Holographic fluids and vorticity in $2 + 1$ dimensions

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(published and forthcoming works with R.G. Leigh and A.C. Petkou)

Highlights

Framework

AdS/CFT → QCD & plethora of strongly coupled systems

- Superconductors and superfluids [Hartnoll, Herzog, Horowitz '08]
- Strange metals [e.g. Faulkner *et al.* '09]
- Quantum-Hall fluids [e.g. Dolan *et al.* '10]

Holography also applied to hydrodynamics i.e. to a regime of local thermodynamical equilibrium for the boundary theory

- Conjectured bound $\eta/s \geq \hbar/4\pi k_B$ – saturated in holographic fluids (nearly-perfect) [Policastro, Son, Starinets '01, Baier *et al.* '07, Liu *et al.* '08]
- More systematic description of fluid dynamics [many authors since '08]

Why vorticity?

Developments in ultra-cold-atom physics: new twists in the physics of near-perfect neutral fluids fast rotating in normal or superfluid phase $\rightarrow$ new challenges in strong-coupling regimes

- Dilute rotating Bose gases in harmonic traps – potentially fractional-quantum-Hall liquids or topological (anyonic) superfluids [e.g. Cooper *et al.* '10, Chu *et al.* '10, Dalibard *et al.* '11]

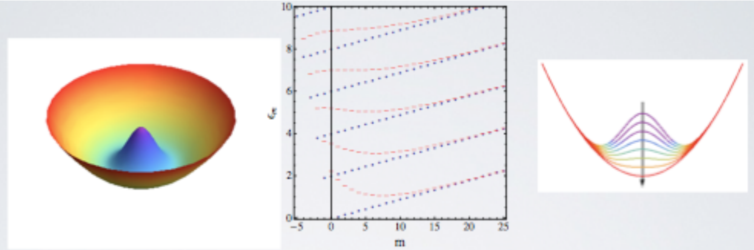

Figure: Trap, rotation and Landau levels – toward a strongly coupled FQH phase for small filling factor (ν = particles/vortices ≈ 1)

- Strongly interacting Fermi gases *above BEC* behave like near-perfect fluids with very low η/s [Shaefer *et al.* '09, Thomas *et al.* '09]

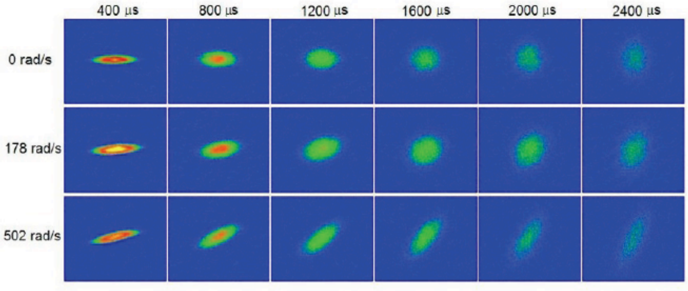

Figure: Irrotational elliptic flow in very small η/s rotating fluid – rotates *faster* as it expands due to *inertia moment quenching*

Foreseeable progress in the measurement of transport coefficients calls for a better understanding of the strong-coupling dynamics of vortices

Developments in analogue-gravity systems for the description of sound/light propagation in moving media *[see e.g. review by M. Visser et al. '05]*

Propagation in $D-1$-dim moving media

$$\Updownarrow$$

Waves or rays in D-dim "analogue" curved space–times

Sometimes in supersonic/superluminal vortex flows: $v_{\text{medium}} > v_{\text{wave}}$

- Horizons & optical or acoustic black holes
- Hawking radiation [Belgiorno *et al.* '10, Cacciatori *et al.* '10]
- Vortices and Aharonov–Bohm effect for neutral atoms [Leonhardt *et al.* '00, Barcelo *et al.* '05]

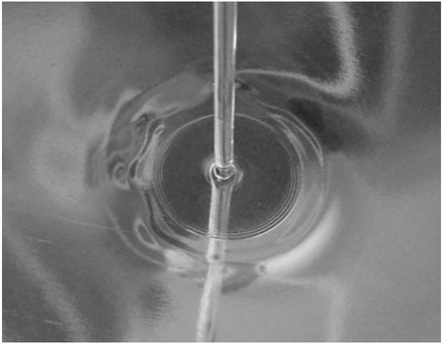

Figure: White hole's horizon in analogue gravity

Holographic description of the D-dim set up?

Aim

Use AdS/CFT to describe rotating fluids viewed

- *either as genuine rotating near-perfect Bose or Fermi gases*
- *or as analogue-gravity set ups for acoustics/optics in rotating media* [see also Schäfer et al. '09, Das et al. '10]

Here

Starting from a $3+1$-dim asymptotically AdS background a $2+1$-dim holographic dual appears as a set of boundary data

- boundary frame
- boundary stress tensor

Within hydrodynamics, data interpreted as a $2+1$-dim fluid moving in a background – generically with vorticity

- Kerr AdS
- Taub–NUT AdS

exact bulk solutions that will serve to illustrate various properties

Highlights

Motivations

Holographic fluids

AdS Kerr & Taub–NUT backgrounds

Alternative interpretations

Outlook

Holographic duality

Applied beyond the original framework – maximal susy YM in $D = 4$ – usually in the classical gravity approximation without backreaction

- Bulk with $\Lambda = -3k^2$: asymptotically AdS $d = D + 1$-dim $\mathcal{M}$
- Boundary at $r \to \infty$: asymptotic coframe E^μ $\mu = 0, \ldots, D-1$

$$\mathrm{d}s^2 \approx \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 \eta_{\mu\nu} E^\mu E^\nu = \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 g_{(0)\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$$

Holography: determination of $\langle \mathcal{O} \rangle_{\text{bry. F.T.}}$ as a response to a boundary source perturbation $\delta\phi_{(0)}$ (momentum vs. field in Hamiltonian formalism – related via some regularity condition)

