolan, Spiridonov, Vartanov '11)*, but requires mathematical identities for 4d SCIs that have not been proven yet).

[however, a better understanding of this connection is needed and work in this direction is underway...]

The duality makes a series of non-trivial predictions for the sphere PF

$$F\left(\frac{2}{n+1};N,k\right)=F\left(\frac{2}{n+1};nk-N,k\right)$$

or in the 't Hooft limit

$$F\left(\frac{2}{n+1};\lambda\right)=F\left(\frac{2}{n+1};n-\lambda\right)\quad \lambda\in(\lambda^*_{n+1},n-\lambda^*_{n+1})\ ,\quad n=1,2,...$$

Checks.

1) *n=1 (topological case)*. The hard $e^{l(...)}$ drop out and direct computation gives

$$Z_{S^3}\left[U(N)^{(2)}_k\right]=e^{-\frac{\pi i}{12}(k^2-6k+2)}Z_{S^3}\left[U(k-N)^{(2)}_{-k}\right]$$

2) *N=2, k=1, n=3 (a simple non-topological case)*. Again, by brute force computation

$$Z_{S^3}\left[U(2)^{(4)}_1\right]=e^{\frac{3\pi i}{8}}Z_{S^3}\left[U(1)^{(4)}_{-1}\right]$$

similar to an $SU(2)_1$ duality by Jafferis, Yin '11

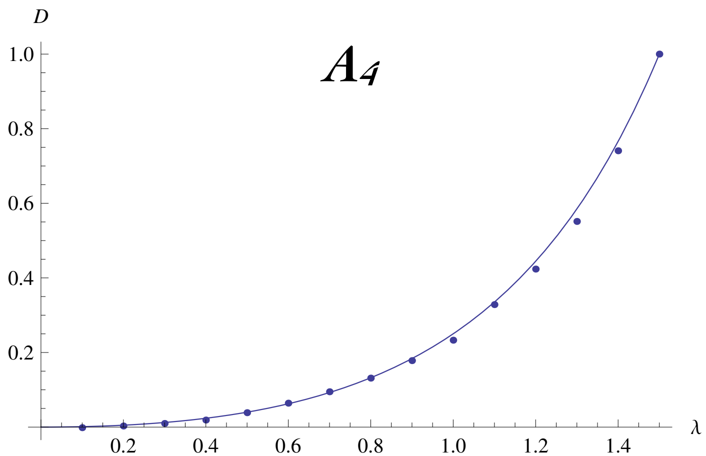
non-topological

topological U(1) CS + free field

3) *Large-N, planar limit.*

$$f(\lambda) = \frac{1}{N^2} F(\lambda, N)$$

$$D(\lambda) := \frac{f(n-\lambda)}{f(\lambda)} = \lambda^2 (n-\lambda)^{-2}$$



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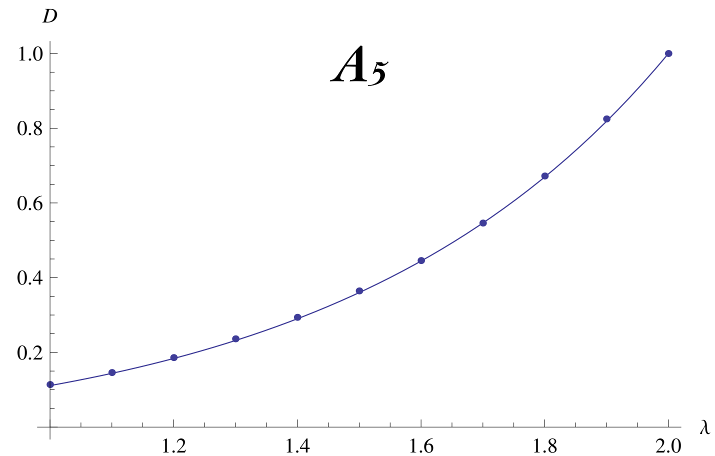

