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On generalized quantum criticality at finite density in Holography

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Towards a classification of IR criticality in strongly-coupled finite-density theories

Bibliography

Based on ongoing work with B. Gouteraux, (APC) and

published recent work with

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C. Charmousis, B. Gouteraux (Orsay), B. S. Kim and R. Meyer (Crete)
. arXiv:1005.4690 [hep-th]

and previous work

U. Gürsoy, E.K. and F. Nitti, arXiv:0707.1324 [hep-th],arXiv:0707.1349 [hep-th]

U. Gürsoy, E.K. L. Mazzanti and F. Nitti, arXiv:0804.0899 [hep-th]

The plan of the talk

- Introduction (Motivation, Tools, Goals, Strategy)

- The notion of Effective Holographic Theory

- Holographic Dynamics at zero charge density (Solutions, thermodynamics, spectra and transport)

- Generalized Criticality

- Holographic Dynamics at finite charge density (Solutions, thermodynamics, spectra and transport)

- A QC non-relativistic holographic system with strange metal behaviour.

- Outlook

Introduction

• Holographic techniques offer a new look into strongly-coupled, semiclassical theories, at finite density.

• The goal is to (a) extend our understanding of known CM mechanisms at strong coupling (b) Look for novel phenomena.

• Like in QFT, a very useful and efficient tool is that of an effective theory: Effective Holographic Theory $\to$ EHT.

The reason is that it is useful to:

(1) Develop intuition

(2) Do efficient model building

(3) Be useful as an intermediary with data.

- Unlike QFT we know much less about EHT.

- The first step is to hierarchically treat, the

(a) Field content

(b) IR classification of interactions

- The next step is to assess which EHTs are sensible and which are not.

- Eventually a calculation of observables (thermodynamics and transport data for example) should be done in EHT.

Effective Holographic Theories

The strategy is:

1. Select the operators expected to be important for the dynamics

2. Write an effective (gravitational) holographic action that captures the (IR) dynamics.

3. Find the saddle points (classical solutions)

4. Study the physics around each acceptable saddle point.

• The bulk metric $g_{\mu\nu} \leftrightarrow T_{\mu\nu}$ is always sourced in any theory. In CFTs it captures all the dynamics of the stress tensor and the solution is AdS_{p+1}.

• In a theory with a conserved U(1) charge, a gauge field is also necessary, $A_\mu \leftrightarrow J_\mu$. If only $g_{\mu\nu}, A_\mu$ are important then we have an AdS-Einstein-Maxwell theory with saddle point solution=AdS-RN.

• The thermodynamics and CM physics of AdS-RN has been analyzed in detail in the last few years, revealing rich physical phenomena

Chamblin+Emparan+Johnson+Myers (1999), Hartnoll+Herzog (2008), Bak+Rey (2009),Cubrovic+Schalm+Zaanen (2009), Faulkner+Liu+McGreevy+Vegh (2009)

1. Emergent AdS_2 scaling symmetry

2. Interesting fermionic correlators

but

3. Is unstable (in N=4) to both neutral and charged scalar perturbations

Gubser+Pufu (2008), Hartnoll+Herzog+Horowitz (2008)

4. Has a non-zero (large) entropy at $T = 0$.

Einstein-Scalar-U(1) theory

• To go beyond RN, we must include the most important (relevant) scalar operator in the IR. This can capture the dynamics of the system.

• The most general 2d action is

$$S = \int d^{p+1}x\sqrt{g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - Z(\phi)F^2\right]$$

involving two arbitrary functions of ϕ. Typically the potential is non-trivial. It may have an UV fixed point (not necessary).

• We assume here it does not have an IR fixed point (maximum of V) (otherwise back to RN). (by engineering V and Z one can have many layers of different physics at different energy scales).

• We will parametrize the IR asymptotics of V, Z using sugra intuition.

$$V(\phi) \sim e^{-\delta\phi} \quad , \quad Z(\phi) \sim e^{\gamma\phi} \quad , \quad \phi \to \pm\infty$$

• We must have $V(\phi) \to \infty$ in the IR (and the inverse in the UV). For Z in the IR

$$\begin{cases} Z \to \infty \quad , \quad \text{weak coupling} \quad , \quad \text{bulk U(1)'s} \\ \\ Z \to 0 \quad , \quad \text{strong coupling} \quad , \quad \text{tachyon condensation} \end{cases}$$

• From now on we set

$$V = \Lambda\, e^{-\delta\phi} \quad , \quad Z = e^{\gamma\phi}$$

• Solutions depend on ($\Lambda \to \Lambda\, e^{\delta\phi_0}$)

$$\phi_0 \quad , \quad Q \quad , \quad T$$

• IMPORTANT: this parametrization is not binding except in some "crossover cases". More later.

On naked holographic singularities

• If no IR AdS/Lifshitz, all Poincaré invariant solutions end up in a naked IR singularity.

• In GR naked singularities are proscribed.

• In holographic gravity some may be acceptable. The reason is that they do not always signal a breakdown of predictability as is the case in GR. They could be resolved by stringy or KK physics, or they could be shielded for finite energy configurations.

• Are they resolvable? Does the near-singularity physics depends on the resolution?

• An important task in EHT is to therefore ascertain when such naked singularities are acceptable and when are reliable (alias "good")
(A priori these are different things)

♠ Gubser gave a criterion for good (acceptable) singularities: They should be limits of solutions with a regular horizon.

Gubser (2000)

• The second criterion amounts to having a well-defined spectral problem for fluctuations around the solution: The second order equations describing all fluctuations are Sturm-Liouville problems (no extra boundary conditions needed at the singularity).

Gursoy+E.K.+Nitti (2008)

• The singularity is "repulsive" (like the Liouville wall). It has an overlap with the previous criterion. It involves the calculation of "Wilson loops"

Gursoy+E.K.+Nitti (2008)

• It is not known whether the list is complete.

Solutions at zero charge density

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Gursoy+Kiritsis+Mazzanti+Nitti (2009)

• The only parameter relevant for the solutions is $\delta \in R$ in $V \sim e^{-\delta\phi}$. Take $p+1=4$.

• $0 \leq |\delta| < 1$. T=0 singularity acceptable. Continuous spectrum/no mass gap. Continuous transition to BH phase at $T > 0$

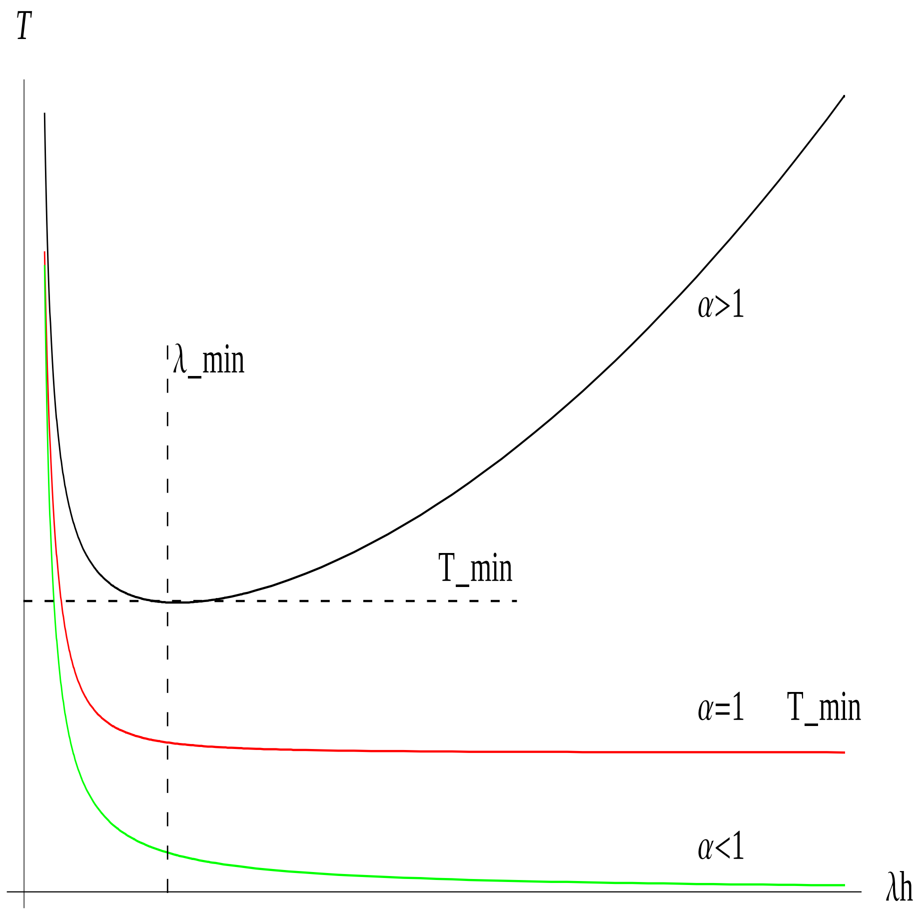
• $1 < |\delta| < \sqrt{3}$. Discrete spectrum/mass gap. BH is thermodynamically subdominant and unstable. $1 < |\delta| < \sqrt{\frac{5}{3}}$. The spin-2 and spin-0 spectral problem is reliable without resolution.

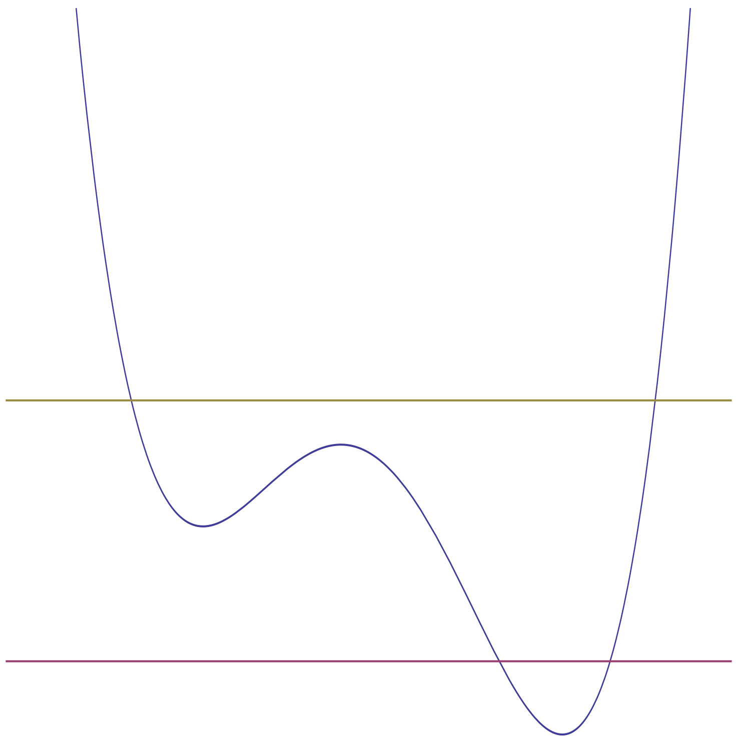
• $|\delta| \geq \sqrt{3}$. Gubser bound violated, singularity→unacceptable.

The crossover value here is $|\delta| = 1$. For all other $\delta \neq 1$, corrections like $V = e^{-\delta\phi}\phi^k + e^{-\delta'\phi}\phi^{k'} + ...$ give subleading corrections.

- $1 < |\delta| < \sqrt{3}$. In the gapped case, the BH is unstable and thermodynamically irrelevant. The complete story at finite T depends on the subleading terms in the potential (aka the UV completion).

- There is a first order phase transition at T_c to a large BH.





- For more complicated potentials multiple phase transitions are possible.

Gursoy+Kiritsis+Mazzanti+Nitti (2009), Alanen+Kajantie+Tuominen (2010)

• $|\delta| = 1$. This is the "marginal" case. It has a good singularity, a continuous spectrum and a gap. A lot of the physics of finite temperature transitions depends on subleading terms in the potential:

♠ If $V = e^{\phi}\left[1 + C\ e^{-\frac{2\phi}{n-1}} + \cdots\right]$, then at $T = T_{min} = T_c$ there is an n-th order continuous transition.

♠ If $V = e^{\phi}\left[1 + C/\phi^k + \cdots\right]$, then at $T = T_{min} = T_c$ there is a generalized KT phase transition

Gursoy (2010)

♠ If $V = e^{\phi}\ \phi^P$, with $P < 0$ this behaves as in $|\delta| < 1$. When $P > 0$ like $|\delta| > 1$.

The spectra depend importantly on P, when $P > 0$.

In particular, we will see that $P = \frac{1}{2}$ is very much like what we expect in 4D large-N YM.

The hidden scale invariance

$$ds^2 = \frac{dr^2}{f} + \frac{(-f dt^2 + dx \cdot dx)}{r^{-\frac{4}{(p-1)\delta^2}}} \quad , \quad f = 1 - \left(\frac{r_0}{r}\right)^{\frac{2p}{(p-1)\delta^2}-1} \quad , \quad e^{\delta\phi} = \frac{V_0 \delta^4 \, r^2}{2\left(\frac{2p}{p-1} - \delta^2\right)}$$

Changing variables

$$w = r^{1-\frac{2}{(p-1)\delta}} \quad , \quad (t, x^i) \to \sqrt{\frac{(p-1)\delta^2}{(p-1)\delta^2 - 2}}\,(t, x^i)$$

$$ds^2 = e^{2\chi(r)}\left[\frac{dw^2}{w^2 f(w)} + \frac{-f(w)dt^2 + dx \cdot dx}{w^2}\right] \quad , \quad e^{2\chi} \sim r^2 \sim e^{\delta\phi} \sim \frac{1}{V(\phi)}$$

- When $\delta^2 < \frac{2}{p-1}$ this is the dimensional reduction of an AdS_{p+1+n} solution on T^n with

$$\delta^2 = \sqrt{\frac{1}{1+\frac{p-1}{n}}} \cdot \frac{2}{p-1} \quad \leq \frac{2}{p-1}$$

Gubser+Nellore, Skenderis+Taylor

• This explains the continuous spectrum and absence of mass gap for $\delta^2 < \frac{2}{p-1}$.

• Therefore, the theory is quantum critical in the IR, despite the non-trivial potential.

• The singularity is resolved by the KK-modes (oxydation).

• Different δ can be obtained by extending to real $n > 0$.

• The crossover value $\delta^2 = \frac{2}{p-1}$ is obtained when $n \to \infty$.

• Dimensional Reduction of AdS_{p+1+n} solution on S^n gives

$$\delta^2 = \frac{2}{p-1} + \frac{2}{n} \quad \geq \frac{2}{p-1}$$

E.K.+Gouteraux

and a naturally discrete spectrum and mass gap.

• Violation of the Gubser bound: $n \leq 1$. Marginal case: $n \to \infty$.