Pure gravity

Holographic data

- Field $g_{rr}, g_{\mu\nu} \to g_{(o)\mu\nu}$: boundary metric – source
- Momentum T_{rr}, $T_{\mu\nu} \to T_{(o)\mu\nu}$: $\langle T_{(o)\mu\nu} \rangle$ – response

Palatini formulation and $3+1$ *split* *[Leigh, Petkou '07, Mansi, Petkou, Tagliabue '08]*

θ^a: orthonormal coframe $\mathrm{d}s^2 = \eta_{ab}\theta^a\theta^b$ $(\eta : + - ++)$

- Vierbein: $\theta^r = N\frac{\mathrm{d}r}{kr} \quad \theta^\mu = N^\mu \mathrm{d}r + \tilde{\theta}^\mu \quad \mu = 0, 1, 2$
- Connection: $\omega^{r\mu} = q^{r\mu}\mathrm{d}r + \mathcal{K}^\mu \quad \omega^{\mu\nu} = -\epsilon^{\mu\nu\rho}\left(Q_\rho \frac{\mathrm{d}r}{kr} + \mathcal{B}_\rho\right)$
- Gauge choice: $N = 1$ and $N^\mu = q^{r\mu} = Q_\rho = 0 \to \tilde{\theta}^\mu, \mathcal{K}^\mu, \mathcal{B}_\rho$

Holography: Hamiltonian evolution from data on the boundary – captured in Fefferman–Graham expansion for large r [Fefferman, Graham '85]

$$\begin{aligned}
\tilde{\theta}^\mu(r,x) &= kr\,E^\mu(x) + {}^1\!/_{kr} F^\mu_{[2]}(x) + {}^1\!/_{k^2r^2} F^\mu(x) + \cdots \\
\mathcal{K}^\mu(r,x) &= -k^2 r\,E^\mu(x) + {}^1\!/_{r} F^\mu_{[2]}(x) + {}^2\!/_{kr^2} F^\mu(x) + \cdots \\
\mathcal{B}^\mu(r,x) &= B^\mu(x) + {}^1\!/_{k^2r^2} B^\mu_{[2]}(x) + \cdots
\end{aligned}$$

Independent $2+1$ boundary data: vector-valued 1-forms E^μ and F^μ

- E^μ: boundary orthonormal coframe – allows to determine

  $$\mathrm{d}s^2_{\mathrm{bry.}} = g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}E^\mu E^\nu,\ B^\mu,\ B^\mu_{[2]},\ F^\mu_{[2]},\ldots$$

- F^μ: stress-tensor current one-form – allows to construct the boundary stress tensor ($\kappa = {}^{3k}\!/_{8\pi G}$)

  $$T = \kappa F^\mu e_\mu = T^\mu{}_\nu E^\nu \otimes e_\mu$$

Highlights

AdS Kerr: the solid rotation

The bulk data

$$
\begin{aligned}
\mathrm{d}s^2 &= (\theta^r)^2 - (\theta^t)^2 + (\theta^\vartheta)^2 + (\theta^\varphi)^2 \\
&= \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r},\vartheta)} - V(\tilde{r},\vartheta)\left[\mathrm{d}t - \frac{a}{\Xi}\sin^2\vartheta\,\mathrm{d}\varphi\right]^2 \\
&+ \frac{\rho^2}{\Delta_\vartheta}\mathrm{d}\vartheta^2 + \frac{\sin^2\vartheta\Delta_\vartheta}{\rho^2}\left[a\,\mathrm{d}t - \frac{r^2+a^2}{\Xi}\mathrm{d}\varphi\right]^2
\end{aligned}
$$

$V(\tilde{r},\vartheta) = {}^{\Delta}/_{\rho^2}$ with

$$
\begin{aligned}
\Delta &= \left(\tilde{r}^2 + a^2\right)\left(1 + k^2\tilde{r}^2\right) - 2M\tilde{r} \\
\rho^2 &= \tilde{r}^2 + a^2\cos^2\vartheta \\
\Delta_\vartheta &= 1 - k^2a^2\cos^2\vartheta \\
\Xi &= 1 - k^2a^2
\end{aligned}
$$

The boundary metric – following FG expansion

$$\begin{aligned} \mathrm{d}s^2_{\text{bry.}} &= \eta_{\mu\nu} E^\mu E^\nu = g_{(0)\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\left(\mathrm{d}t - \frac{a \sin^2\vartheta}{\Xi} \mathrm{d}\varphi\right)^2 + \frac{1}{k^2 \Delta_\vartheta} \left(\mathrm{d}\vartheta^2 + \left(\frac{\Delta_\vartheta \sin\vartheta}{\Xi}\right)^2 \mathrm{d}\varphi^2\right) \end{aligned}$$

- $E^t = \mathrm{d}t - \frac{a \sin^2\vartheta}{\Xi} \mathrm{d}\varphi$ and $e_t = \partial_t$
- $\nabla_{\partial_t} \partial_t = 0$: observers at rest are *inertial*
- note: conformal to Einstein universe in a *rotating frame* (requires $(\vartheta, \varphi) \to (\vartheta', \varphi')$)

The boundary stress tensor $\kappa F^\mu \mathrm{e}_\mu$ [see also Caldarelli, Dias, Klemm '08]

$$T = T_{\mu\nu} E^\mu E^\nu = \frac{\kappa M k}{3}\left(2(E^t)^2 + (E^\vartheta)^2 + (E^\varphi)^2\right)$$

perfect-fluid-like ($T = (\varepsilon + p) u \otimes u + p \eta_{\mu\nu} E^\mu \otimes E^\nu$)

- traceless: conformal fluid with $\varepsilon = 2p = {}^{2\kappa Mk}/_{3}$
- velocity one-form: $u = -E^t = -\mathrm{d}t + b$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial

Fluid without expansion and shear but with vorticity

$$\omega = \frac{1}{2}\mathrm{d}u = \frac{1}{2}\mathrm{d}b = \frac{a\cos\vartheta\sin\vartheta}{\Xi}\mathrm{d}\vartheta \wedge \mathrm{d}\varphi = k^2 a\cos\vartheta E^\vartheta \wedge E^\varphi$$

Reminder [Ehlers '61]

Vector field $\mathbf{u}$ *with* $u_\mu u^\mu = -1$ *and space–time variation* $\nabla_\mu u_\nu$

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \sigma_{\mu\nu} + \frac{1}{D-1}\Theta h_{\mu\nu} + \omega_{\mu\nu}$$

- $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$: projector/metric on the orthogonal space
- $a_\mu = u^\nu \nabla_\nu u_\mu$: acceleration – transverse
- $\sigma_{\mu\nu}$: symmetric traceless part – shear
- $\Theta = \nabla_\mu u^\mu$: trace – expansion
- $\omega_{\mu\nu}$: antisymmetric part – vorticity

$$\omega = \frac{1}{2}\omega_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}(\mathrm{d}u + u \wedge a)$$

Notes

The fluid may be perfect or not

$$T_{\text{visc}} = -\left(2\eta\sigma^{\mu\nu} + \zeta h^{\mu\nu}\Theta\right) e_\mu \otimes e_\nu$$

$T_{\text{visc}} = 0$ if the congruence is shear- and expansion-less

A shear- and expansion-less isolated fluid is geodesic if [Caldarelli et al. '08]

$$\nabla_{\mathbf{u}}\varepsilon = 0$$

$$\nabla p + u\nabla_{\mathbf{u}} p = 0$$

fulfilled here with ε, p csts.

Only $\delta g_{(o)\mu\nu}$ *give access to* η *and* ζ *via* $\langle \delta T_{(o)\mu\nu} \rangle$

How does vorticity i.e. rotation get manifest?

Boundary geometries are stationary of Randers form [Randers '41]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - b\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

and the fluid is at rest: $\mathbf{u} = \partial_t$

- $\nabla_{\partial_t}\partial_t = 0$: the fluid is inertial and carries vorticity $\omega = \frac{1}{2}\mathrm{d}b$
- $\nabla_{\partial_t}\partial_i = \omega_{ij}a^{jk}\left(\partial_k + b_k\partial_t\right)$: frame and fluid dragging

Other privileged frames exist where the observers experience differently the rotation of the fluid – e.g. Zermelo dual frame

AdS Taub–NUT: the nut charge

The bulk data *[Taub '51, Newman, Tamburino, Unti '63]*

$$
\begin{aligned}
\mathrm{d}s^2 &= (\theta^r)^2 - (\theta^t)^2 + (\theta^\vartheta)^2 + (\theta^\varphi)^2 \\
&= \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r})} - V(\tilde{r})\left[\mathrm{d}t - 2n\cos\vartheta\,\mathrm{d}\varphi\right]^2 + \rho^2\left[\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi\right]^2
\end{aligned}
$$

$V(\tilde{r}) = \Delta/\rho^2$ with

$$
\begin{aligned}
\Delta &= \left(\tilde{r}^2 - n^2\right)\left(1 + k^2\left(\tilde{r}^2 + 3n^2\right)\right) + 4k^2n^2\tilde{r}^2 - 2M\tilde{r} \\
\rho^2 &= \tilde{r}^2 + n^2
\end{aligned}
$$

No rotation parameter a *but nut charge* n *– one of the most peculiar solutions to Einstein's Eqs.* *[Misner '63]*

Parenthesis: Kerr vs. Taub–NUT (Lorentzian time)

Taub–NUT: rich geometry – foliation over squashed 3-spheres with $SU(2) \times U(1)$ isometry (homogeneous and axisymmetric)

- horizon at $r = r_+ \neq n$: 2-dim fixed locus of $-2n\partial_t \to$ *bolt* (Killing becoming light-like)
- extra fixed point of $\partial_\varphi - 4n\partial_t$ on the horizon at $\vartheta = \pi$

nut at $r = r_+, \vartheta = \pi$ from which departs a *Misner string* (coordinate singularity if $t \not\approx t + 8\pi n$) [Misner '63]

Kerr: stationary (rotating) black hole

- horizon at $r = r_+$: fixed locus of $\partial_t + \Omega_{\mathrm{H}}\partial_\varphi \to$ bolt
- pair of nut–anti-nut at $r = r_+, \vartheta = 0, \pi$ (fixed points of ∂_φ) connected by a Misner string [Argurio, Dehouck '09]

Pictorially: nuts and Misner strings

Figure: Kerr vs. Taub–NUT

How is Taub–NUT related to rotation?

Back to Taub–NUT

Following FG → boundary metric and stress tensor

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{bry.}} &= \eta_{\mu\nu} E^\mu E^\nu = g_{(0)\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\left(\mathrm{d}t - 2n(\cos\vartheta - 1)\mathrm{d}\varphi\right)^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \end{aligned}$$

$$T = T_{\mu\nu} E^\mu E^\nu = \frac{\kappa M k}{3}\left(2(E^t)^2 + (E^\vartheta)^2 + (E^\varphi)^2\right)$$

Fluid interpretation: perfect-like stress tensor

- conformal with $\varepsilon = 2p = {}^{2\kappa M k}/_{3}$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial

Same fluid: no expansion, no shear but *vorticity*

The vorticity on the boundary of AdS Taub–NUT

$$b = -2n(1-\cos\vartheta)\mathrm{d}\varphi$$
$$\omega = \tfrac{1}{2}\mathrm{d}b = -n\sin\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi = -nk^2E^{\vartheta}\wedge E^{\varphi}$$

- Dirac-monopole-like *vortex* (“hedgehog” or homogeneous)
- created by the nut charge (equivalently by the Misner string)

$$n = -\frac{1}{4\pi}\int_{S^2}\omega$$

Kerr produces a dipole without nut charge: $\int\omega = 0$ *– solid rotation*
Taub–NUT is well designed to describe “monopolar” vortices