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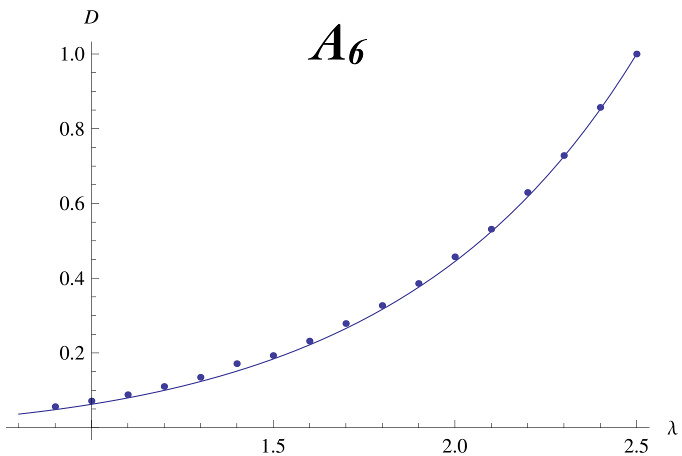

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4) *Other aspects of duality.* The duality itself is rather interesting & nontrivial.

In the $U(N)_k{}^{(n+1)}$ theory there are several regimes as we vary λ

The deforming op. is irrelevant in elec. theory: A-theory fixed point

Deforming op. is highly relevant in elec. theory, but irrelevant in magn. theory

spont. SUSY breaking

\`Conformal window': the deforming op. is relevant in both electric and magnetic descriptions

naively still $m = \left[\frac{n+1}{4}\right]$

and m (+ R-charge assignment) is not duality invariant. Naive elec. description is not valid.

$$m = \left[\frac{1}{2R(\lambda)}\right]$$

$$m = \left[\frac{n+1}{4}\right]$$

<operator recoupling>

m : number of decoupled operators, i.e. decoupled ops TrX, TrX²,..., TrXᵐ

F-theorem

A web of RG flows.

Deformations by the general superpotential $W = \sum_{i=0}^{n} \frac{g_i}{n+1-i} \mathrm{Tr} X^{n+1-i}$ generate RG flows of the type

$$U(N)_k^{(n+1)} \to U(N_1)_k^{(n_1+1)} \times \cdots \times U(N_\ell)_k^{(n_\ell+1)} \ , \quad \sum_{i=1}^{\ell} N_i = N \ , \quad \sum_{i=1}^{\ell} n_i = n$$