• The theory is again quantum critical in the IR,

Charged near-extremal scaling solutions

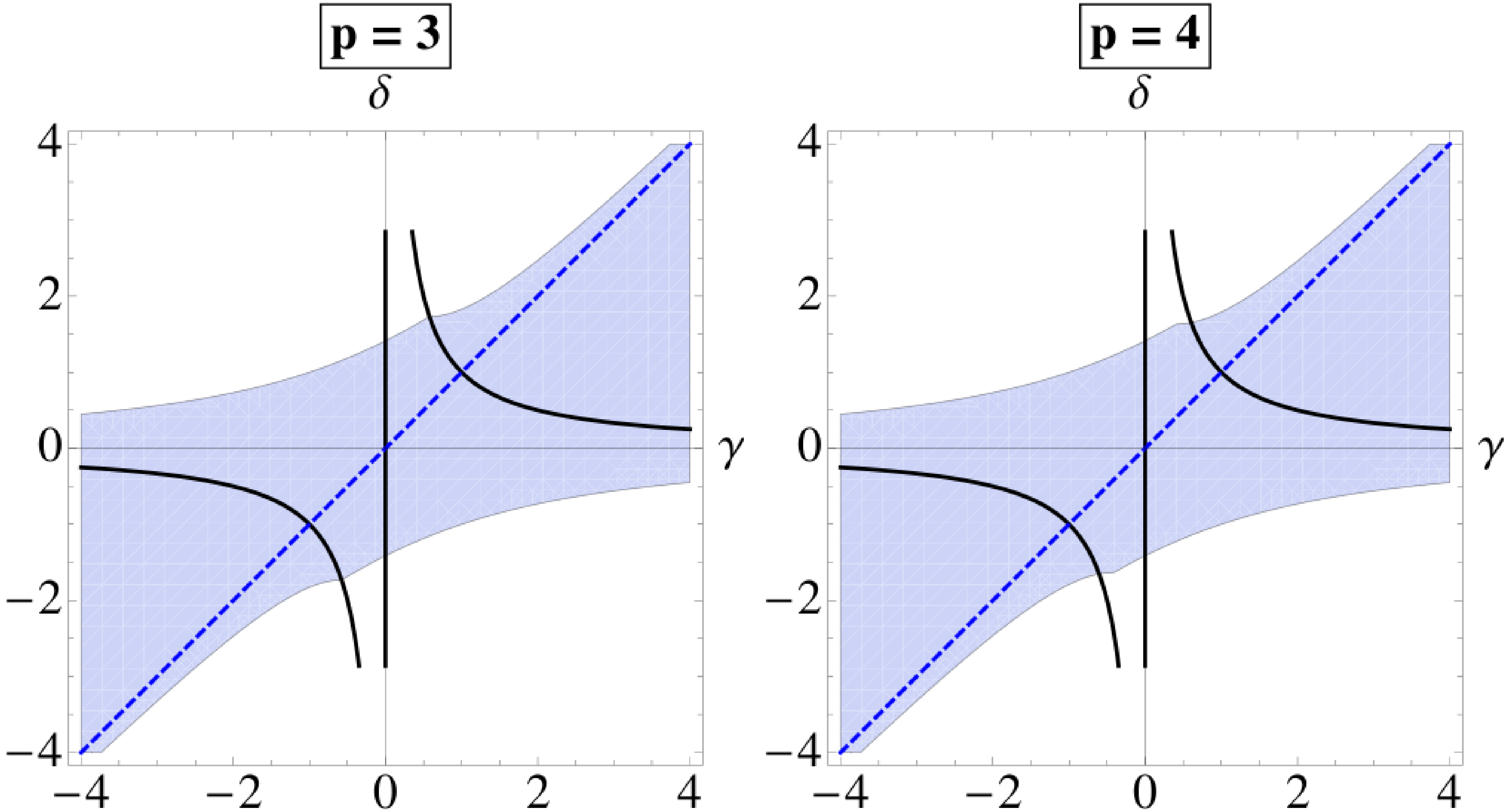
$$ds^2 = r^{\frac{(\gamma-\delta)^2}{2}}\left[dx^2 + dy^2 - f(r)dt^2\right] + \frac{dr^2}{f(r)}$$

$$f(r) = \frac{16\,(-\Lambda)}{wu^2}e^{-\delta\phi_0}\quad r^{1-\frac{3}{4}(\gamma-\delta)^2+\frac{wu}{4}}\quad\left(1-\frac{2m}{r^{\frac{wu}{4}}}\right),$$

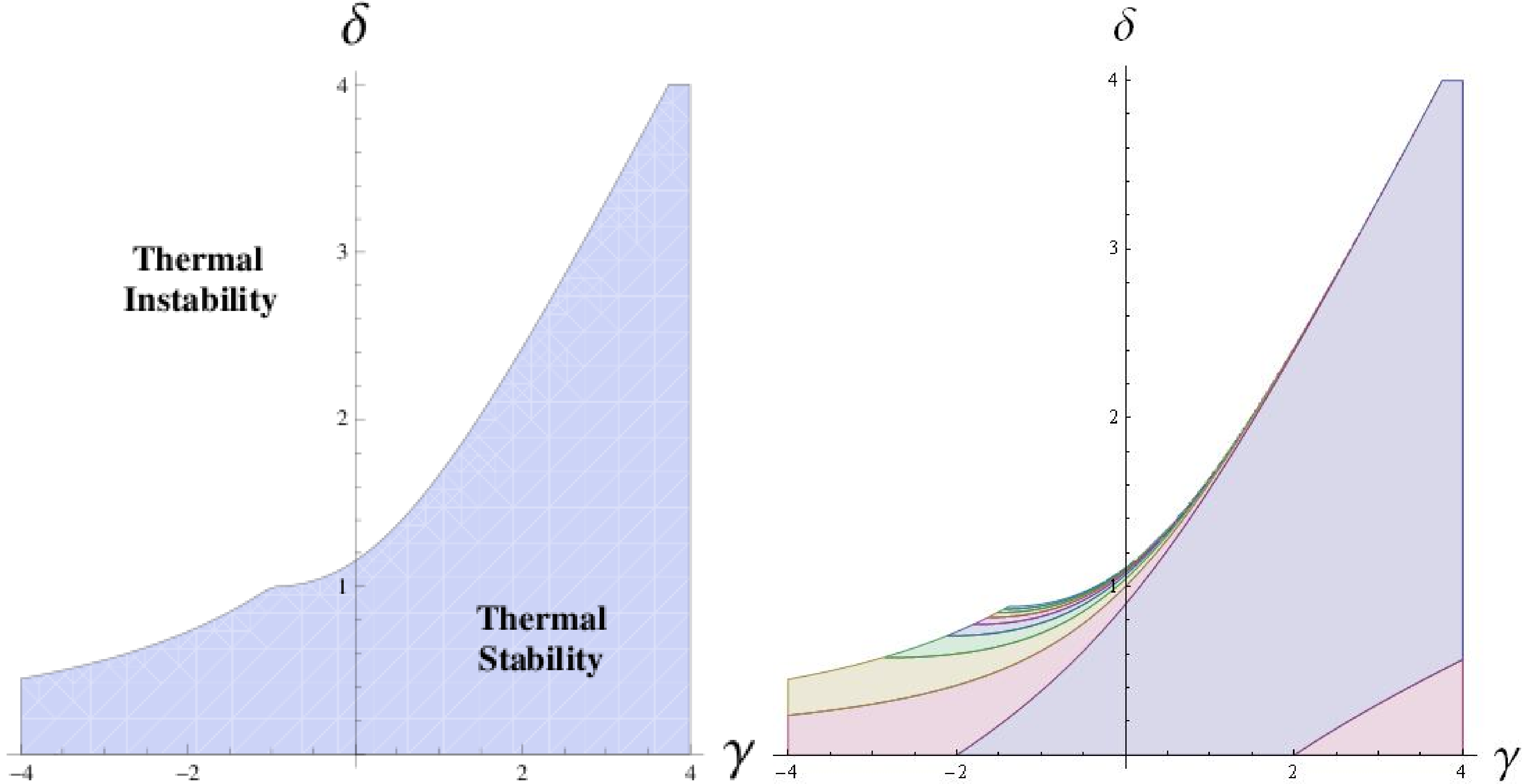
$$e^{\phi} = e^{\phi_0}r^{-(\gamma-\delta)}\quad,\quad \mathcal{A} = \frac{8}{wu}\sqrt{\frac{v\Lambda}{u}}e^{-\frac{(\gamma+\delta)}{2}\phi_0}\quad\left[r^{\frac{wu}{4}} - 2m\right]dt$$

$$wu = 3\gamma^2 - \delta^2 - 2\gamma\delta + 4 > 0\,,\quad u = \gamma^2 - \gamma\delta + 2\,,\quad v = \delta^2 - \gamma\delta - 2\,, \delta^2 \leq 3$$

- These are near extremal solutions (the charge density is fixed).
- The Entropy vanishes at extremality if $\gamma \neq \delta$.
- If $\gamma = \delta$ the extremal solution is $AdS_2 \times R^2$.
- The charge entropy dominates the $Q = 0$ entropy almost everywhere.
- When $\frac{dS}{dT} < 0$ the BH is unstable$\rightarrow$ gapped spectra.



This graph shows the Gubser bounds on the near extremal solution on the whole of the (γ, δ) plane for $p = 3$ and $p = 4$. The blue regions are the allowed regions where the near extremal solutions are black-hole like. The white regions are solutions of a cosmological type and therefore fail the Gubser bound. The dashed blue line is the $\gamma = \delta$ solutions while the solid black line corresponds to the $\gamma\delta = 1$ solutions.



On the left: region of local stability of the near extremal black hole. Right: The variety of phase transitions of the near extremal black hole to the background at zero temperature. In the blue region continuous transitions occur, in the purple region adjacent to the blue one the transitions are of third-order. The stripes starting with yellow to the left of the blue and purple regions depicts transitions of fourth-(yellow) up to tenth-order. Above them all higher-order transitions also occur.

Hidden scaling at finite density

• The extremal solutions for all (γ, δ) are simple powers, and therefore scaling.

• The metric can always be written as

$$ds^2 = e^{\chi}\ d\hat{s}^2 \quad , \quad e^{\chi} \sim e^{\delta\phi} \quad , \quad d\hat{s}^2 = -\frac{dt^2}{w^{2z}} + \frac{dw^2 + dx^i dx^i}{w^2}$$

with

$$z = \frac{(\gamma-\delta)(\gamma+(2p-3)\delta)+2(p-1)}{(\gamma-\delta)(\gamma+(p-2)\delta)}$$

• Most of these can be lifted to solutions in higher dimensions with generalized scaling symmetry (Boosted AdS black-holes or black AdS q-branes).

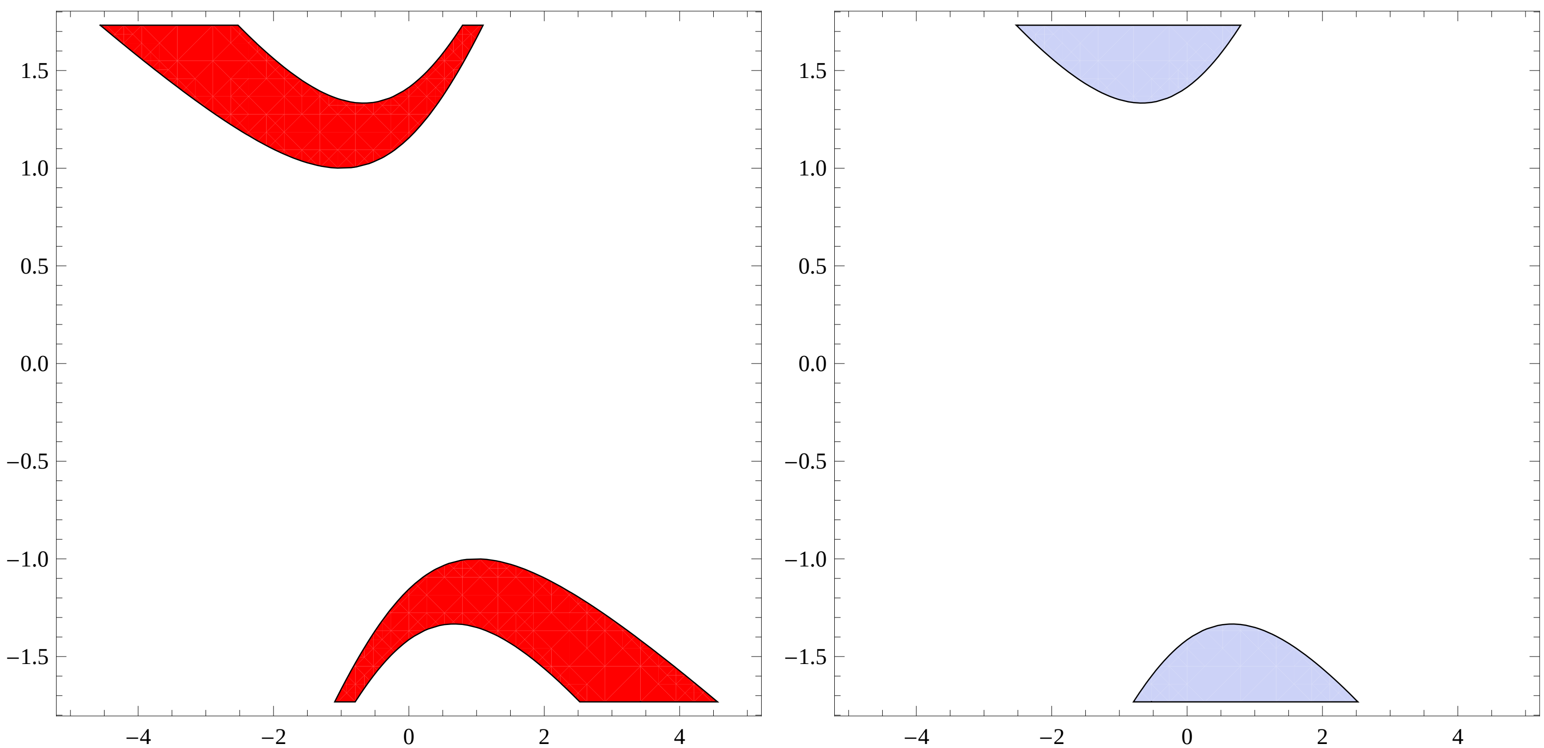
E.K.+Gouteraux

• They represent the most general critical behavior at zero temperature, generalizing the AdS and Lifshitz geometries.

• The dimensional oxydation resolves the IR singularity.

• Note that at $\gamma + (p-2)\delta$ we obtain an $AdS_2 \times R^2$ geometry at extremality but with $S = 0$.

Mott-like spectra



Left: The region on the $(\gamma\delta)$ plane where the IR black holes are unstable and $c > 0$. Here the extremal finite density system has a mass gap and a discrete spectrum of charged excitations, when $\Delta < 1$. This resembles a Mott insulator and the figure provides the Mott insulator "islands" in the (γ, δ) plane. Right:The region where the IR black holes are unstable, and $c < 0$. In this region the extremal finite density system has a gapless continuous spectrum at zero temperature. In both figures the horizontal axis parametrizes γ, whereas the vertical axis δ.

A similar system was analyzed independently by Mc Greevy and Balasubramanian

Exact charged solutions

- The full set of solutions for $\gamma\delta = 1$ and $\gamma = \delta$ are known.

- $\delta^2 < 3$ otherwise the solutions are De-Sitter like (cf Gubser).