Remark

Rotation in flat space (spherical coordinates)

Data: $\vec{v} \quad \vec{\omega} = {}^1\!/\!{}_2 \vec{\nabla} \times \vec{v}$

- Solid rotation ($\ell = 2$):
  - $\vec{v} = \Omega \partial_\varphi$ and $\|\vec{v}\| = \Omega r \sin\vartheta$
  - $\vec{\omega} = \Omega \cos\vartheta \partial_r - \frac{\Omega \sin\vartheta}{r} \partial_\vartheta = \Omega \partial_z$ (parallel to Oz)
- Dirac-monopole vortex ($\ell = 1$):
  - $\vec{v} = \alpha \frac{1-\cos\vartheta}{r^2 \sin^2\vartheta} \partial_\varphi$ and $\|\vec{v}\| = \alpha \frac{1-\cos\vartheta}{r \sin\vartheta}$
  - $\vec{\omega} = \frac{\alpha}{2r^2} \partial_r$ (hedgehog)
- Ordinary vortex ($\ell = 0$):
  - $\vec{v} = \frac{\beta}{r^2 \sin^2\vartheta} \partial_\varphi$ and $\|\vec{v}\| = \frac{\beta}{r \sin\vartheta}$
  - $\vec{\omega} = 0$ (irrotational) – up to a δ-function contribution

More general vortices on the boundary

$$b = 2(-1)^\ell \alpha \left(1 - P_\ell(\cos\vartheta)\right) \mathrm{d}\varphi$$
$$\omega = (-1)^\ell \alpha \, P'_\ell(\cos\vartheta) \sin\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$$

- for odd ℓ there is indeed a vortex around the track of the Misner string at the south pole with a nut-like charge

$$n = -\frac{1}{4\pi} \int \omega = \alpha$$

- for even ℓ the Misner string does not reach the poles and the total charge vanishes – e.g. Kerr as a dipole with $\alpha = {}^{a}/_{3\Xi}$

Bulk realization for $\ell \geq 3$: generalization of Weyl multipoles [Weyl '19]
($\ell = 0$ is Schwarzschild with $\mathrm{d}t \to \mathrm{d}t + \mathrm{d}\varphi$)

AdS Taub–NUT: more on the boundary and CTCs

Homogenous boundary space–time: Lorentzian squashed 3-sphere

$$\begin{aligned} ds^2_{\text{bry.}} &= \frac{1}{k^2}\left((\sigma^1)^2 + \left(\sigma^2\right)^2\right) - 4n^2\left(\sigma^3\right)^2 \\ &= \frac{1}{k^2}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) - \left(dt - 2n(\cos\vartheta - 1)d\varphi\right)^2 \end{aligned}$$

- Gödel-like space (sourced by dust distribution) [classification in Raychaudhuri *et al.* '80, Rebouças *et al.* '83]
- Stationary foliation in 2-spheres with a *time fiber*
- CTCs of angular opening $< 2\vartheta_0$ ($g_{\varphi\varphi}(\vartheta_0) = 0$) – *no closed time-like geodesics*
- Special point: south pole of the 2-sphere – track of the Misner string – can be moved anywhere by homogeneity

Any observer is the center of a circular horizon of azimuthal radius $\pi - \vartheta_0$ beyond which he cannot send any ray

Highlights

Randers forms and Zermelo metrics [Zermelo '31, Randers '41]

The boundary geometries describing vorticity are stationary metrics of the Randers form

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - b\right)^2 + a_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

Properties: magnetic paradigm and CTCs

- The projection of geodesics onto the base space with metric $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$ provides trajectories for a non-relativistic charged particle in a magnetic field $\tilde{F} = \mathrm{d}b$
- CTCs can appear for $b^2 > 1$ $(b^2 = a^{ij}b_ib_j)$
  - Kerr: none
  - Taub–NUT: $\exists$ CTCs $\longrightarrow$ horizon around the vortex

Equivalently recast as Zermelo metrics $(a, b) \leftrightarrow (h, W)$

$$\mathrm{d}s^2 = \frac{1}{c^2 - W^2}\left(-c^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i - W^i\mathrm{d}t\right)\left(\mathrm{d}x^j - W^j\mathrm{d}t\right)\right)$$

- *Originally: navigation on* $h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ *in a drift current* $W^i\partial_i$
- *Here: analogue-gravity geometries originating from bulk solutions of Einstein's equations via holography*

- Zermelo metrics are acoustic: null geodesics describe sound propagation in (non-)relativistic fluids moving on geometries $h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ with velocity field $\mathbf{W} = W^i\partial_i$ [see e.g. Visser '97]
- CTCs capture physical effects: sound propagation in supersonic-flow regions $(W^2 > c^2) \rightarrow$ horizons

Similar approaches exist for light propagation in moving media or sound propagation in (non-)relativistic (conformal) fluids

Highlights

Class of bulk solutions describing conformal fluids in $2+1$ dim with vorticity – backgrounds still to be unravelled for $\ell \geq 3$ and most importantly perturbations to be understood [see e.g. Bakas '08]

- Spectrum of bulk excitations $\rightarrow$ *anyons* on the boundary – like in exotic BEC phases (under experimental investigation)
- Transport coefficients like shear viscosity (nearly-perfect fluids)
- Investigation of the analogue-gravity interpretation

More ambitious: recast the superfluid phase transition and the appearance of vortices

Combine Kerr and nut charge in AdS Kerr Taub–NUT thermodynamics ($M \longrightarrow$ temperature, $\{a, n\} \longrightarrow$ rotation)

- add a $U(1)$ and a scalar field
- analyse the phase diagramme, identify the order parameter
- study the potential transition as nut–anti-nut dissociation