SUSY vacuum for each of the factors iff $N_i \leq n_i k$

***F*-theorem predictions.**

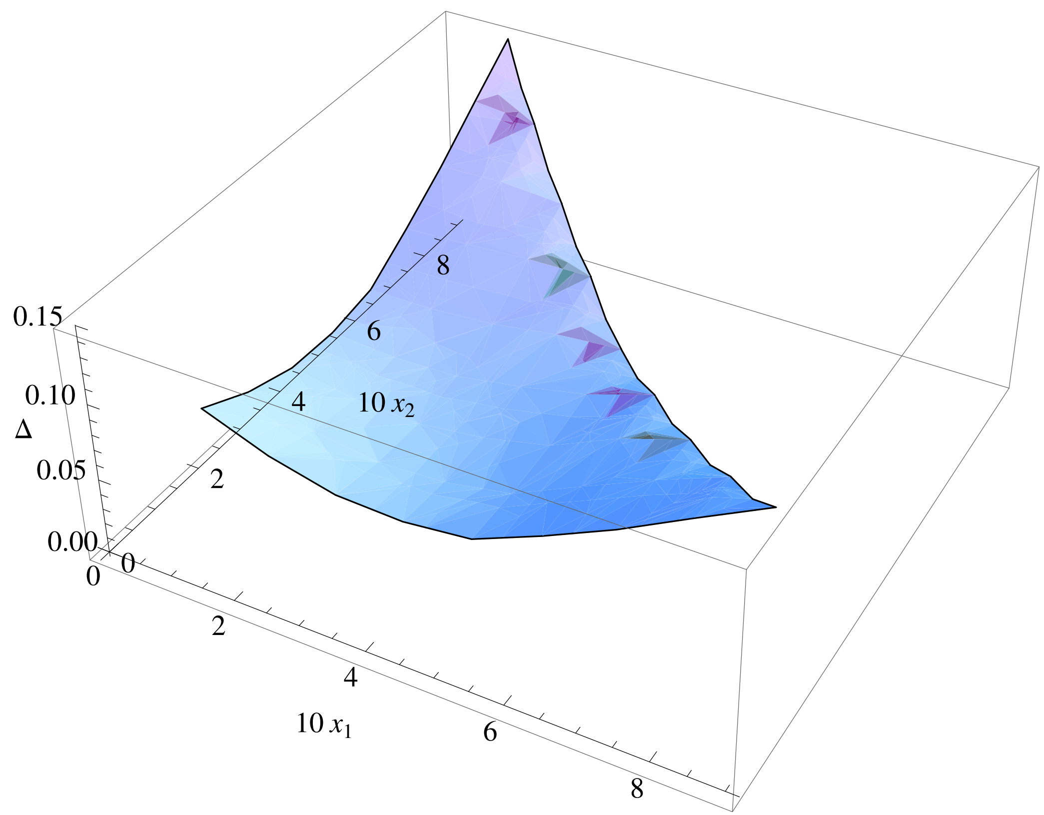
$$F_{UV} > F_{IR} \ \Leftrightarrow \ F_{N,k}^{(n+1)} > \sum_{i=1}^{\ell} F_{N_i,k}^{(n+1)}$$

In the 't Hooft limit set of non-trivial inequalities:

$$f\left(\sum_{i=1}^{\ell} \lambda_i ; \sum_{i=1}^{\ell} n_i\right) > \sum_{i=1}^{\ell} x_i^2 f(\lambda_i, n_i) \ , \quad N_i = x_i N \ , \quad \lambda_i = x_i \lambda \ , \quad 0 < x_i < 1 \ , \quad \sum_{i=1}^{\ell} x_i = 1$$

Shown to hold in several examples using numerical results.

Example: $A_{11} \rightarrow A_4 \otimes A_4 \otimes A_5$



In this case 2 independent parameters: x_1, x_2 . Plot for λ=3.
F-theorem requires Δ>0, which is verified.

Matrix models and SSB in 3d QFT

Consider a general 3d classically superconformal QFT.

Place the Euclidean version of this theory on a 3-sphere and compute the partition function

$$Z_{S^3} = \int e^{S+t\{Q,V\}}$$

S: action of the theory,
Q: one of the supercharges (that generates a symmetry of S),
V: a suitably chosen fermionic interaction, so that $\{Q,V\}$ is positive definite.

$Z_{S^3}(t=0)$ is the standard 3-sphere PF of the theory.
In a theory with a SUSY vacuum a standard argument shows

$$\frac{dZ_{S^3}}{dt} = \int \{Q,V\} e^{S+t\{Q,V\}} = 0$$

Hence, $$Z_{S^3}(t=0) = \lim_{t\to\infty} Z_{S^3}(t) := Z^{(loc)}_{S^3}$$

In $Z_S{}^{3(loc)}$ the path integral is localized to configurations that solve the equation $\{Q,V\}$=0 and reduces (as shown by *Kapustin et al '09* for general 3D gauge theories) to a matrix integral.

For general N=2 theories $Z_S{}^{3(loc)}$ depends also on trial R-charges. One can fix this ambiguity by F-maximization.

W/o any apriori knowledge about the fate of SUSY at the quantum level $Z_S{}^{3(loc)}$ is the quantity one would naturally compute in any case.