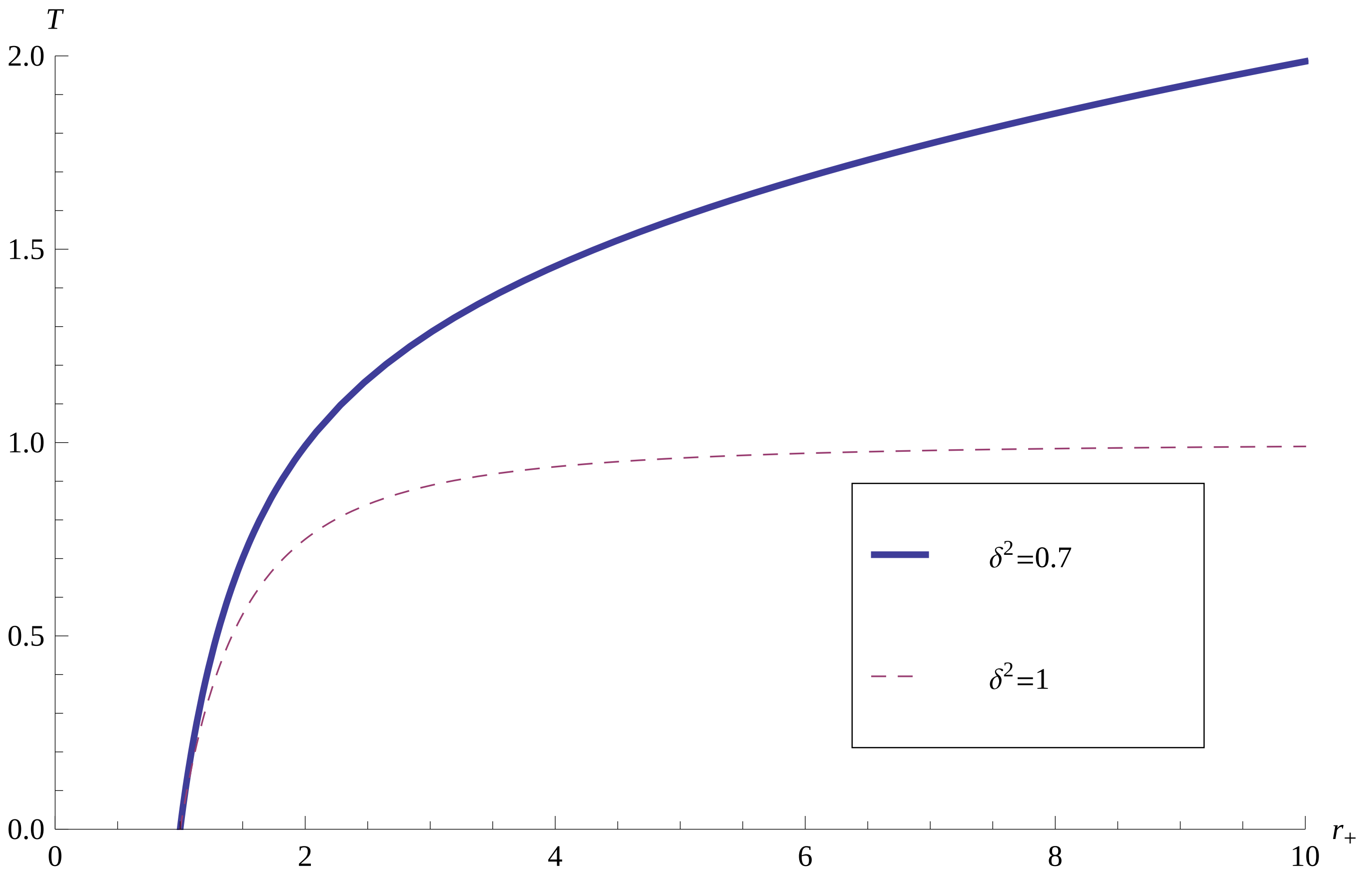
- For $\gamma\delta = 1$ there are three distinct classes of dynamics:
  $\delta^2 \quad \in \quad [0, 1] \quad \cup \quad [1, 1 + \frac{2}{\sqrt{3}}] \quad \cup \quad [1 + \frac{2}{\sqrt{3}}, 3)$

- At $Q = 0$ all $|\delta| > 1$ systems were insulators. Now this range is split in two in $\gamma\delta = 1$.

- $\gamma\delta = 1$ has zero entropy but $\gamma = \delta$ has finite entropy at extremality .

$\gamma\delta = 1$ solutions

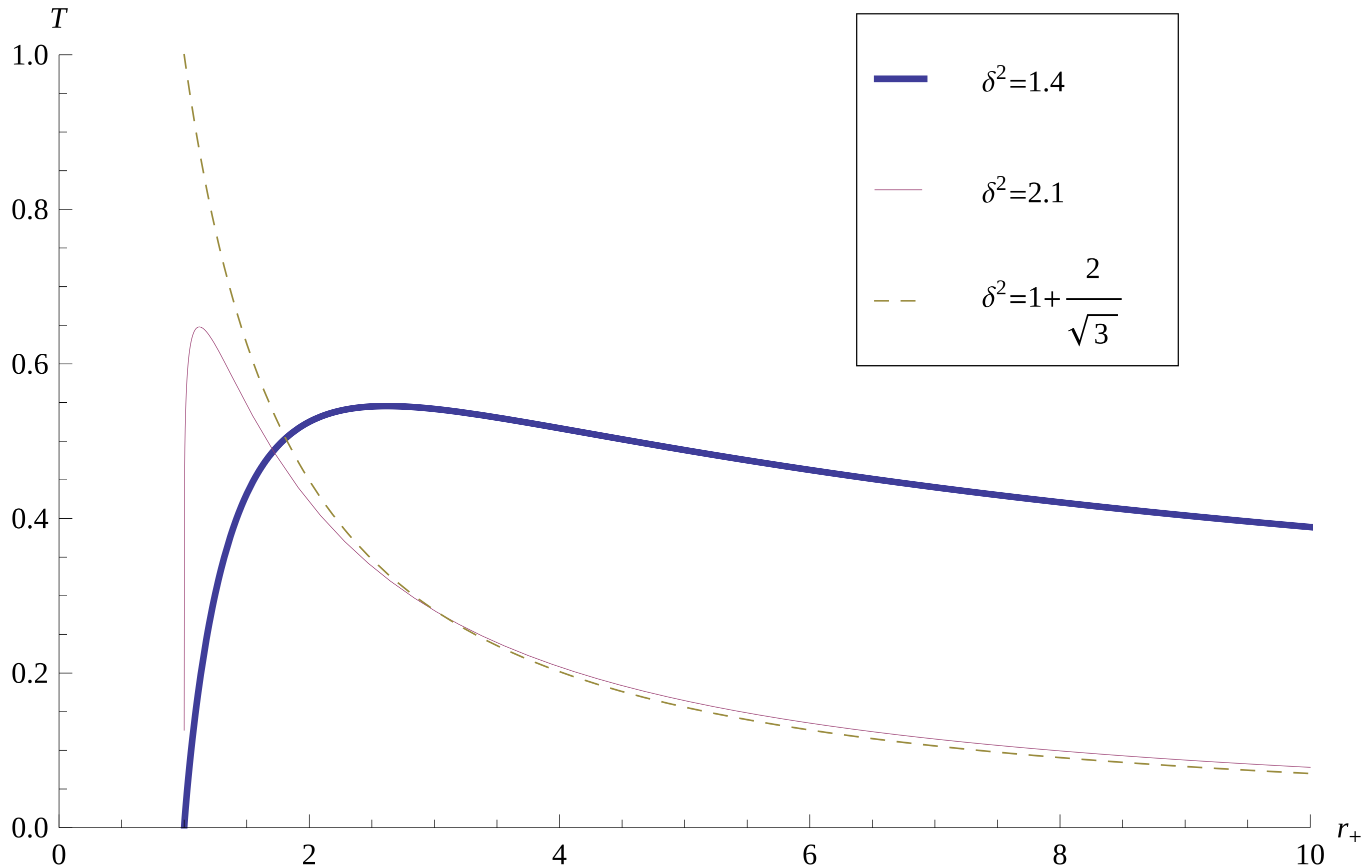
- $0 \leq |\delta| < 1$



- A single branch of BH that dominate at $T > 0$. The transition at $T = 0^+$ is between 2nd and 3rd order.

- The system is a conductor.

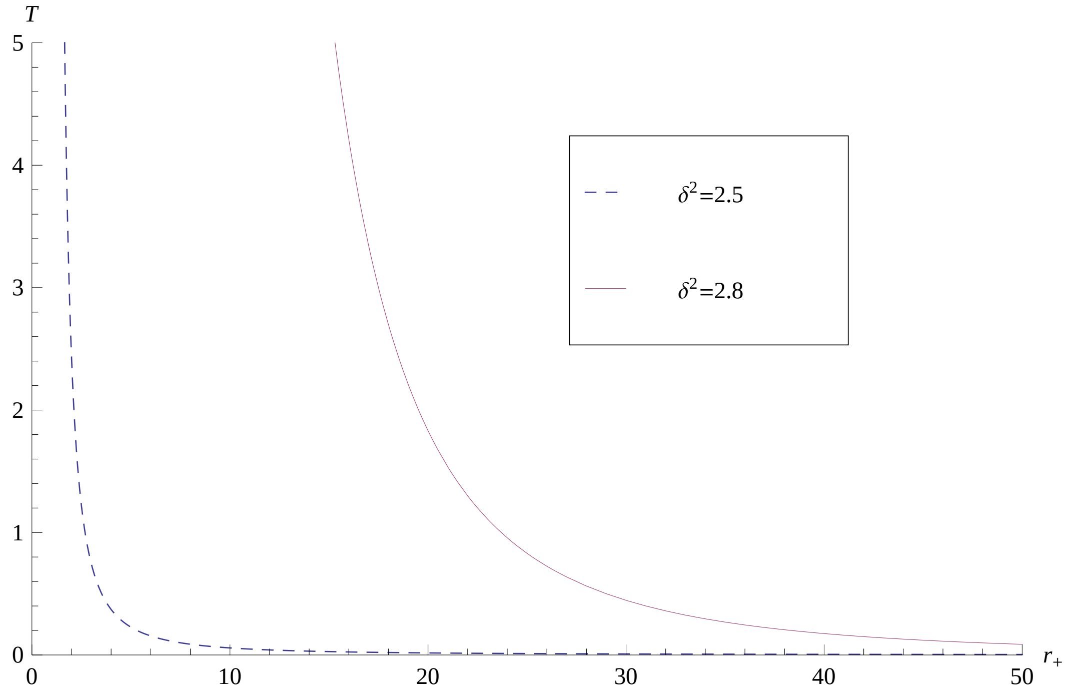
- $\delta^2 \in [1, 1 + \frac{2}{\sqrt{3}}]$



- There are two black holes at a given temperature $T < T_{max}$.

- At $T_{max} > T > 0$ it is the small black hole branch that dominates thermodynamically. The transition at $T = 0^+$ is continuous of any order. Upon UV completion, at $T_c \sim T_{max}$ a transition is expected to an RN-BH.

• $\delta^2 \in [1+\frac{2}{\sqrt{3}}, 3]$



• The BH solution is unstable and never dominant. This is like the $\delta^2 > 1$ case at zero density.

• For $1+\frac{2}{\sqrt{3}} \leq \delta^2 \leq \frac{5+\sqrt{33}}{4}$ the system has a mass gap and discrete spectrum in the current correlator if $\Delta < 1$. It is a Mott-like insulator.

• Upon UV completion a RN-like new stable BH solution is expected to appear for $T > T_{min}$. There will be a first or second order phase transition to a conducting phase at $T_c > T_{min}$.

• For $\frac{5+\sqrt{33}}{4} \leq \delta^2 < 3$ The system has a continuous spectrum and is again a conductor.

QC systems with Schröndiger symmetry

• The solutions found, can be put in a different coordinate system that realizes $z = 2$ Schröndiger symmetry.

Son, Balasubramanian+McGreevy

• Consider the simplest example: AdS-Schwarzschild Black hole in light-cone coordinates boosted by an arbitrary boost.

$$ds^2 = \frac{\ell^2}{r^2}\left[\frac{(1-f(r))}{4b^2}(dx^+)^2 - (1+f(r))dx^+dx^- + (1-f(r))b^2(dx^-)^2 + dx^2 + dy^2 + \frac{dr^2}{f(r)}\right]$$

• This realizes $z = 2$ non-relativistic Schrödinger symmetry in 2 spatial dimensions.

Golberger (08), Barbon+Fuertes(08), Maldacena+Martelli+Tachikawa (08)

• One can compute the conductivities using the Karch-O'Bannon formalism applied in this context

Kim+Yamada (10)

The conductivity in the absence of magnetic field (but with light-cone electric field) reads

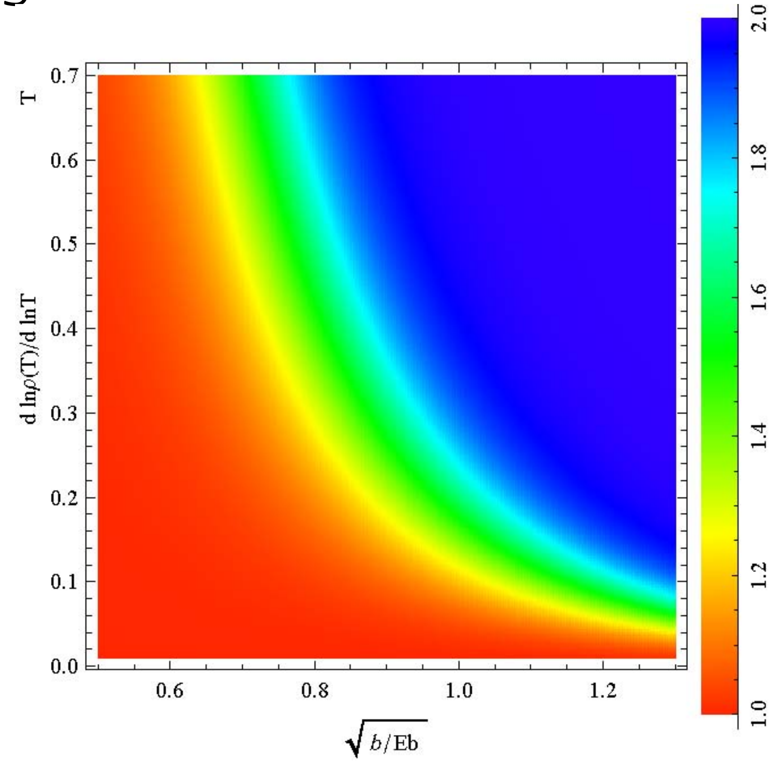
$$\rho = \frac{\rho_0}{\sqrt{\frac{J^2}{t^2 A(t)} + \frac{t^3}{\sqrt{A(t)}}}}\ ,\ A(t) = t^2 + \sqrt{1+t^4} \quad , \quad t = \frac{\pi \ell T b}{\sqrt{2b\tilde{E}_b}}\ ,\ J^2 = \frac{64\sqrt{2}\langle J^+\rangle^2}{(\tilde{\mathcal{N}} b \cos^3\theta)^2(2b\tilde{E}_b)^3}.$$

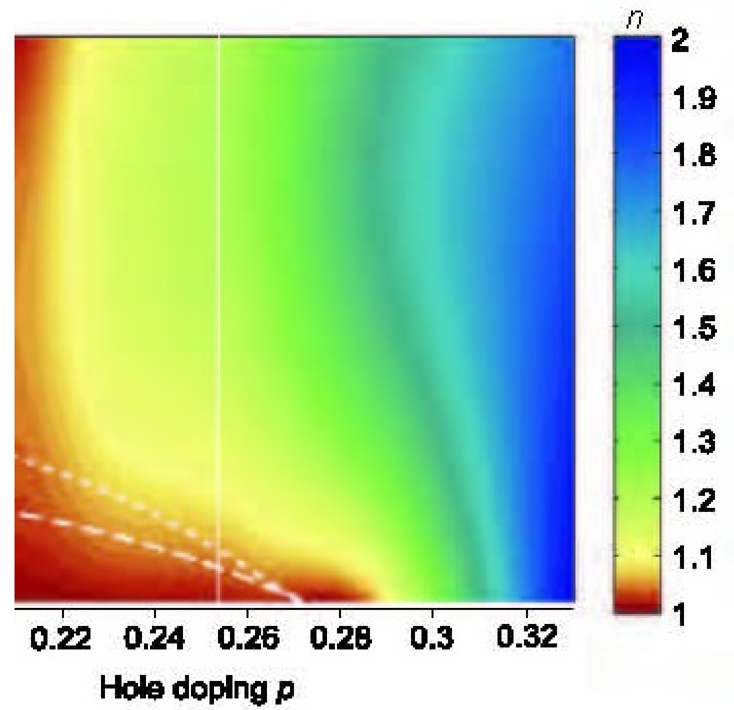
Kim+E.K.+Panagopoulos

When the "drag" term dominates

$$\rho \sim t\sqrt{t^2 + \sqrt{1+t^4}}$$

showing a transition from linear to quadratic behavior.