Formation of a vortex: nut–anti-nut dissociation

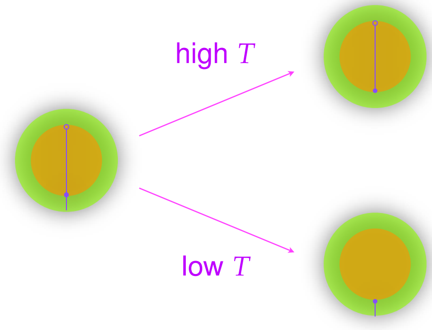

Figure: high-T vs. low-T stable phase

Highlights

Holography in a nutshell

More on AdS Taub–NUT

Sailing in a drift current

Randers vs. Zermelo pictures and analogue gravity

Holography

Applied beyond the original framework – maximal susy YM in $D = 4$ – usually in the classical gravity approximation without backreaction

- Bulk: "asymptotically AdS" d-dim $\mathcal{M}$ $(d = D + 1)$

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 H(kr) \left(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2\right)$$

- Boundary at $r \to \infty$: $\mathrm{d}s^2 \approx \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 g_{(0)\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$
- Dynamical field ϕ with action $I\left[\phi\right]$ and boundary value $\phi_{(0)}(x)$

The basic relation

$$Z_{\mathsf{bulk}}[\phi] = \langle 1 \rangle_{\mathsf{bry.\ F.T.}}$$

gives access to the data of the boundary theory

$$\left\langle \exp i \int_{\partial\mathcal{M}} \mathrm{d}^D x \sqrt{-g_{(0)}}\delta\phi_{(0)}\, \mathcal{O} \right\rangle_{\mathsf{bry.\ F.T.}} = Z_{\mathsf{bulk}}[\phi + \delta\phi_{(0)}]$$

- $\delta\phi_{(0)}$: boundary perturbation $\longrightarrow$ source
- $\mathcal{O}$: observable functional of $\phi_{(0)}$ $\longrightarrow$ response
- $\phi_{(0)} \leftrightarrow \mathcal{O}$: conjugate variables

Semi-classically around a classical solution $\phi_\star$

$$Z_{\mathrm{bulk}}[\phi] = \exp - I_{\mathrm{E}}\left[\phi_\star\right]$$

$$\langle \mathcal{O} \rangle = \left. \frac{\delta I}{\delta \phi_{(0)}} \right|_{\phi_\star}$$

Hamiltonian interpretation of $\langle \mathcal{O} \rangle$

- $\pi = \frac{\partial \mathcal{L}}{\partial \partial_r \phi} \Rightarrow I = \int \mathrm{d}r \int \mathrm{d}^D x \left[\pi \partial_r \phi - \mathcal{H}(\pi, \phi, \partial_\mu \phi)\right]$
- on-shell variation

$$\delta I|_{\phi_\star} = \int_{\partial \mathcal{M}} \mathrm{d}^D x \, \pi_{(0)} \, \delta \phi_{(0)} \Rightarrow \langle \mathcal{O} \rangle = \pi_{(0)}$$

What is holography? How do we get $\pi_{(0)} = \pi_{(0)}\left[\phi_{(0)}\right]$?

$$\partial\mathcal{M} = \begin{cases} \text{boundary } r \to \infty \\ \text{horizon } r_{\mathrm{H}} \end{cases}$$

- $\phi_{(0)}(x)$ and $\pi_{(0)}(x)$ are *independent* data set at large r

$$\phi(r) = r^{\Delta-d}\phi_{(0)}(x) + \frac{r^{-\Delta}}{k(2\Delta - D)}\pi_{(0)}(x) + \cdots$$

  (non-normalizable and normalizable modes)

- become related if a *regularity condition* is imposed at r_{H}

$$\langle\mathcal{O}\rangle = \pi_{(0)}\left[\phi_{(0)}\right]$$

In summary

Holography: computation of $\langle \mathcal{O} \rangle_{\text{bry. F.T.}}$ as a response to a boundary source perturbation $\delta\phi_{(0)}$

- Dynamical field ϕ with action $I\left[\phi\right]$ and boundary value $\phi_{(0)}(x)$
- Momentum $\pi(r, x)$ with boundary value $\pi_{(0)}(x)$
- On-shell variation

$$\delta I|_{\phi_\star} = \int_{\partial\mathcal{M}} \mathrm{d}^D x \, \pi_{(0)} \, \delta\phi_{(0)}$$

- Holography: regularity on $r_{\mathrm{H}} \Rightarrow \pi_{(0)} = \pi_{(0)}\left[\phi_{(0)}\right] \longrightarrow$

$$\langle \mathcal{O} \rangle = \pi_{(0)}\left[\phi_{(0)}\right]$$

Examples

Electromagnetic field in $d = 4$, $D = 3$

- Field $A_r, A_\mu \longrightarrow A_{(o)\mu}$: boundary electromagnetic field – source
- Momentum $\mathcal{E}_\mu \longrightarrow \mathcal{E}_{(0)\mu}$: $\langle \varrho \rangle$, $\langle j_i \rangle$ – response
- Bulk gauge invariance $\longrightarrow$ continuity equation

Gravitation in $d = D + 1$

- Field $g_{rr}, g_{\mu\nu} \longrightarrow g_{(o)\mu\nu}$: boundary metric – source
- Momentum $T_{\mu\nu} \longrightarrow T_{(o)\mu\nu}$: $\langle T_{(o)\mu\nu} \rangle$ – response
- Bulk diffeomeorphism invariance $\longrightarrow$ conservation equation

Gravity in $d = 4$

Palatini formulation and $3+1$ *split* [Leigh, Petkou '07, Mansi, Petkou, Tagliabue '08]

$$I_{\text{EH}} = -\frac{1}{32\pi G}\int_{\mathcal{M}} \epsilon_{abcd}\left(\mathcal{R}^{ab} - \frac{\Lambda}{6}\theta^a \wedge \theta^b\right)\wedge \theta^c \wedge \theta^d$$