When SUSY is spontaneously broken the above logic breaks down

$$\frac{dZ_{S^3}}{dt} \neq 0 \;\Rightarrow\; Z_{S^3}(t=0) \neq Z^{(loc)}_{S^3}$$

Still, we propose that $Z_{S^{3}}^{(loc)}$ is an interesting quantity to compute and the above breakdown occurs in a fashion that gives a characteristic signal of SSB.

Conjecture: SSB occurs if and only if $Z_{S^{3}}^{(loc)}=0$.

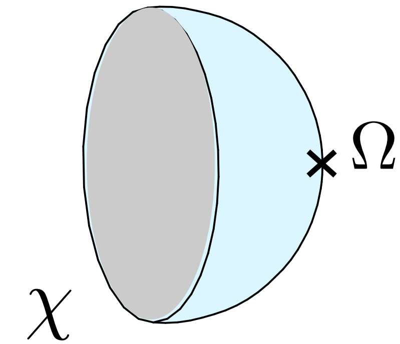
A heuristic argument.

- **SSB implies $Z_{S^{3}}^{(loc)}=0$**

Three useful facts:

i) For a theory on a 3-manifold M (here $M=S^3$) with an S^2 boundary there is a natural way one can associate a Hilbert space H with the boundary S^2 *(Witten '89, see also a review in Ginsparg, Moore 9304011)*.

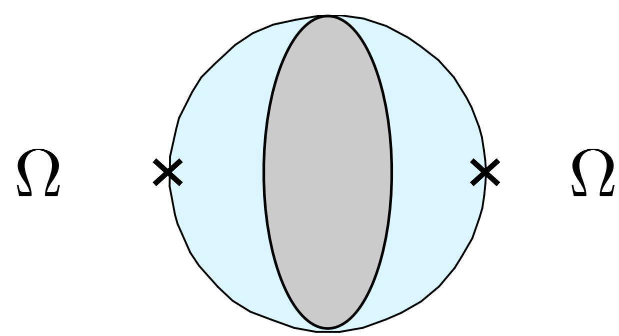
The path integral $$Z(\chi) = \int_{\Phi|_{S^2}=\chi} e^S = \langle \Omega | \chi \rangle$$



defines a vector $|\chi\rangle \in \mathcal{H}$. $\langle \Omega |$ is the insertion of the vacuum operator at the pole of the hemi-3sphere.

With an insertion of the identity (here interested in the Q-deformed path integral that computes $Z_{S^{3(loc)}}$)

$$Z_{S^3} = \langle \Omega | \Omega \rangle = \int d\chi Z(\chi) Z^*(\chi) = \int d\chi \langle \Omega | \chi \rangle \langle \chi | \Omega \rangle$$



$$\sum_{\chi}$$

ii) Localization is based on the supercharge Q. There is an additional supercharge Q^+ with anticommutator

$$\{Q, Q^\dagger\} = M + R$$

M: rotation on S^3 that can be viewed as a translation along the Hopf fiber

$$S^1 \hookrightarrow S^3 \to S^2$$

R: R-symmetry operator.

In Lorentzian signature one would have

$$\langle\Omega|M|\Omega\rangle = \langle\Omega|\{Q, Q^\dagger\}|\Omega = |Q|\Omega\rangle|^2 + \left|Q^\dagger|\Omega\rangle\right|^2$$

which would imply that a ground state (has R=0) is supersymmetric iff it has M=0. In Euclidean signature Q^+ is not the Hermitian conjugate of Q so this argument is more subtle. Nevertheless, ***assume*** that the conclusion holds.

iii) One can show (using a simple argument that Witten made in *'82* in the context of the Witten index) that the M=0 ground states are precisely the R=0 states $|\alpha\rangle$ obeying the eqs

$$Q|\alpha\rangle = 0 \ , \quad |\alpha\rangle \neq Q|\beta\rangle \ , \quad \text{for any state } |\beta\rangle$$

■

We can now proceed to the argument:

1) The states $|\chi\rangle$ that contribute to $Z_S^{3(loc)}$ solve the equation

$$\{Q, V\}|\chi\rangle = 0$$

2) So there are states of the form $|\alpha\rangle = VQ|\chi\rangle$ that are annihilated by Q.