$La_{2-x}Sr_xCuO_4$ in R. A. Cooper et al., Science **323**, *603 (2009).*

- This transition can be achieved by decreasing the light-cone electric field, E_b. It interpolates between AdS and z=2 Lifshitz scaling.

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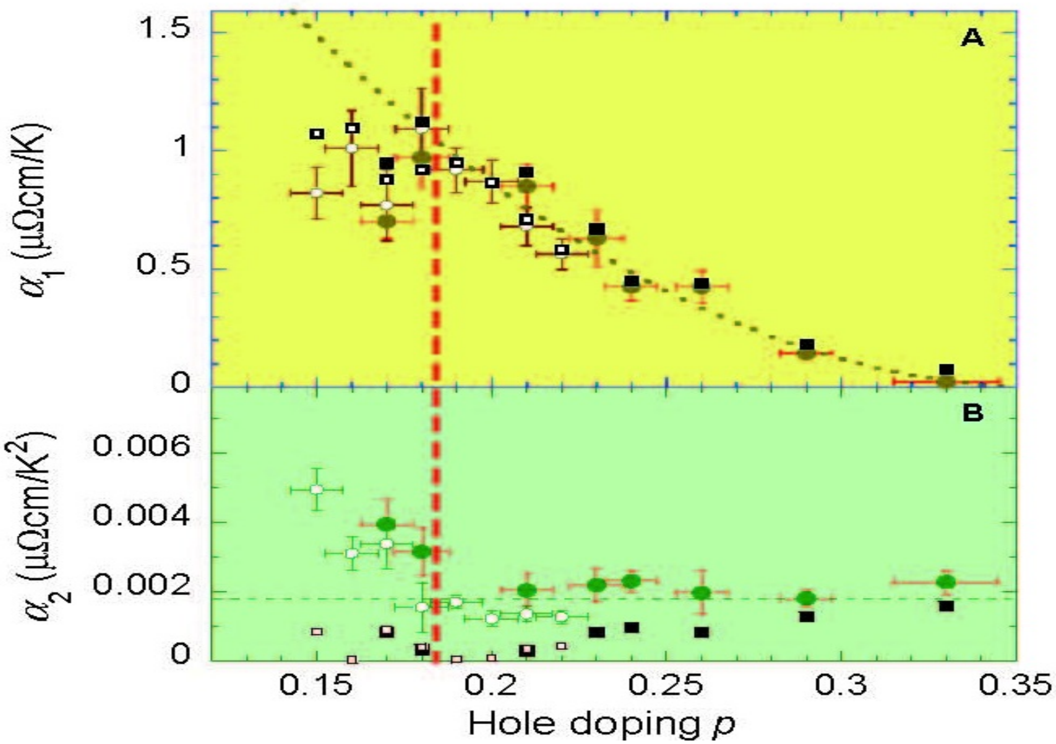
- By parametrizing $\rho = a_1 T + a_2 T^2$ we obtain $\alpha_1 \sim \sqrt{E_b}$ and $\rho_2 = constant$.



$La_{2-x}Sr_xCuO_4$ in R. A. Cooper et al., Science **323***, 603 (2009).*

Resistivity at non-zero magnetic field

At finite magnetic field

$$\sigma^{yy} = \sigma_0 \frac{\sqrt{\mathcal{F}_+(t) J^2 + t^4 \sqrt{\mathcal{F}_+(t)} \mathcal{F}_-(t)}}{\mathcal{F}_-(t)} \quad , \quad \sigma^{yz} = \bar{\sigma}_0 \frac{\mathcal{B}}{\mathcal{F}_-(t)}$$

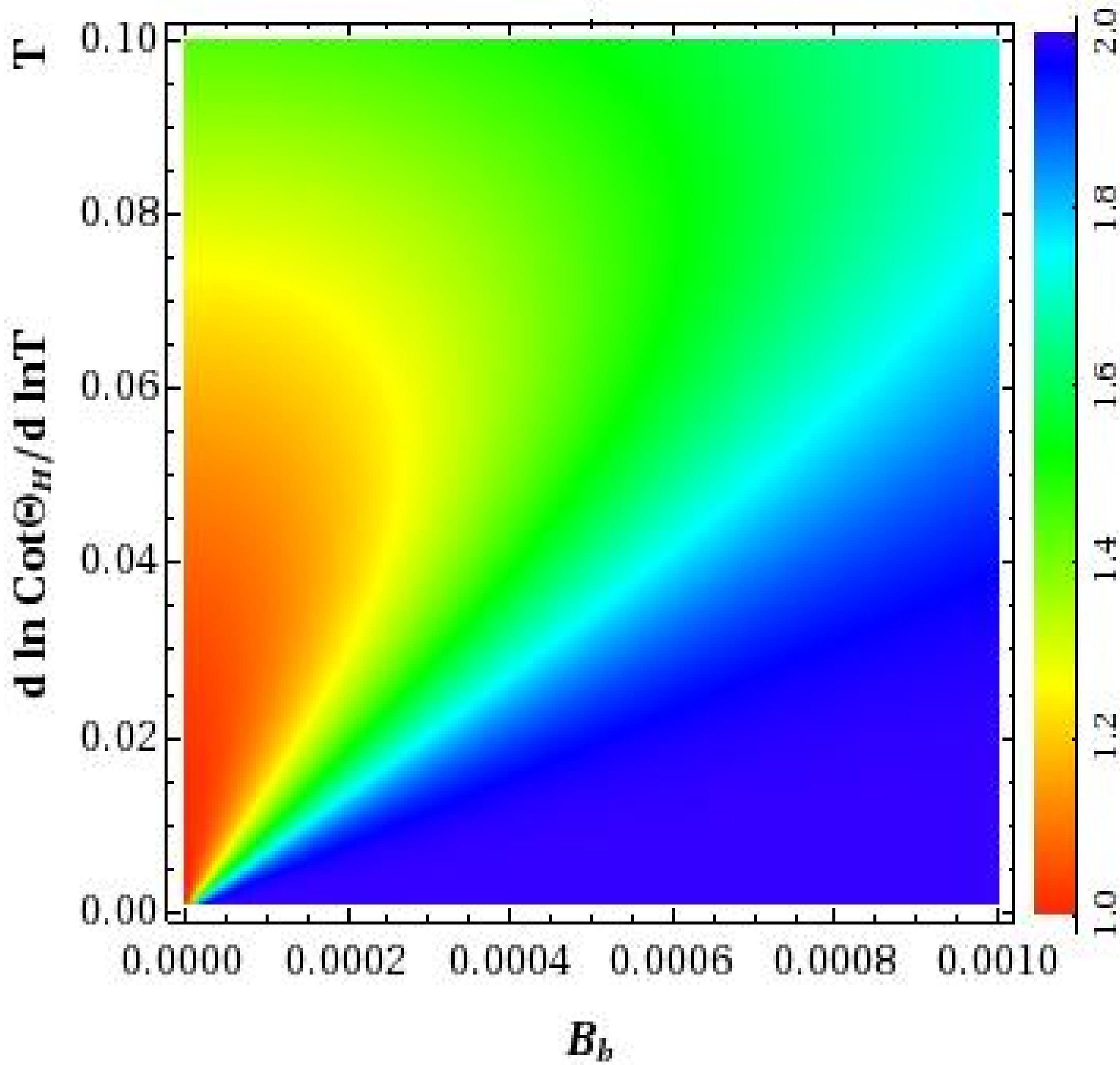
$$\mathcal{F}_\pm = \sqrt{\left(\mathcal{B}^2 + t^4\right)^2 + t^4} \mp \mathcal{B}^2 + t^4 \ , \ \mathcal{B} = \frac{\tilde{B}_b}{2b\tilde{E}_b}$$

• The scaling variable $\mathcal{B} = \frac{\tilde{B}_b}{2b\tilde{E}_b}$ seems to be in agreement with experimental data

$Tl_2Ba_2CuO_{6+\delta}$ in A. W. Tyler et al., Phys. Rev. B **57***, R278 (1998).*

• The inverse Hall angle is defined as the ratio between Ohmic conductivity and Hall conductivity as

$$\cot \Theta_H = \frac{\sigma^{yy}}{\sigma^{yz}}$$


d ln CotΘ_H/d lnT
0.00
0.02
0.04
0.06
0.08
0.10
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010
B_b
1.0
1.2
1.4
1.6
1.8
2.0

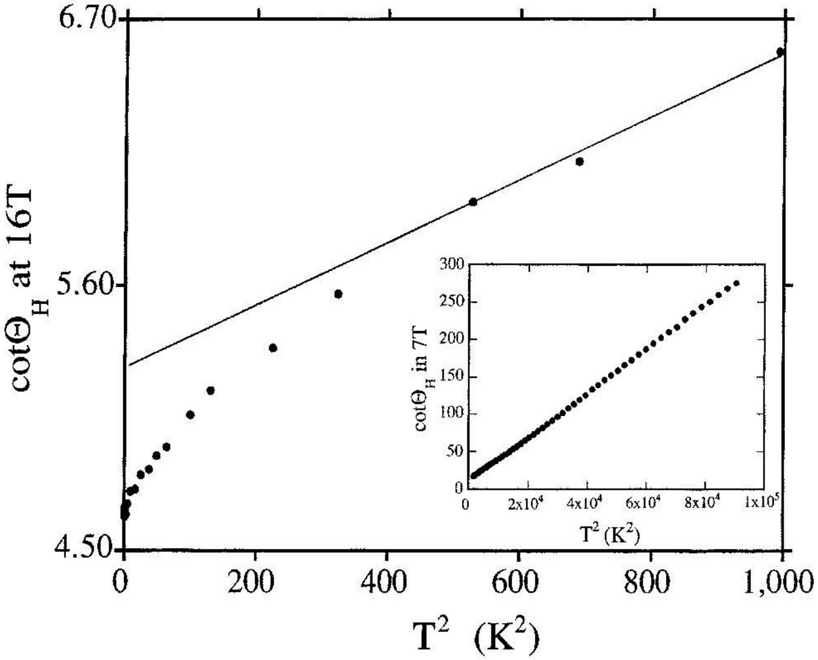
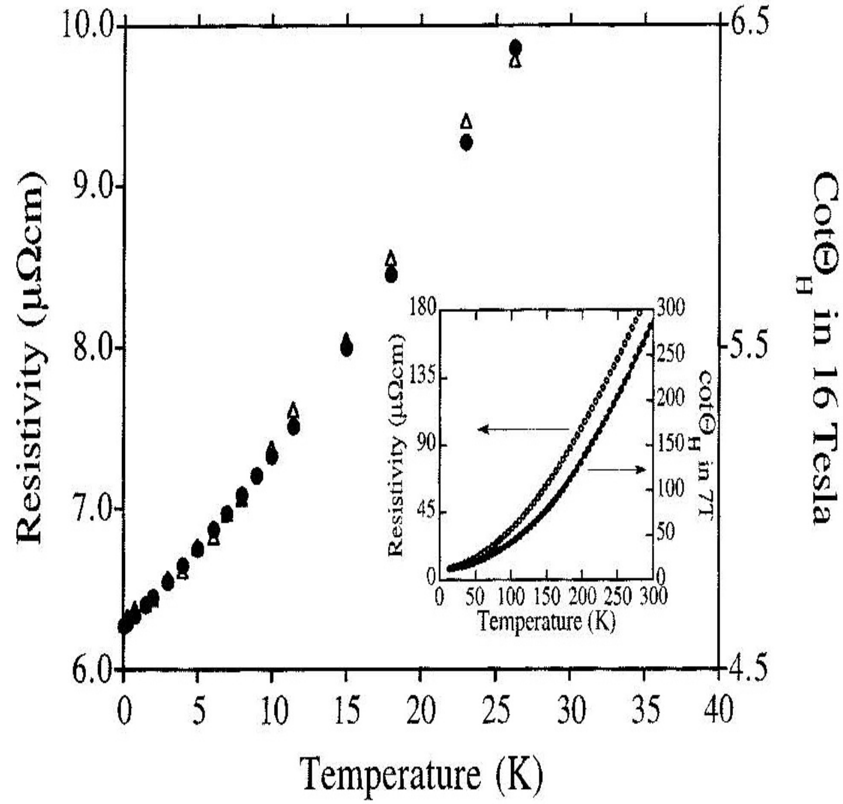


FIG. 8. The cotangent of the Hall angle plotted against T^2 below 30 K. The low-temperature data deviate significantly from the $A+BT^2$ dependence seen at high temperatures (inset), whose extrapolation is shown by the solid line.



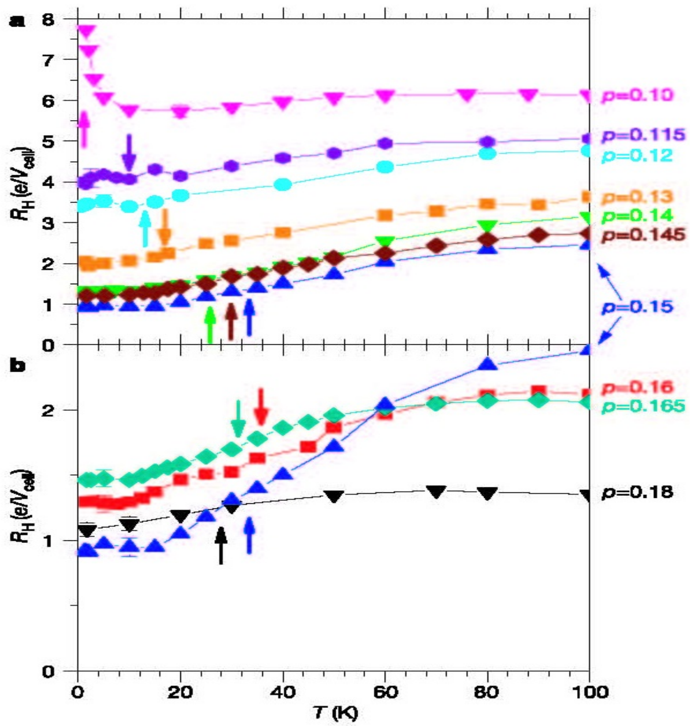
The resistivity and $cot\Theta_H$ are correlated at low temperatures in $Tl_2Ba_2CuO_{6+\delta}$

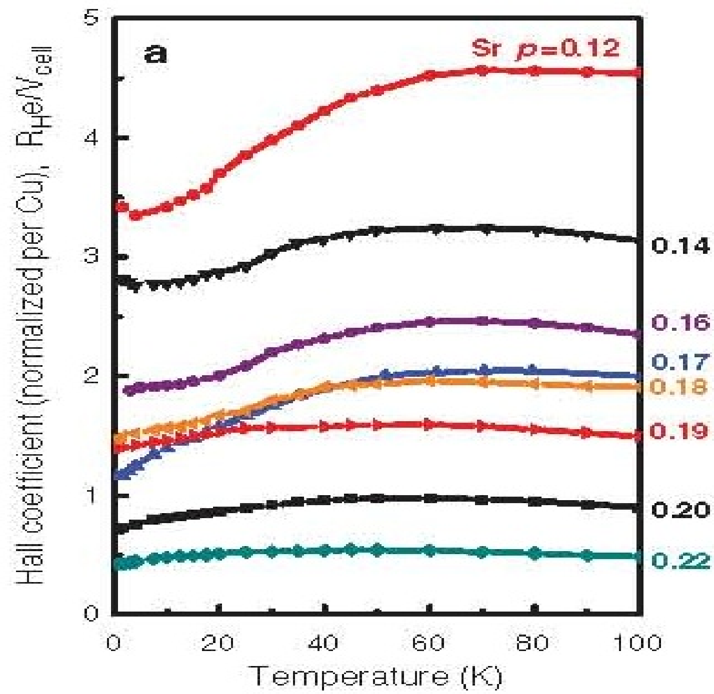
Mackenzie et al. Phys. Rev. B ***53****, 5848 (1996).*

The Hall Conductivity $R_H = \left.\frac{\rho_{yz}}{B}\right|_{B=0}$ is constant in the two different regimes (linear and quadratic)

$$R_H \simeq \frac{\bar{\sigma}_0}{\sigma_0^2 J^2} \sim E_b$$

and decreases with doping.