θ^a an orthonormal frame $\mathrm{d}s^2 = \eta_{ab}\theta^a\theta^b$ ($\eta : +-++$)

- Vierbein: $\theta^r = N\frac{\mathrm{d}r}{kr} \quad \theta^\mu = N^\mu \mathrm{d}r + \tilde{\theta}^\mu \quad \mu = 0, 1, 2$

$$\mathrm{d}s^2 = N^2\frac{\mathrm{d}r^2}{k^2r^2} + \eta_{\mu\nu}\left(N^\mu \mathrm{d}r + \tilde{\theta}^\mu\right)\left(N^\nu \mathrm{d}r + \tilde{\theta}^\nu\right)$$

- Connection: $\omega^{r\mu} = q^{r\mu}\mathrm{d}r + \mathcal{K}^\mu \quad \omega^{\mu\nu} = -\epsilon^{\mu\nu\rho}\left(Q_\rho\frac{\mathrm{d}r}{kr} + \mathcal{B}_\rho\right)$

(note: $\Lambda = -3k^2$)

Aim: Hamiltonian evolution from data on the boundary $r \to \infty$
Question: what are the field and momentum variables?

- Gauge choice: $N = 1$ and $N^\mu = q^{r\mu} = Q_\rho = 0$

$$ds^2 = \frac{dr^2}{k^2 r^2} + \eta_{\mu\nu}\tilde{\theta}^\mu \tilde{\theta}^\nu$$

- Fields and momenta: $\tilde{\theta}^\mu, \mathcal{K}^\mu, \mathcal{B}_\rho$ one-forms

What are the independent boundary data? Answer in asymptotically AdS: Fefferman–Graham expansion for large r [Fefferman, Graham '85]

$$
\begin{aligned}
\tilde{\theta}^\mu(r,x) &= kr\,E^\mu(x) + \tfrac{1}{kr}F^\mu_{[2]}(x) + \tfrac{1}{k^2r^2}F^\mu(x) + \cdots \\
\mathcal{K}^\mu(r,x) &= -k^2r\,E^\mu(x) + \tfrac{1}{r}F^\mu_{[2]}(x) + \tfrac{2}{kr^2}F^\mu(x) + \cdots \\
\mathcal{B}^\mu(r,x) &= B^\mu(x) + \tfrac{1}{k^2r^2}B^\mu_{[2]}(x) + \cdots
\end{aligned}
$$

Independent $2+1$ boundary data: E^μ and F^μ

Upon canonical transformations (i.e. boundary terms or holographic renormalization)

$$
\delta I_{\mathrm{EH}}\big|_{\mathrm{on-shell}} = \int_{\partial\mathcal{M}} T^\mu \wedge \delta\Sigma_\mu
$$

- $\Sigma_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho}E^\nu \wedge E^\rho$: field – source
- $T^\mu = \kappa F^\mu$: momentum – response

Application: Schwartzschild AdS

The bulk data

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r})} - V(\tilde{r})\mathrm{d}t^2 + \tilde{r}^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)$$

- $V(r) = 1 + k^2\tilde{r}^2 - {}^{2M}/_{\tilde{r}}$
- $\theta^r = {}^{\mathrm{d}\tilde{r}}/_{\sqrt{V(\tilde{r})}} = {}^{\mathrm{d}r}/_{kr}$

The Fefferman–Graham expansion

$$\begin{aligned}
\theta^t &= \sqrt{V(\tilde{r})}\mathrm{d}t = \left(kr + \tfrac{1}{4kr} - \tfrac{2M}{3kr^2} + \mathrm{O}\left(\tfrac{1}{r^3}\right)\right)\mathrm{d}t \\
\theta^\vartheta &= \tilde{r}\,\mathrm{d}\vartheta = \left(r - \tfrac{1}{4k^2r} + \tfrac{M}{3k^2r^2} + \mathrm{O}\left(\tfrac{1}{r^3}\right)\right)\mathrm{d}\vartheta \\
\theta^\varphi &= \tilde{r}\sin\vartheta\,\mathrm{d}\varphi = \left(r - \tfrac{1}{4k^2r} + \tfrac{M}{3k^2r^2} + \mathrm{O}\left(\tfrac{1}{r^3}\right)\right)\sin\vartheta\,\mathrm{d}\varphi
\end{aligned}$$

The boundary data

- coframe: $E^t = \mathrm{d}t \quad E^\vartheta = \frac{\mathrm{d}\vartheta}{k} \quad E^\varphi = \frac{\sin\vartheta\,\mathrm{d}\varphi}{k}$
- stress current: $F^t = -\frac{2Mk}{3}\mathrm{d}t \quad F^\vartheta = \frac{M}{3}\mathrm{d}\vartheta \quad F^\varphi = \frac{M}{3}\sin\vartheta\,\mathrm{d}\varphi$

The boundary metric

$$\begin{aligned}\mathrm{d}s^2_{\text{bry.}} &= \eta_{\mu\nu}E^\mu E^\nu = g_{(0)\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\mathrm{d}t^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)\end{aligned}$$

- Einstein universe
- $e_t = \partial_t$
- $\nabla_{e_t} e_t = 0$: observers at rest are inertial

The boundary stress tensor $\kappa F^{\mu} e_{\mu}$

$$T = T_{\mu\nu} E^{\mu} E^{\nu} = \frac{\kappa M k}{3} \left(2(E^t)^2 + (E^{\vartheta})^2 + (E^{\varphi})^2 \right)$$

- traceless: conformal fluid with $\varepsilon = 2p = {}^{2\kappa Mk}/{}_{3}$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial
- velocity one-form: $u = -E^t = -\mathrm{d}t$

Static fluid without expansion, shear or vorticity

More general examples

We can exhibit backgrounds with stationary boundaries and fluids

$$T = (\varepsilon + p)\mathbf{u} \otimes \mathbf{u} + p\eta^{\mu\nu} e_\mu \otimes e_\nu$$