3) If these states have R=0 and are not Q-exact then we infer from *ii)* and *iii)* that they are supersymmetric ground states.

We conclude that:

SSB implies no SUSY ground states.

So, no states with M=R=0 including states of the form $|\alpha\rangle = VQ|\chi\rangle$.

Hence, no states annihilated by $\{Q,V\}$ that can contribute to $Z_S{}^{3(loc)}$ and therefore $\boldsymbol{Z_S{}^{3(loc)}=0}$.

Loose ends:

•Is it true that the states $|\alpha\rangle = VQ|\chi\rangle$ are not Q-exact?

Seems plausible but we have not been able to prove it.

•Is it true that the states $|\alpha\rangle = VQ|\chi\rangle$ have R=0? Solving the localization equations in general 3D gauge theories one finds that the path integral reduces to an integration over the zero R-charge vector multiplet scalar. This may imply that $|\chi\rangle$ have R=0. Since $R(VQ) = 0$ the same would hold for the states $|\alpha\rangle = VQ|\chi\rangle$

- **$Z_{S^{3(loc)}}$=*0* implies SSB**

This is also natural for the following reason.

If there is a SUSY vacuum, $Z_{S^{3(loc)}}$ computes the physical 3-sphere PF. Being a rough `measure of degrees of freedom' F is naturally expected to be a finite quantity. Hence, finding $Z_{S^{3(loc)}}$=0 implies F diverges and cannot be the 3-sphere free energy of the theory. This implies

$$\frac{dZ_{S^3}}{dt} \neq 0 \;\Rightarrow\; Z_{S^3}(t=0) \neq Z_{S^3}^{(loc)}$$

which can be attributed to SSB.

A potential pitfall: The computation of $Z_{S^{3(loc)}}$ in terms of a matrix integral relies on the assumption that the UV Lagrangian captures correctly the quantum IR physics.

If this assumption fails then one can encounter situations where a SUSY vacuum exists but the matrix model computation gives falsely $Z_{S^3}^{(loc)}=0$.

An example was presented recently by *Benini et al.* In 3d SQCD with N_f (anti)fundamental multiplets. For $N_f = N_c-1$ this theory is believed to have a deformed moduli space of SUSY vacua (*Aharony et al. '97*) but the matrix model gives $Z_{S^3}^{(loc)}=0$. The IR theory is a free theory of neutral chiral multiplets and

$$Z_{S^3} = Z_{S^3}^{(loc)} \neq 0$$

in accordance with our conjecture.

Other checks of the conjecture.

vanishes as expected in the SUSY breaking regime N>k

1) $\mathcal{N}$=2 Chern-Simons theory:

$$|Z_{S^3}(\lambda)| = \frac{1}{N!}\left|\int \left(\prod_{j=1}^{N} e^{i\pi k t_j^2} dt_j\right)\prod_{i<j}^{N}(2\sinh(\pi t_{ij}))^2\right| = \frac{1}{k^{N/2}}\prod_{m=1}^{N-1}\left(2\sin\frac{\pi m}{k}\right)^{N-k}$$

2) (Chiral) U(N_c) CS-SQCD theories with N_f superfields in the fundamental and $\tilde{N}_f$ superfields in the anti-fundamental

Using properties of hyperbolic Γ-functions one can show analytically that $Z_{S^{3}}^{(loc)}$ vanishes when SUSY breaks spontaneously.

3) U(N_c) CS theories with an adjoint and N_f superfields in the fundamental and N_f superfields in the anti-fundamental

These are generalization of the theories we discussed in this talk. Again, one can show analytically that $Z_{S^{3}}^{(loc)}$ vanishes when SUSY breaks spontaneously. Our large-N numerics are consistent with this.

It would be interesting to explore the potential implications of our conjecture within the 3d-4d connection...