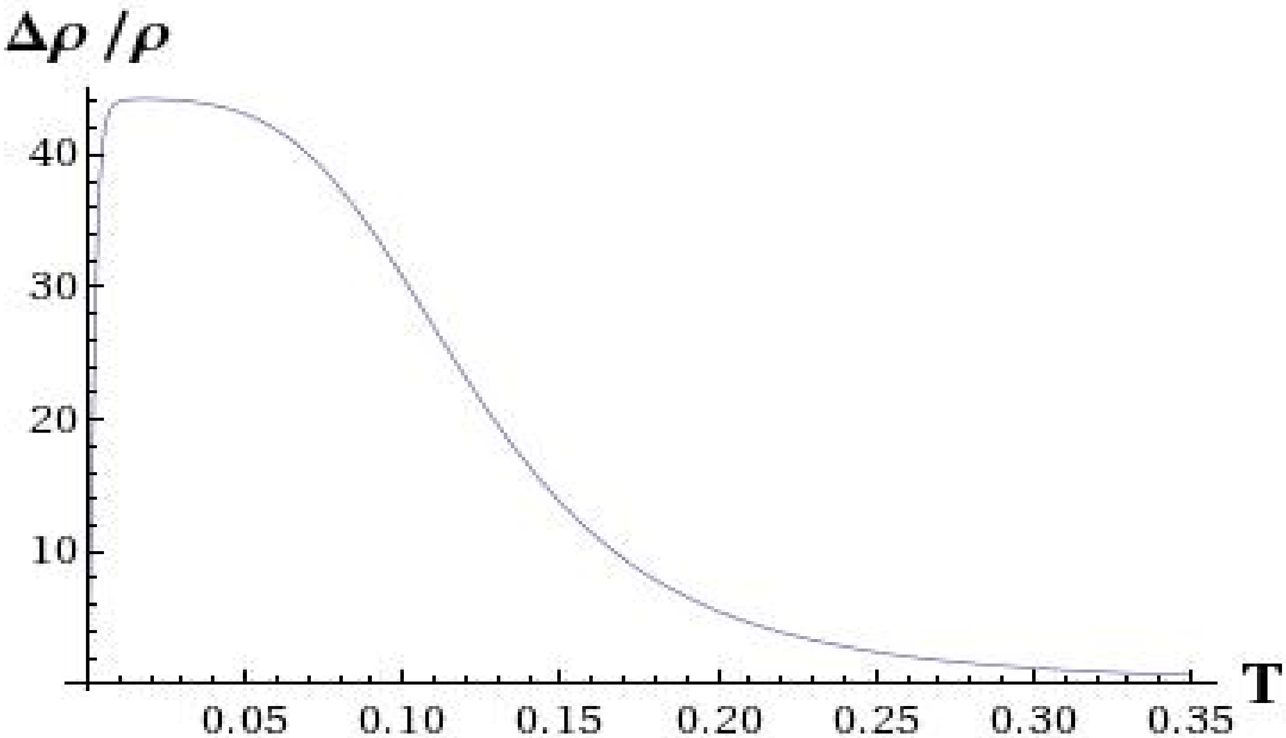




$Bi_2Sr_{2-x}La_xCuO_{6+\delta}$ from F. F. Balakirev et al., NATURE 424 (2003) 912; Phys. Rev. Lett. **102**, *017004 (2009).*

• The magnetoresistance

$$\frac{\Delta\rho}{\rho} = \frac{\rho_{yy}(B) - \rho_{yy}(0)}{\rho_{yy}(0)}$$



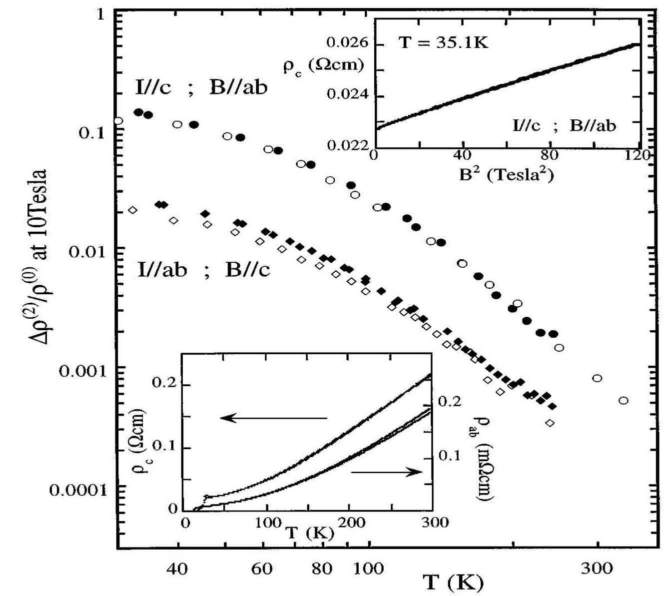


FIG. 1. T dependences of the B^2 terms $\Delta\rho^{(2)}/\rho^{(0)}$ at 10 T for c-axis MR (circles) and a-b plane MR (diamonds) in overdoped $Tl_2Ba_2CuO_6$. Data for two crystals are shown in each case. Bottom inset: Zero-field $\rho_c(T)$ and $\rho_{ab}(T)$ for the crystals shown in the main figure. Top inset: MR field sweep at 35.1 K for $\mathbf{I} \parallel c$, $\mathbf{B} \parallel ab$.

N. E. Hussey et al., Phys. Rev. Lett, ***76****, 122 (1996).*

• We find that the modified Köhler rule

$$\tilde{K} = (\cot\Theta_H)^2 \frac{\Delta\rho}{\rho} \simeq \text{temperature} \quad \text{independent}$$

is valid in regions (linear+quadratic), as demanded by data,

J. M. Harris et al., Phys. Rev. Lett, **75**, *1391 (1995).*

• We also find that the Köhler rule

$$K = \rho^2 \frac{\Delta\rho}{\rho} \simeq \text{temperature} \quad \text{independent}$$

is approximately valid in the same regions.

This is not supported by the data at high temperatures but is valid at low temperatures.

Outlook

• We have used the concept of EHT to study IR asymptotics of a class of theories involving a scalar, a graviton and a vector.

• This is a part of an ordered investigation of EHTs that can be extended to more fields and more interactions.

• The behaviors we find are rich and calculable, giving a wide set of different transport behaviors.

• Some of these systems have observables that bear a remarkable resemblance to what is seen in strange metals.

• A more detailed analysis of this physics and its link to microscopics is in need.

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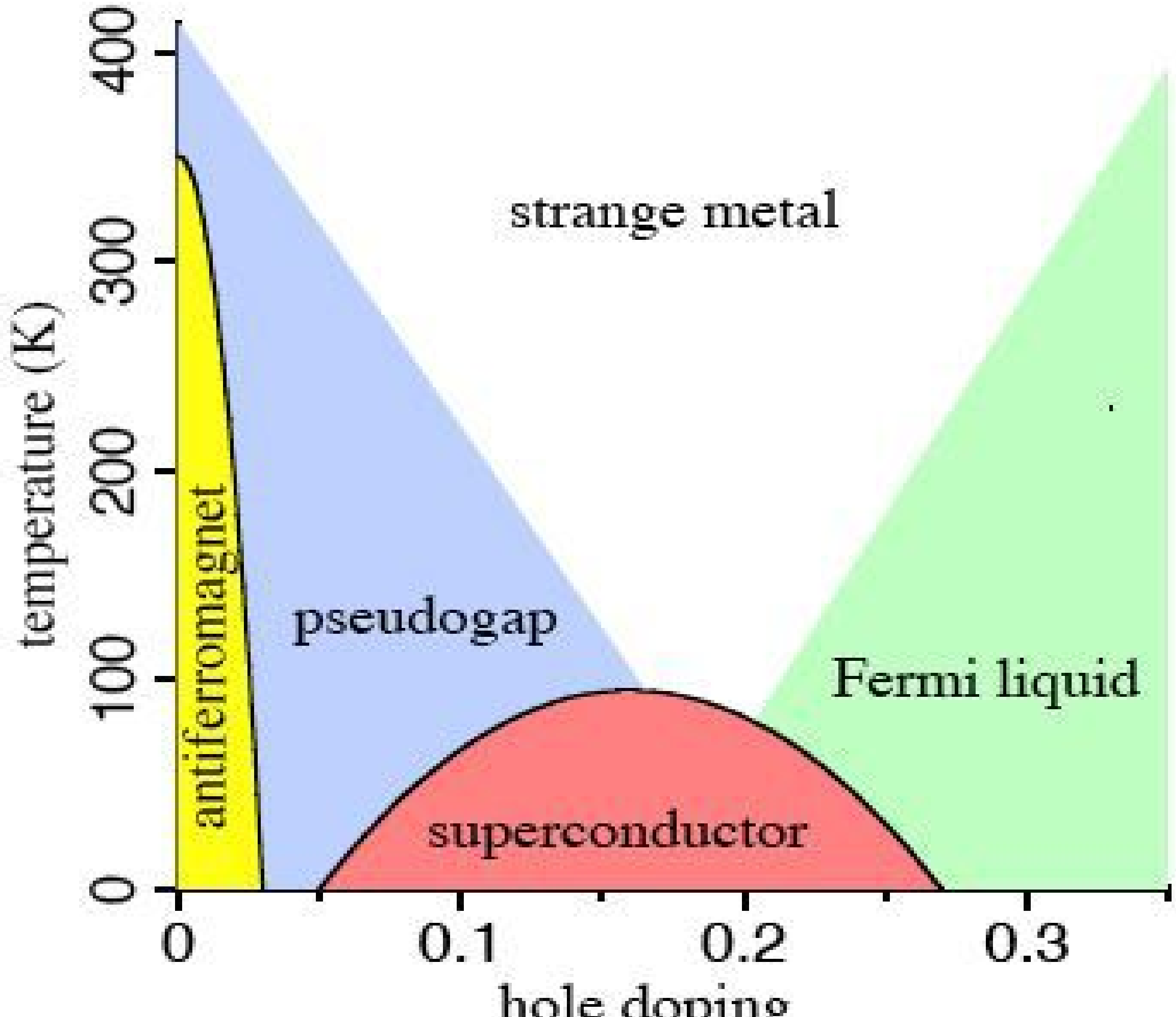
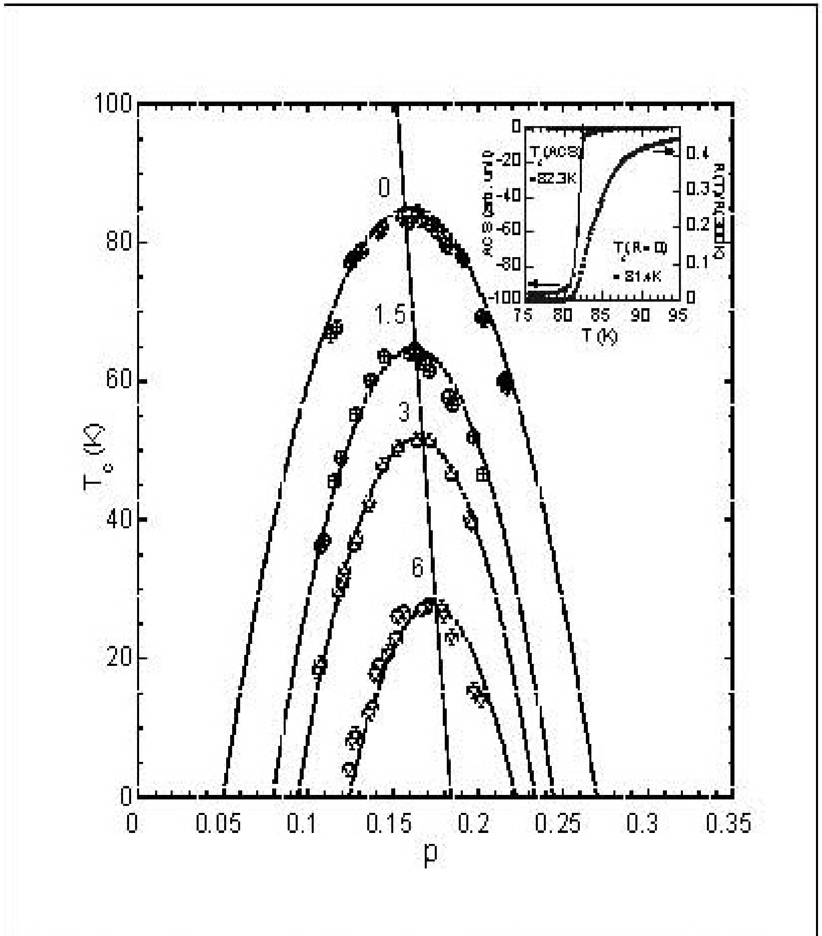
THANK YOU

A typical phase diagram

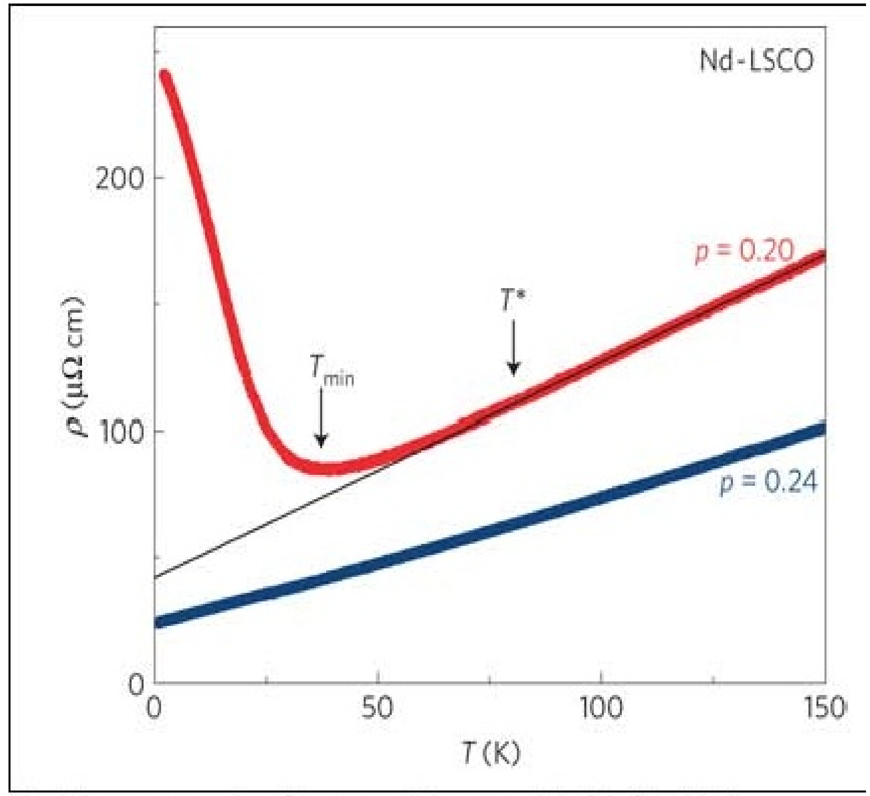


Phase diagram of hole-doped cuprates.In other systems the pseudogap region is much smaller, the superconducting region can shrink to almost nothing etc.