- $\varepsilon = 2p$: conformal
- $\nabla_{\mathbf{u}}\mathbf{u} = 0$: inertial
- $\mathbf{u} = e_0$: at rest (comoving)

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AdS Taub–NUT: the nut charge

Reminder: the bulk data *[Taub '51, Newman, Tamburino, Unti '63]*

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tilde{r}^2}{V(\tilde{r})} - V(\tilde{r})\left[\mathrm{d}t - 2n\cos\vartheta\,\mathrm{d}\varphi\right]^2 + \rho^2\left[\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi\right]^2$$

$V(\tilde{r}) = \Delta/\rho^2$ with

$$\begin{aligned} \Delta &= \left(\tilde{r}^2 - n^2\right)\left(1 + k^2\left(\tilde{r}^2 + 3n^2\right)\right) + 4k^2n^2\tilde{r}^2 - 2M\tilde{r} \\ \rho^2 &= \tilde{r}^2 + n^2 \end{aligned}$$

The Fefferman–Graham expansion with r s.t. $\mathrm{d}r/kr = \mathrm{d}\tilde{r}/\sqrt{V(\tilde{r})}$

- boundary coframe and frame

$$E^t = \mathrm{d}t - b \quad E^\vartheta = \frac{\mathrm{d}\vartheta}{k} \quad E^\varphi = \frac{\sin\vartheta\,\mathrm{d}\varphi}{k}$$
$$e_t = \partial_t \quad e_\vartheta = k\,\partial_\vartheta \quad e_\varphi = -\frac{2kn(1-\cos\vartheta)}{\sin\vartheta}\partial_t + \frac{k}{\sin\vartheta}\partial_\varphi$$

$$b = -2n(1-\cos\vartheta)\mathrm{d}\varphi$$

- boundary stress current

$$F^t = -\frac{2Mk}{3}E^t \quad F^\vartheta = \frac{Mk}{3}E^\vartheta \quad F^\varphi = \frac{Mk}{3}E^\varphi$$

For comparison: AdS Kerr

The Fefferman–Graham expansion of $\theta^t, \theta^\vartheta, \theta^\varphi$

- boundary orthonormal coframe and frame

$$E^t = \mathrm{d}t - b \quad E^\vartheta = \frac{\mathrm{d}\vartheta}{k\sqrt{\Delta_\vartheta}} \quad E^\varphi = \frac{\sqrt{\Delta_\vartheta}\sin\vartheta\,\mathrm{d}\varphi}{k\Xi}$$

$$e_t = \partial_t \quad e_\vartheta = k\sqrt{\Delta_\vartheta}\,\partial_\vartheta \quad e_\varphi = \frac{ka\sin\vartheta}{\sqrt{\Delta_\vartheta}}\partial_t + \frac{k\Xi}{\sin\vartheta\sqrt{\Delta_\vartheta}}\partial_\varphi$$

$$b = \frac{a\sin^2\vartheta}{\Xi}\mathrm{d}\varphi$$

- boundary stress current

$$F^t = -\frac{2Mk}{3}E^t \quad F^\vartheta = \frac{Mk}{3}E^\vartheta \quad F^\varphi = \frac{Mk}{3}E^\varphi$$

The boundary metric and stress tensor

$$\begin{aligned} \mathrm{d}s^2_{\text{bry.}} &= \eta_{\mu\nu} E^\mu E^\nu = g_{(0)\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -\left(\mathrm{d}t + 2n(1-\cos\vartheta)\mathrm{d}\varphi\right)^2 + \tfrac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \end{aligned}$$

$$T = T_{\mu\nu} E^\mu E^\nu = \frac{\kappa M k}{3}\left(2(E^t)^2 + (E^\vartheta)^2 + (E^\varphi)^2\right)$$

Fluid interpretation: perfect-like stress tensor

- conformal fluid with $\varepsilon = 2p = {}^{2\kappa Mk}/_{3}$
- velocity field $\mathbf{u} = e_t = \partial_t$: comoving & inertial

Fluid without expansion and shear but *with vorticity*

$$\omega = \frac{1}{2}\mathrm{d}b = -n \sin\vartheta \mathrm{d}\vartheta \wedge \mathrm{d}\varphi = -k^2 n E^\vartheta \wedge E^\varphi$$

AdS Taub–NUT: more on the boundary

Homogenous boundary space–time: Lorentzian squashed 3-sphere

$$\mathrm{d}s^2_{\text{bry.}} = \frac{1}{k^2}\left((\sigma^1)^2 + \left(\sigma^2\right)^2 - 4n^2\left(\sigma^3\right)^2\right)$$

- Gödel-like space (sourced by dust distribution) [classification in Raychaudhuri *et al.* '80, Rebouças *et al.* '83]
- Stationary foliation in 2-spheres with a *time fiber*
- CTCs of angular opening $< 2\vartheta_0$ ($g_{\varphi\varphi}(\vartheta_0) = 0$) – *no closed time-like geodesics*
- Special point: south pole of the 2-sphere – track of the Misner string

Around the poles: Som–Raychaudhuri and cosmic spinning string

- North pole: Som–Raychaudhuri space – sourced by rigidly rotating charged dust [Som, Raychaudhuri '68]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + \Omega\varrho^2 \mathrm{d}\varphi\right)^2 + \varrho^2 \mathrm{d}\varphi^2 + \mathrm{d}\varrho^2$$

  $\Omega = k^2 n$ and $\varrho = \vartheta/k$

- South pole: spinning cosmic string [vortex in analogue gravity]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + A\mathrm{d}\varphi\right)^2 + \varrho^2 \mathrm{d}\varphi^2 + \mathrm{d}\varrho^2$$

  $A = 4n - \Omega\varrho^2$ and $\varrho = \pi - \vartheta/k$

Around the poles of Kerr: Som–Raychaudhuri with $\Omega = -k^2 a$

Kerr vs. Taub–NUT "rotation" *[Dowker '74, Bonnor '75, Hunter '98]*

- Kerr: rigid rotation with angular momentum and velocity
  - horizon at $r = r_+$: fixed locus of $\partial_t + \Omega_H \partial_\varphi \rightarrow$ bolt
  - pair of nut–anti-nut at $r = r_+, \vartheta = 0, \pi$ (fixed points of ∂_φ) connected by a Misner string [Argurio, Dehouck '09]

  asymptotically $\Omega_\infty = -ak^2$
- Taub–NUT: "non-rigid rotation" with angular momentum distribution along the Misner string (vanishing integral) – asymptotically:
  - north pole: angular velocity $\Omega_\infty = nk^2$
  - south pole: no angular velocity

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The Zermelo problem

What is the minimal-time trajectory of a non-relativistic ship sailing on a space with positive-definite metric $\mathrm{d}t^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ *and velocity* $U^i = {}^{\mathrm{d}x^i}/_{\mathrm{d}t}$ *s.t.* $\|\mathbf{U}\|^2 = 1$*?*

- time functional is

$$T = \int \mathrm{d}t \sqrt{h_{ij} U^i U^j}$$

- minimization is realized with geodesics of $\mathrm{d}t^2$

What happens in the presence of a lateral drifting flow $\mathbf{W} = W^i \partial_i$ *("wind" or "tide")?* [Zermelo '31]

- velocity: $U^i = {}^{\mathrm{d}x^i}/_{\mathrm{d}t} = V^i + W^i$
  - $\mathbf{U}$: vector tangent to the trajectory
  - $\mathbf{V}$: "propelling" velocity with $\|\mathbf{V}\|^2 = 1$
    - no longer aligned with the trajectory
    - instantaneous navigation road – velocity of the ship with respect to a local frame dragged by the drifting flow
- norm: $\mathbf{U}^2 = 1 + \mathbf{W}^2 + 2\mathbf{V} \cdot \mathbf{W}$

- time functional is

$$\begin{aligned} T &= \int \mathrm{d}t \left(\sqrt{\frac{\mathbf{U}^2}{1-\mathbf{W}^2} + \left(\frac{\mathbf{W}\cdot\mathbf{U}}{1-\mathbf{W}^2}\right)^2} - \frac{\mathbf{W}\cdot\mathbf{U}}{1-\mathbf{W}^2} \right) \\ &= \int \mathrm{d}t \left(\sqrt{\left(\frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda^2}\right) U^i U^j} - \frac{W_k U^k}{\lambda} \right) \end{aligned}$$

  with $\lambda = 1 - \mathbf{W}^2$
- minimization is realized with null geodesics of the Zermelo metric

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left(-\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i - W^i \mathrm{d}t\right)\left(\mathrm{d}x^j - W^j \mathrm{d}t\right)\right)$$

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Note: the time functional is of Randers type with Finsler Lagrangian

$$T = \int \mathrm{d}t \, F(x^i, U^i)$$

with

$$F(x^i, U^i) = \sqrt{a_{ij} U^i U^j} + b_i U^i$$

and

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i W_j}{\lambda^2} \quad b_i = -\frac{h_{ij} W^j}{\lambda}$$

the data of the Randers form

Equivalently Randers stationary forms are recast as Zermelo metrics

$$\mathrm{d}s^2 = \frac{1}{\lambda}\left(-\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i - W^i\mathrm{d}t\right)\left(\mathrm{d}x^j - W^j\mathrm{d}t\right)\right)$$

with

$$\begin{aligned} h_{ij} &= \lambda\left(a_{ij} - b_i b_j\right) \\ \lambda &= 1 - b_i b_j a^{ij} \\ W^i &= -\frac{a^{ij} b_j}{\lambda} \end{aligned}$$

Null geodesics in Zermelo metric are minimal-time curves for sailing in the base space of metric $\mathrm{d}t^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ *under the influence of a drifting "wind"* $\mathbf{W} = W^i\partial_i$ *[Zermelo '31]*

Analogue gravity picture

Zermelo metrics are acoustic [see e.g. Visser '97, Chapline, Mazur '04]

$$\mathrm{d}s^2 = \frac{\varrho}{c_\mathrm{s}}\left(-c_\mathrm{s}^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i - W^i\mathrm{d}t\right)\left(\mathrm{d}x^j - W^j\mathrm{d}t\right)\right)$$

Null geodesics describe sound propagation in non-relativistic fluids moving on geometries $h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ *with velocity fields* $\mathbf{W} = W^i\partial_i$

- inviscid, isolated, barotropic ($\mathrm{d}h = {}^{\mathrm{d}p}/_{\varrho}$)
- local mass density ϱ and pressure p
- local sound velocity $c_\mathrm{s} = {}^1/_{\sqrt{\partial\rho/\partial p}}$

Alternatively the whole boundary set up could be a gravitational analogue of sound propagating in moving fluids or light in moving dielectrics – acoustic/optical black holes

As such our examples fall in a larger class of backgrounds studied in analogue systems [Gibbons et al. '08] – here equipped with a stress tensor

Randers & Zermelo backgrounds address the problems of

- motion of charged particles in magnetic fields
- sailing in the presence of a drift force
- sound propagation in moving media

and are dual to each other

Where are we?

Exploratory tour of some properties of conformal holographic fluids moving in non-trivial gravitational backgrounds

- inertial
- carrying vorticity

Vorticity appears in various fashions

- Kerr $\longrightarrow$ solid rotation on the boundary: dipole
- Taub–NUT $\longrightarrow$ vortex on the boundary: monopole

More general multipoles?

Bonus

Alternative analogue interpretation of the boundary backgrounds: propagation of sound/light in moving media (Randers & Zermelo)

- provides holographic AdS/analogue-gravity correspondence
- evades the CTCs caveats within supersonic/superluminal flows

Bulk for general Randers geometries?