Linear Resistivity



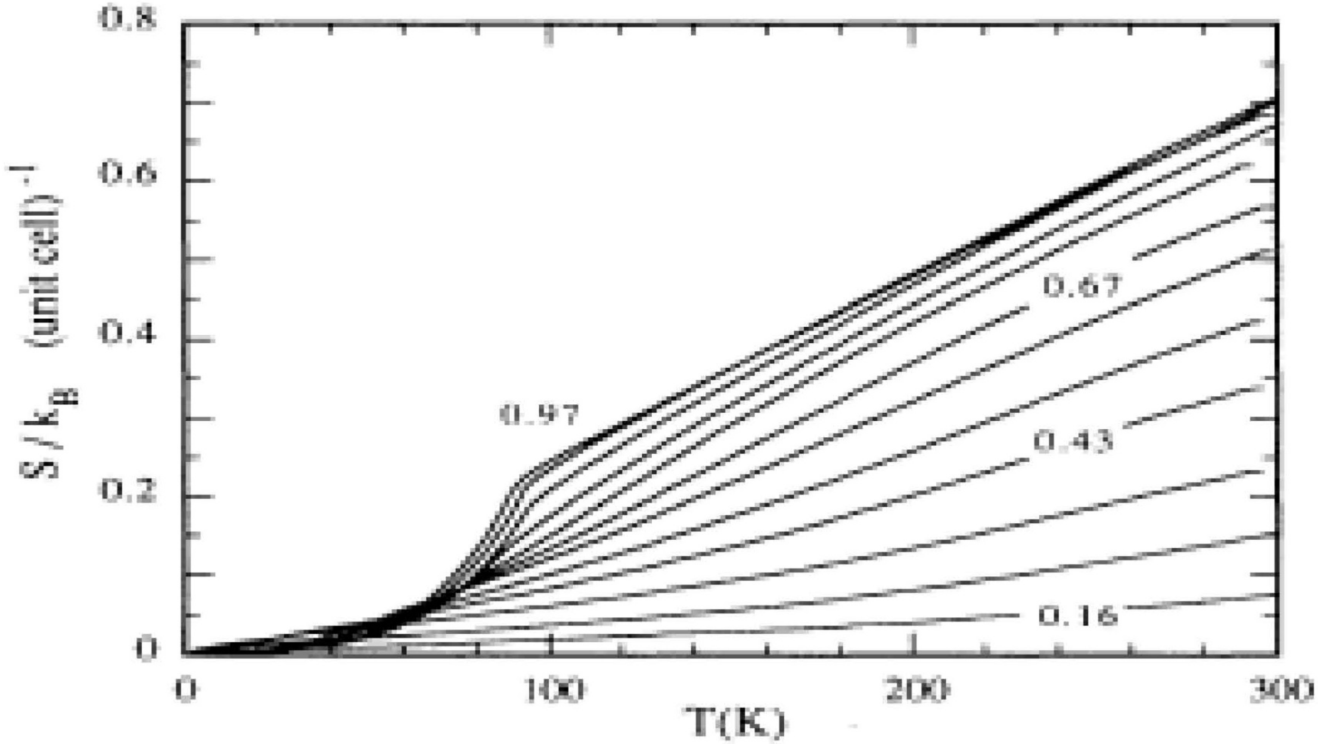
S. H. Naqib et. al., Physica C 387, 365 (2003)



R. Daou et. al., Nature Physics 5, 31 (2009)
& R. A. Cooper, et. al., Science 323, 603(2009)
Nicolas Doiron-Leyraud, et. al., arXiv:0905.0964

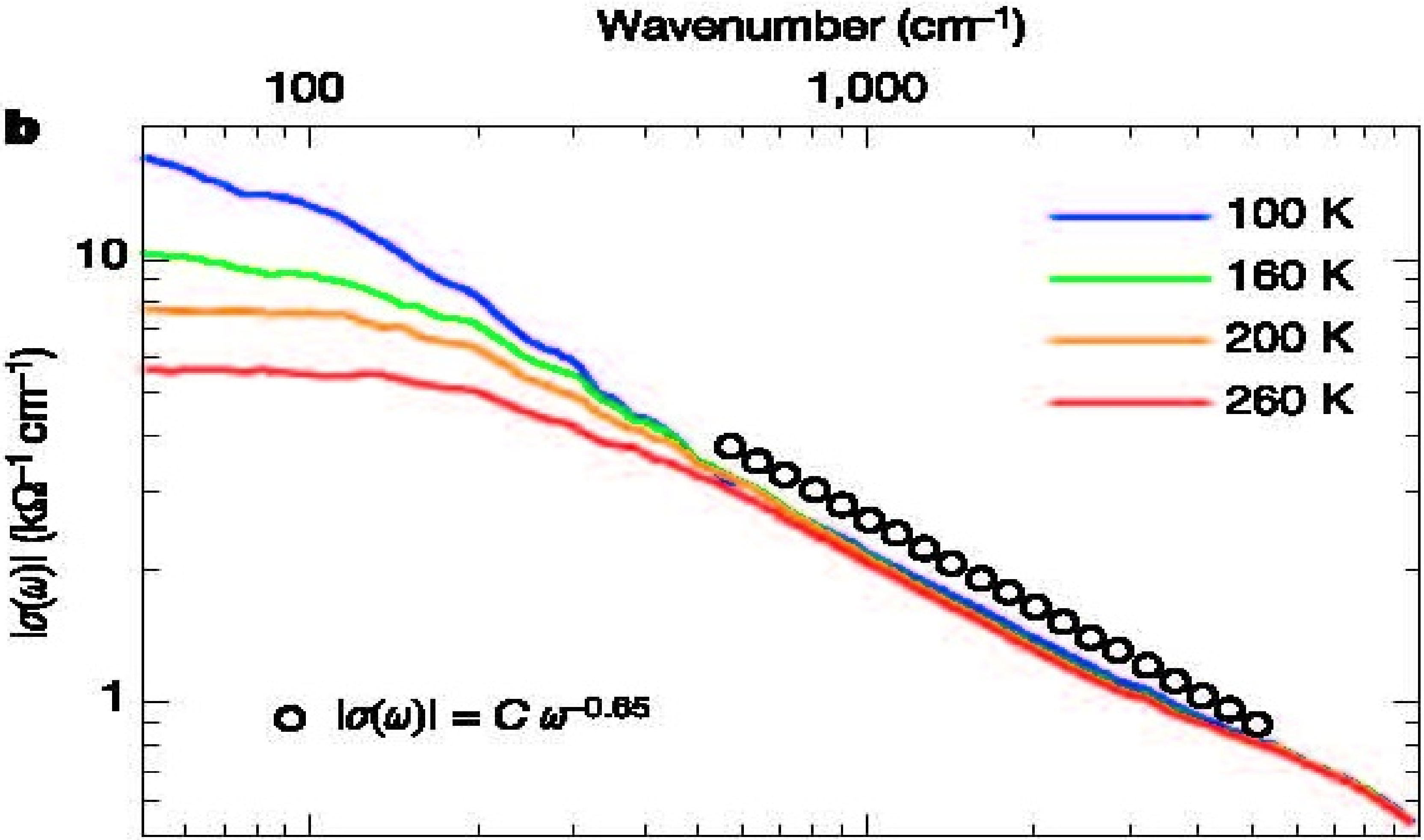
- Suppress superconducting dome with *Zn* substitution or large magnetic field
- Linear temperature dependence of resistivity around the critical point

Linear Heat Capacity



[Loram et. al. PRL 71, 11, 1993]

AC conductivity



van der Marel+Molegraaf+Zaanen+Nussinov+Carbone+Damascelli+Eisaki+Greven+Kes+Li, Nature 425 (2003) 271

Conductivity

- It is main characteristic transport coefficient in a finite density system.

$$J^i(\omega,\vec{k}) = \sigma^{ij}(\omega,\vec{k})\ E_j(\omega,\vec{k})$$

- Can be calculated from a Kubo formula

$$\sigma^{ij}(\omega,\vec{k}) = \frac{G_R^{ij}(\omega,\vec{k})}{i\omega} \quad , \quad G_R^{ij} \equiv \langle J^i\ J^j\rangle$$

- Various limits are of experimental importance

$$\vec{k}\to 0 \quad \to \quad \sigma^{ij}(\omega,T) \quad \to \quad \text{AC conductivity}$$

$$\omega\to 0 \ \text{ and } \ \vec{k}\to 0 \quad \to \quad \sigma^{ij}(T) \quad \to \quad \text{DC conductivity}$$

- The limits $\omega\to 0$ and $\vec{k}\to 0$ do not commute.

Romatchke+Son (2009)

- We can use the drag calculation to calculate the DC conductivity for massive carriers

$$\rho = \frac{T_f}{J^t}\ g_{xx}^E(r_h)\ e^{k\phi(r_h)}$$

RETURN

AC Conductivity: derivation

To compute the frequency depended current correlator we perturb we start with a general diagonal metric ansatz

$$ds^2 = -D(r)dt^2 + B(r)dr^2 + C(r)(dx_i dx^i) \quad , \quad A_t' = q\frac{\sqrt{D(r)B(r)}}{Z(\phi)C(r)^{\frac{p-1}{2}}}$$

In the backreacted case we must turn on perturbations

$$A_i = a_i(r)e^{i(\omega t)} \quad , \quad g_{ti}(r,t) = z_i(r)e^{i\omega t}$$

From the r, x_i Einstein equation we obtain

$$z_i' - \frac{C'}{C}z_i = -ZA_t' \; a_i$$

while from the gauge field equations

$$\partial_r\left(ZC^{\frac{p-3}{2}}\sqrt{\frac{D}{B}}a_i'\right) + ZC^{\frac{p-3}{2}}\sqrt{\frac{B}{D}}\omega^2 \; a_i = \frac{q}{C}\left(z_i' - \frac{C'}{C}z_i\right)$$

Substituting we obtain

$$\partial_r\left(ZC^{\frac{p-3}{2}}\sqrt{\frac{D}{B}}a_i'\right) + ZC^{\frac{p-3}{2}}\left(\sqrt{\frac{B}{D}}\omega^2 - \frac{q^2\sqrt{DB}}{ZC^{p-1}}\right)a_i = 0$$

We can map to a Schrödinger problem

$$\frac{dz}{dr} = \sqrt{\frac{B}{D}} \quad , \quad a_i = \frac{\Psi}{\sqrt{\bar{Z}}} \quad , \quad \bar{Z} = ZC^{\frac{p-3}{3}}$$

$$-\frac{d^2\Psi}{dz^2}+V_{eff}\Psi=\omega^2\Psi \quad , \quad V_{eff}=\frac{q^2 D}{ZC^{p-1}}+\frac{1}{4}\left(\frac{\partial_z\bar{Z}}{\bar{Z}}\right)^2+\frac{1}{2}\partial_z\frac{\partial_z\bar{Z}}{\bar{Z}}$$

Near an AdS boundary the potential asymptotes to

$$V_{eff}\simeq\frac{(p-1)(p-3)}{4z^2}+\frac{q^2}{Z_b}\left(\frac{z}{\ell}\right)^{2(p-2)}+\cdots$$

When $p=3$ the leading behavior is given by

$$V_{\perp,p=3}=-\frac{k}{2}\Delta(2\Delta-1)\ r^{2\Delta-2}+\cdots$$

The frequency dependent conductivity is given by

$$\sigma(\omega)=\frac{1-\mathcal{R}}{1+\mathcal{R}}-\frac{i}{2\omega}\frac{\dot{Z}}{Z}\Big|_{\text{boundary}}$$

Roberts+Horowitz (2009), Goldstein+Kachru+Prakash+Trivedi (2009)

At extremality, near the singularity at $r=r_0$, $D=c_D(r-r_0)^2$, $B=c_B/(r-r_0)^2$ and

$$V\simeq\frac{\nu^2-\frac{1}{4}}{z^2}+\cdots \quad , \quad \nu^2-\frac{1}{4}=\frac{q^2c_B}{Z_0C_0^{p-1}}$$

Calculation of the reflection coefficient then gives

$$\sigma\sim\omega^{2\nu-1}$$

Goldstein+Kachru+Prakash+Trivedi (2009)

Brief summary of results

• We will describe the IR asymptotics of strongly coupled systems at finite density driven by a leading relevant operator.

• To do this we will have to parametrize the gravitational EHT and it will depend on two real constants (γ, δ).

• For zero charge density we will scan the IR landscape and characterize theories by the nature of their spectra and their low temperature thermodynamics. Both 1st order and continuous transitions exist.

• At finite charge density we will find all near-extremal solutions and calculate the low-temperature conductivity, in order to characterize the dynamics. We will also analyze two families of exact solutions.

• We will find that some regions in the (γ, δ) plane will be excluded as unphysical.

• In another large region the has continuous spectra, we will find the most general quantum critical behavior, generalizing AdS_2 and Lifshitz backgrounds.

• For all (γ, δ) except when $\gamma = \delta$ the entropy vanishes at extremality.

• There is a codimension-one space, where the IR resistivity is linear in the temperature

• When the scalar operator is not the dilaton, then in 2+1 dimensions, the IR resistivity has the same scaling as the entropy (and heat capacity).

• We will find the first holographic examples of Mott insulators at finite density.

• Generically the charge-induced entropy dominates the one without charge carriers.

The charged spectra, at zero density and conductivity

• We can also analyze the spectrum of current fluctuations that now depends on γ.

♠ $\frac{\gamma}{\delta} > \frac{3}{2}$ or $\frac{\gamma}{\delta} < -\frac{1}{2}$: When the UV dimension of the scalar $\Delta < 1$ then the potential diverges both in the UV and the IR and the spectrum is discrete and gapped. This resembles to an insulator. Otherwise it is a conductor.

♠ $-\frac{1}{2} < \frac{\gamma}{\delta} < \frac{3}{2}$. The spectral problem is unacceptable and therefore the spin-1 spectrum unreliable.

• The AC Conductivity at zero charge density:

When $|\delta| < 1$ the effective potential is

$$V_{eff} \simeq \frac{c}{z^2} \quad , \quad c = \frac{(\gamma\delta + 1 - \delta^2)\gamma\delta}{(1-\delta^2)^2} \quad , \quad \sigma \sim \omega^n \quad , \quad n = \sqrt{4c+1} - 1$$

• It becomes $n=-\frac{2}{3}$ iff

$$\gamma=\frac{\delta^2-1}{3\delta}\quad \text{or} \quad \gamma=\frac{2(\delta^2-1)}{3\delta}$$

• The DC conductivity can calculated (using Karch-O'Bannon) to be

$$\sigma=e^{-k\phi_0}(\kappa T)^{\frac{2k\delta+2}{\delta^2-1}}\sqrt{\langle J^t\rangle^2+e^{2(\gamma+k)\phi_0}(\kappa T)^{\frac{4[1+(\gamma+k)\delta]}{1-\delta^2}}}\,,$$

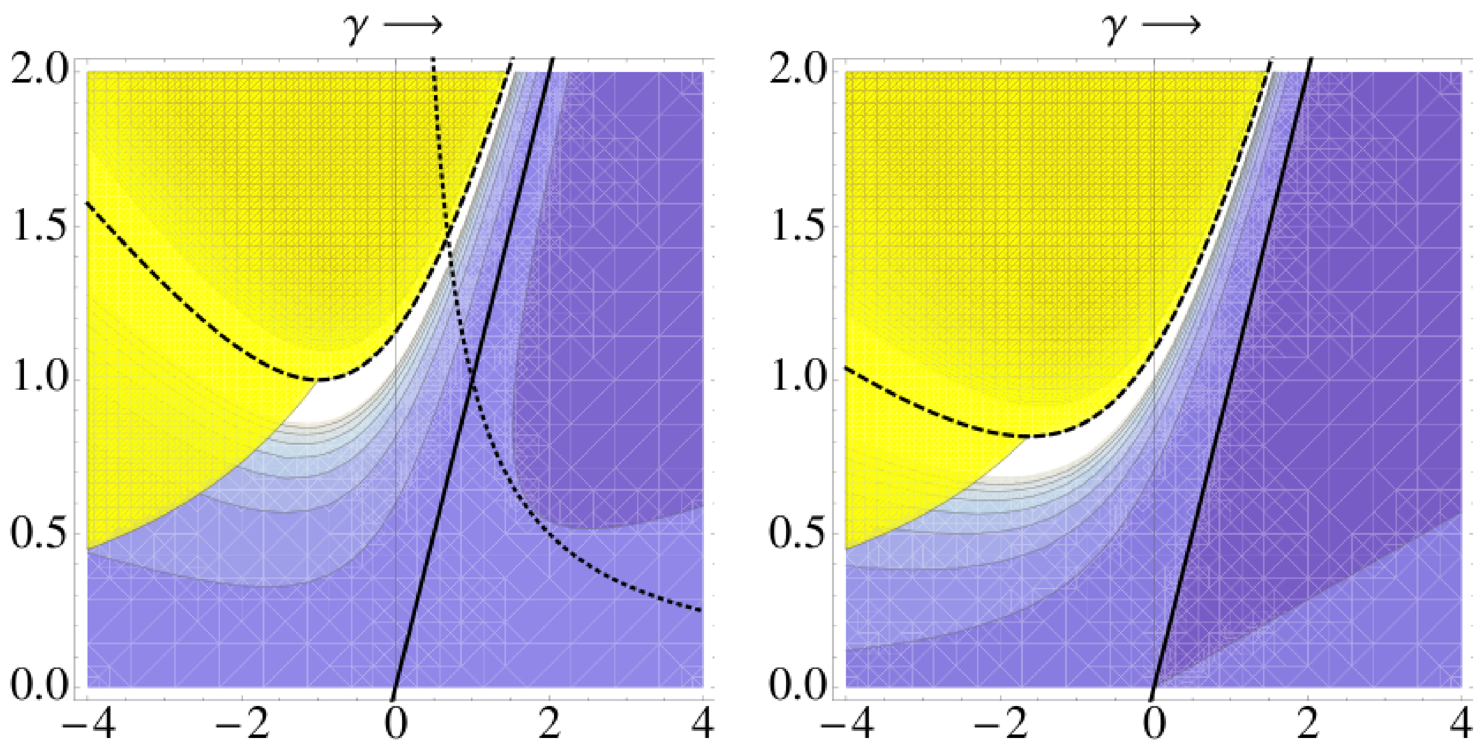
$$\rho_{\text{light}}\sim T^{\frac{2\gamma\delta}{\delta^2-1}}\quad , \quad \rho_{\text{drag}}\sim\frac{T^{\frac{2k\delta+2}{1-\delta^2}}}{\langle J^t\rangle}$$

• In the first case we can attain linear resistivity when

$$\gamma=\gamma_{\text{linear}}\equiv\frac{\delta^2-1}{2\delta}\,.$$

The extremal AC conductivity

$$p = 3 \quad \sigma \sim \omega^n , \qquad n = \left| \frac{(\delta - \gamma)(3\gamma + 5\delta) - 12}{(\delta - \gamma)(\gamma + 3\delta) - 4} \right| - 1 \,.$$



Contour plot of the scaling exponent n in the (γ, δ) upper half plane for $p = 3$ (left figure $0 \leq \delta \leq \sqrt{\frac{5}{3}}$) and $p = 4$ (right figure, $0 \leq \delta \leq \sqrt{\frac{4}{3}}$). Left figure: Contours correspond to $n = 1.52, \ldots, 8.36$, starting with $n = 1.52$ in the upper right corner and increasing in steps of 0.76. The black solid line $\gamma = \delta$ is $n = 2$, and brighter colors correspond to larger n. Right figure: Contours correspond to $n = 2.2, \ldots, 12.1$, starting with $n = 2.2$ in the lower right corner and increasing in steps of 1.1. The black solid line $\gamma = \delta$ is again at

$n = 2$. In the yellow regions the computation of n cannot be trusted, since an explicit AdS completion of the space-time is needed to render the thermodynamics well-defined. The scaling exponent diverges to $+\infty$ along the dashed black line in both cases.

The near-extremal DC conductivity

- For massive charge carriers

$$\rho \sim T^m, \qquad \frac{4k(\delta-\gamma)+2(\delta-\gamma)^2}{4(1-\delta(\delta-\gamma))+(\delta-\gamma)^2}$$

- The exponent becomes unity for two values of γ

$$\gamma_\pm = 3\delta + 2k \pm 2\sqrt{1+(\delta+k)^2}\,.$$

- For a non-dilatonic scalar, $k = 0$ and the temperature dependence of the entropy and the resistivity are the same. Therefore, the entropy also scales linearly with T.
- For the Lifshitz solutions, we must take $\delta = 0$ and $\gamma = -\sqrt{\frac{4}{(z-1)}}$. In this case we obtain that

$$m_p = \frac{2+k\sqrt{4(z-1)}}{z},$$

- When $k = 0$ this is in agreement with Hartnoll+Polchinski+Silverstein+Tong

Drag calculation of DC conductivity

.

Gubser (2005), Karch+O'Bannon (2007)

$$S_{NG} = T_f \int d^2\xi \sqrt{\hat{g}} + \int d\tau A_\mu \dot{x}^\mu \,, \qquad \hat{g}_{\alpha\beta} = g_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu \,,$$

In a direction with translation invariance we have the following world-sheet Poincaré conserved currents

$$\pi^\alpha_\mu = \bar{\pi}^\alpha_\mu + A_\mu \eta^{\alpha\tau} = T_f \sqrt{\hat{g}}\hat{g}^{\beta\alpha} g_{\nu\mu}\partial_\beta x^\nu + A_\mu \eta^{\alpha\tau} \,,$$

The bulk and boundary equations are

$$\partial_\alpha \bar{\pi}^\alpha_\mu = 0 \quad , \quad T_f \sqrt{\hat{g}}\hat{g}^{\sigma\beta} g_{\mu\nu}\partial_\beta x^\nu + qF_{\mu\nu}\dot{x}^\nu = 0 \,.$$

We now consider a space-time metric in a generic coordinate system and a bulk gauge field

$$ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)dx^i dx^i \quad , \quad A_{x^1} = -Et + h(r) \quad , \quad A_t(r)$$

We choose a static gauge with $\sigma = r$ and $\tau = t$ and make the ansatz

$$x^1 = X = vt + \xi(r) \,,$$

which is motivated by the expectation that the motion of the string will make it have a profile that is dragging on one side as it lowers inside the bulk space.
The boundary equation for $\mu = t$ and $\mu = x$ are equivalent and become

$$T_f \frac{\hat{g}_{\sigma\tau}}{\sqrt{-\hat{g}}} g_{tt} + Ev = 0 \quad \rightarrow \quad \bar{\pi}_x = E \,.$$

Solving we obtain

$$\xi' = \sqrt{\frac{g_{rr}}{g_{tt}g_{xx}}}\frac{\sqrt{g_{tt}-g_{xx}v^2}}{\sqrt{T_f^2 g_{tt}g_{xx}-\bar{\pi}_x^2}}\bar{\pi}_x\,.$$

To ensure we have a real solution, there must be a turning point at $r=r_s$

$$v^2 = \frac{g_{tt}(r_s)}{g_{xx}(r_s)}\,, \qquad \bar{\pi}_x = -T_f\sqrt{g_{tt}(r_s)g_{xx}(r_s)}$$

Finally as v is constant we obtain

$$T_f\sqrt{g_{tt}(r_s)g_{xx}(r_s)} = -E \quad , \quad \frac{dp}{dt} = -\bar{\pi}_x + qE\,,$$

and the steady state solution is $\bar{\pi}_x = E$. For small velocities we obtain

$$\bar{\pi}_x \simeq -T_f g_{xx}(r_h)\ v + \mathcal{O}(v^2) \quad , \quad J^x = J^t\ v \simeq \frac{J^t\bar{\pi}_x}{T_f g_{xx}(r_h)} \simeq \frac{J^t}{T_f g_{xx}(r_h)}E\,,$$

and we obtain the DC conductivity and related resistivity as

$$\sigma \simeq \frac{J^t}{T_f g_{xx}(r_h)}\,, \qquad \rho \simeq \frac{T_f g_{xx}(r_h)}{J^t} = \frac{T_f g_{xx}^E(r_h)e^{k\phi(r_h)}}{J^t}\,.$$

In the case that $k=0$

$$\frac{\rho(T)}{S(T)^{\frac{2}{p-1}}} = \text{constant}\,.$$

Vacuum solutions in the Einstein-Dilaton theory

$$V(\lambda) \sim V_0 \lambda^{2Q} \quad , \quad \lambda \equiv e^{\phi} \to \infty$$

- The solutions can be parameterized in terms of a fake superpotential

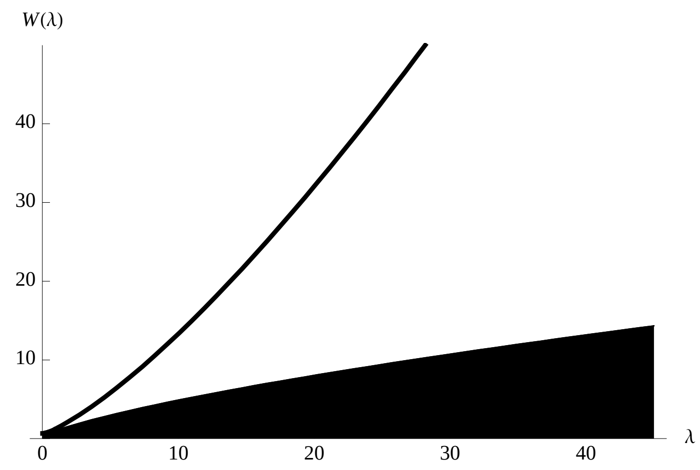
$$V = \frac{64}{27} W^2 - \frac{4}{3}\lambda^2 \; W'^2 \quad , \quad W \geq \frac{3}{8}\sqrt{3V}$$

The crucial parameter resides in the solution to the diff. equation above. There are three types of solutions for $W(\lambda)$:

Gursoy+E.K.+Mazzanti+Nitti

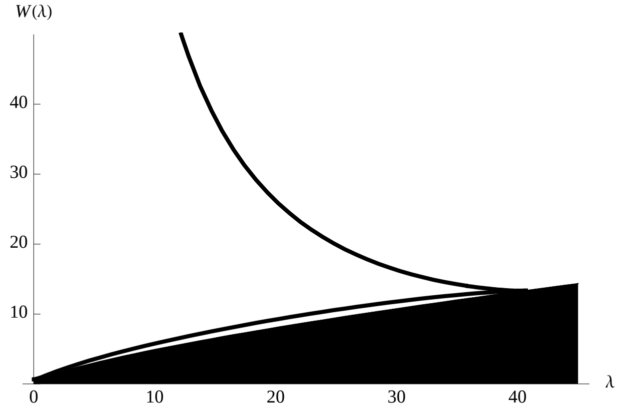
1. Generic Solutions (bad IR singularity)

$$W(\lambda) \sim \lambda^{\frac{4}{3}} \quad , \quad \lambda \to \infty$$



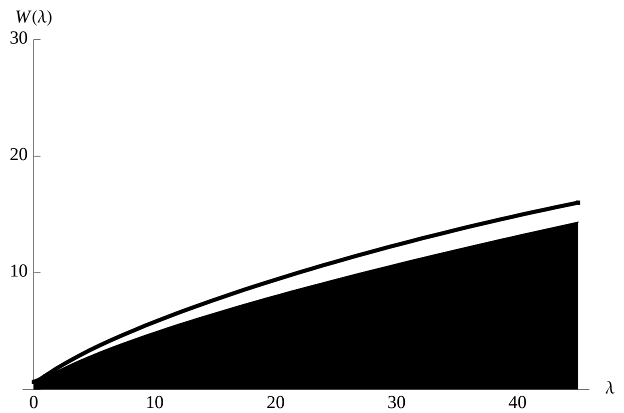
2. Bouncing Solutions (bad IR singularity)

$$W(\lambda) \sim \lambda^{-\frac{4}{3}} \quad , \quad \lambda \to \infty$$



3. The "special" solution.

$$W(\lambda) \sim W_\infty \lambda^Q \quad , \quad \lambda \to \infty \quad , \quad W_\infty = \sqrt{\frac{27V_0}{4(16-9Q^2)}}$$



Good+repulsive IR singularity if $Q < \frac{4\sqrt{2}}{3}$

• For $Q > \frac{4}{3}$ all solutions are of the bouncing type (therefore bad).

• There is another special asymptotics in the potential namely $Q = \frac{2}{3}$. Below $Q = \frac{2}{3}$ the spectrum changes to continuous without mass gap.

In that region a finer parametrization of asymptotics is necessary

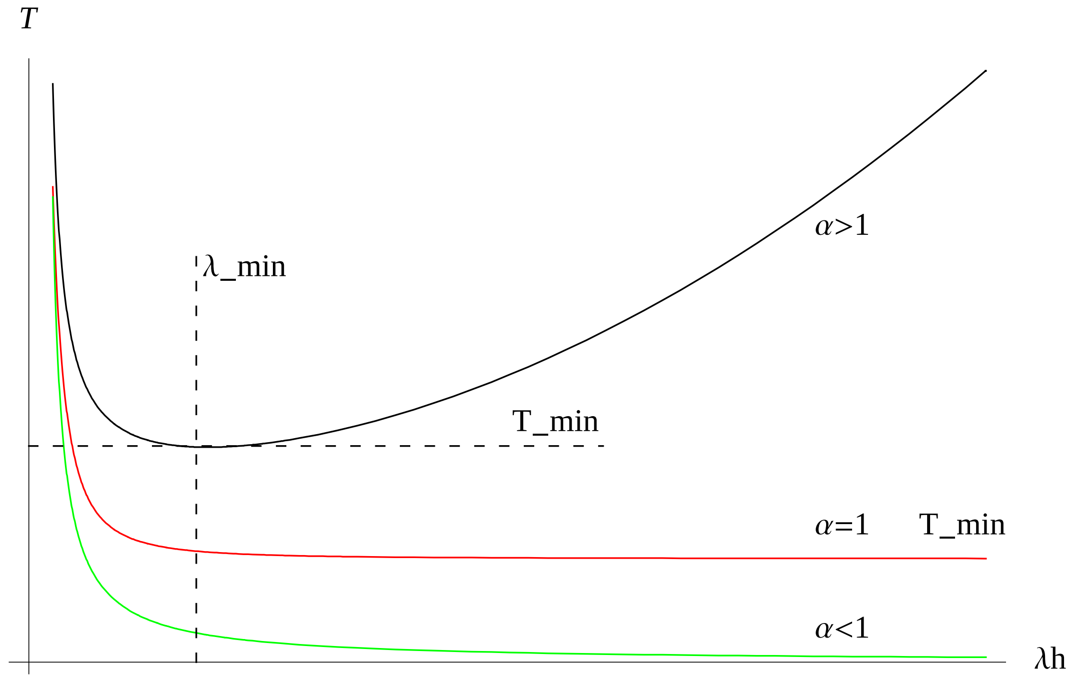
$$V(\lambda) \sim V_0 \; \lambda^{\frac{4}{3}} \; (\log \lambda)^P$$

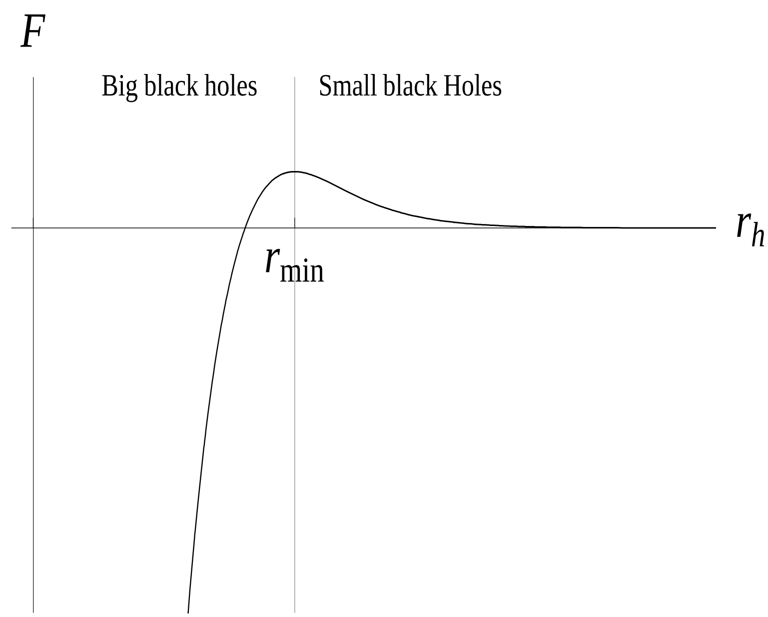
• For $P > 0$ there is a mass gap, discrete spectrum and confinement of charges. There is also a first order deconfining phase transition at finite temperature.

• For $P < 0$, the spectrum is continuous, without mas gap, and there is a transition at T=0 (as in N=4 sYM).

• At $P = 0$ we have the linear dilaton vacuum. The theory has a mass gap but continuous spectrum. The order of the deconfining transition depends on the subleading terms of the potential and can be of any order larger than two.

Gurdogan+Gursoy+E.K.


T
λ_min
α>1
T_min
α=1
T_min
α<1
λh


F
Big black holes
Small black Holes
r_h
r_{min}

Classification of zero temperature solutions

For any positive+monotonic potential $V(\lambda)$, $\lambda \equiv e^{\phi}$ with the asymptotics :

$$V(\lambda) = V_0 + V_1\lambda + V_2\lambda^2 + \ldots \quad V_0 > 0, \qquad \lambda \to 0$$

$$V(\lambda) = V_\infty \lambda^{2Q} (\log \lambda)^P, \quad V_\infty > 0, \qquad \lambda \to \infty$$

the zero-temperature superpotential equation has three types of solutions, that we name the *Generic*, the *Special*, and the *Bouncing* types:

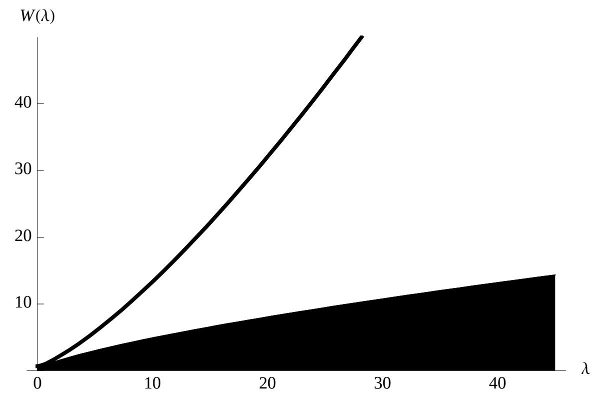
• A continuous one-parameter family that has a fixed power-law expansion near $\lambda = 0$, and reaches the asymptotic large-λ region where it grows as

$$W \simeq C_b \; \lambda^{4/3} \qquad \lambda \to \infty \quad , \quad C_b > 0$$

These solutions lead to backgrounds with "bad" (i.e. non-screened) singularities at finite r_0,

$$b(r) \sim (r_0 - r)^{1/3}, \qquad \lambda(r) \sim (r_0 - r)^{-1/2}$$

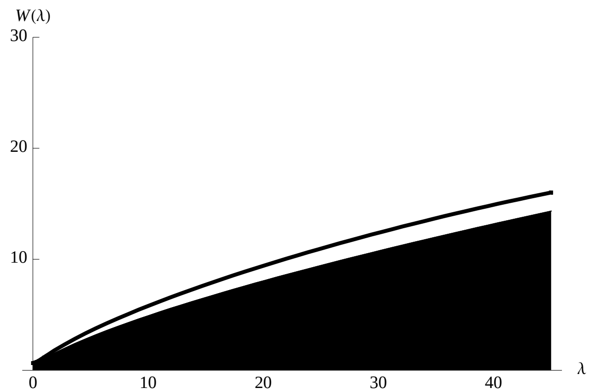
We call this solution *generic*.



• A unique solution, which also reaches the large-λ region, but slower:

$$W(\lambda) \sim W_\infty \lambda^Q (\log \lambda)^{P/2}, \qquad W_\infty = \sqrt{\frac{27V_\infty}{4(16-9Q^2)}}$$

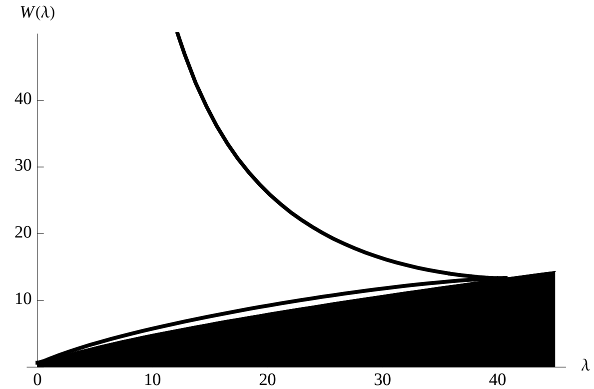
This leads to a repulsive singularity, provided $Q < 2\sqrt{2}/3$ [**?**]. We call this the *special* solution.



• A second continuous one-parameter family where $W(\lambda)$ does not reach the asymptotic region. These solutions have two branches that both reach $\lambda = 0$ (one in the UV, the other in the IR) and merge at a point λ_* where $W(\lambda_*) = \sqrt{27V(\lambda_*)/64}$. The IR branch is again a "bad" singularity at a finite value r_0, where $W \sim \lambda^{-4/3}$, and

$$b(r) \sim (r_0 - r)^{1/3}, \qquad \lambda(r) \sim (r_0 - r)^{1/2}.$$

We call this solution *bouncing*.



The special solution marks the boundary between the generic solutions, that reach the asymptotic large-λ region as $\lambda^{4/3}$ and the bouncing ones, that don't reach it.

If $Q > 4/3$, only bouncing solutions exist.

In all types of solutions the UV corresponds to the region $\lambda \to 0$ on the W_+ branch. There the behavior of W_+ is universal: a power series in λ with *fixed* coefficients, plus a subleading non-analytic piece which depends on an arbitrary integration constant C_w:

$$W = \sum_{i=1}^{\infty} W_i \lambda^i + C_w \lambda^{16/9} e^{-\frac{16W_0}{9W_1}\frac{1}{\lambda}} [1 + O(\lambda)]$$

All the power series coefficients W_i are completely determined by the coefficients in the small λ expansion of $V(\lambda)$, the first few being:

$$W_0 = \frac{\sqrt{27 V_0}}{8}, \quad W_1 = \frac{V_1}{16}\sqrt{\frac{27}{V0}}, \quad W_2 = \frac{\sqrt{27}(64 V_0 V_2 - 7 V1^2)}{1024 V_0^{3/2}}$$

RETURN

The $\gamma\delta = 1$ solutions

$$ds^2 = -\frac{V(r)dt^2}{\left[1-\left(\frac{r_-}{r}\right)^{3-\delta^2}\right]^{\frac{4(1-\delta^2)}{(3-\delta^2)(1+\delta^2)}}} + e^{\delta\phi}\frac{dr^2}{V(r)} + r^2\left[1-\left(\frac{r_-}{r}\right)^{3-\delta^2}\right]^{\frac{2(\delta^2-1)^2}{(3-\delta^2)(1+\delta^2)}}\left(dx^2+dy^2\right),$$

$$V(r) = \left(\frac{r}{\ell}\right)^2 - 2\frac{m\ell^{-\delta^2}}{r^{1-\delta^2}} + \frac{(1+\delta^2)q^2\ell^{2-2\delta^2}}{4\delta^2(3-\delta^2)^2 r^{4-2\delta^2}} \quad , \quad (r_\pm)^{3-\delta^2} = \ell^{2-\delta^2}\left[m \pm \sqrt{m^2 - \frac{(1+\delta^2)q^2}{4\delta^2(3-\delta^2)^2}}\right]$$

$$e^{\phi} = \left(\frac{r}{\ell}\right)^{2\delta}\left[1-\left(\frac{r_-}{r}\right)^{3-\delta^2}\right]^{\frac{4\delta(\delta^2-1)}{(3-\delta^2)(1+\delta^2)}} \quad , \quad \mathcal{A} = \left(\Phi - \frac{q\ell^{2-\delta^2}}{(3-\delta^2)r^{3-\delta^2}}\right)dt \quad , \quad \Phi = \frac{q\ell^{2-\delta^2}}{(3-\delta^2)r_+^{3-\delta^2}}$$

where the parameters m and q are integration constants linked to the gravitational mass and the electric charge. There is an overall scale ℓ

$$\ell^2 = \frac{\delta^2-3}{\Lambda}.$$

RETURN

The $\gamma = \delta$ solutions

$$ds^2 = -V(r)dt^2 + e^{\delta\phi}\frac{dr^2}{V(r)} + r^2\Big(dx^2 + dy^2\Big)$$

$$V(r) = \left(\frac{r}{\ell}\right)^2 - 2m\ell^{-\delta^2}r^{\delta^2-1} + \frac{q^2}{4(1+\delta^2)r^2}$$

$$e^{\phi} = \left(\frac{r}{\ell}\right)^{2\delta} \quad , \quad \mathcal{A} = \left(\Phi - \frac{\ell^{\delta^2}q}{(1+\delta^2)r^{1+\delta^2}}\right)dt \quad , \quad \Phi = \frac{q\ell^{\delta^2}}{(1+\delta^2)r_+^{1+\delta^2}}$$

- There is a "BPS condition" for the existence of a horizon

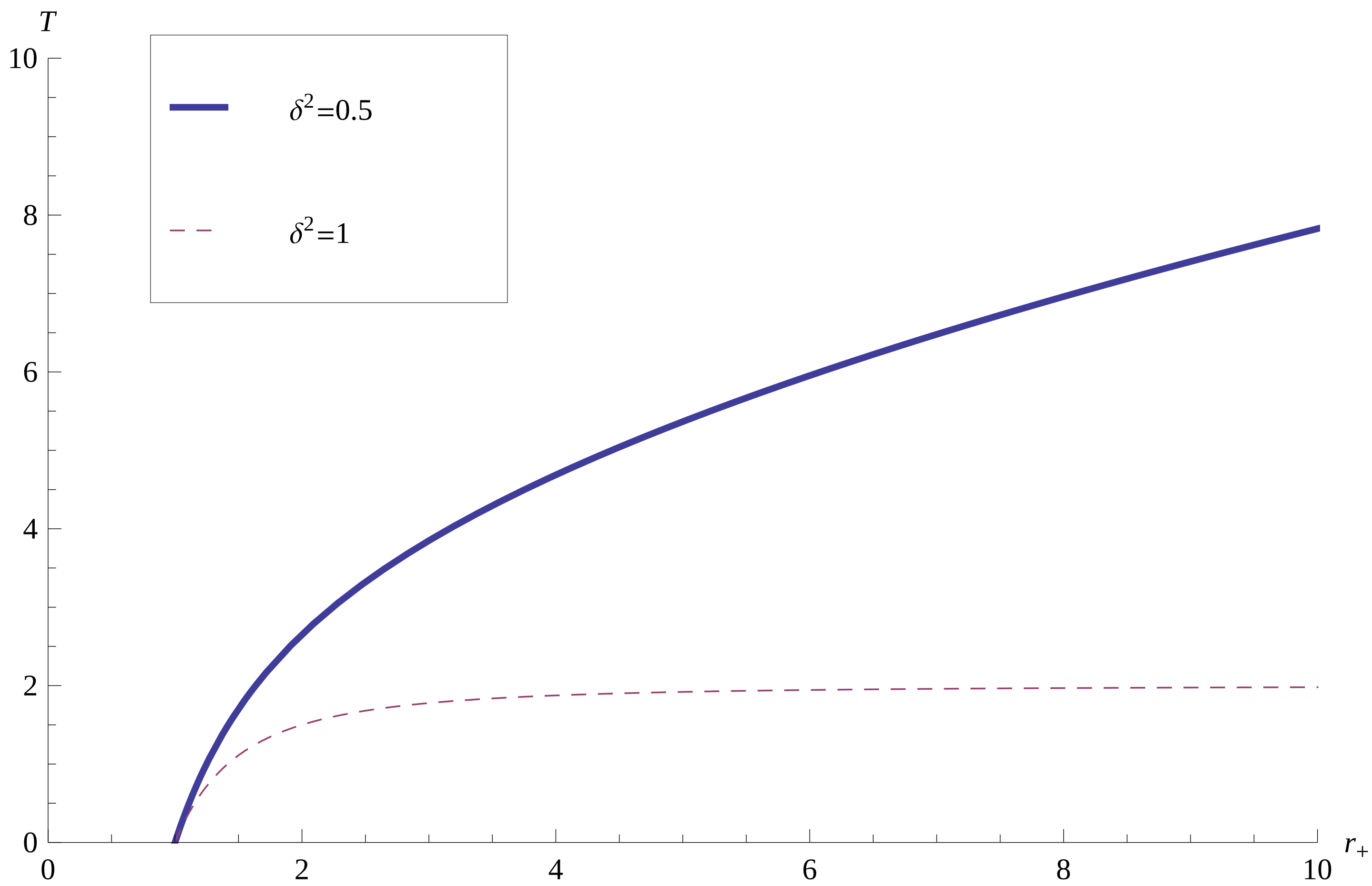
$$m \geq \frac{2q^{\frac{3-\delta^2}{2}}}{1+\delta^2}$$

- U(r) has two roots $0 < r^- < r^+$. The two coincide at the extremality limit, $(1+\delta^2)m = 2q^{\frac{3-\delta^2}{2}}$.

• There are two distinct regimes:

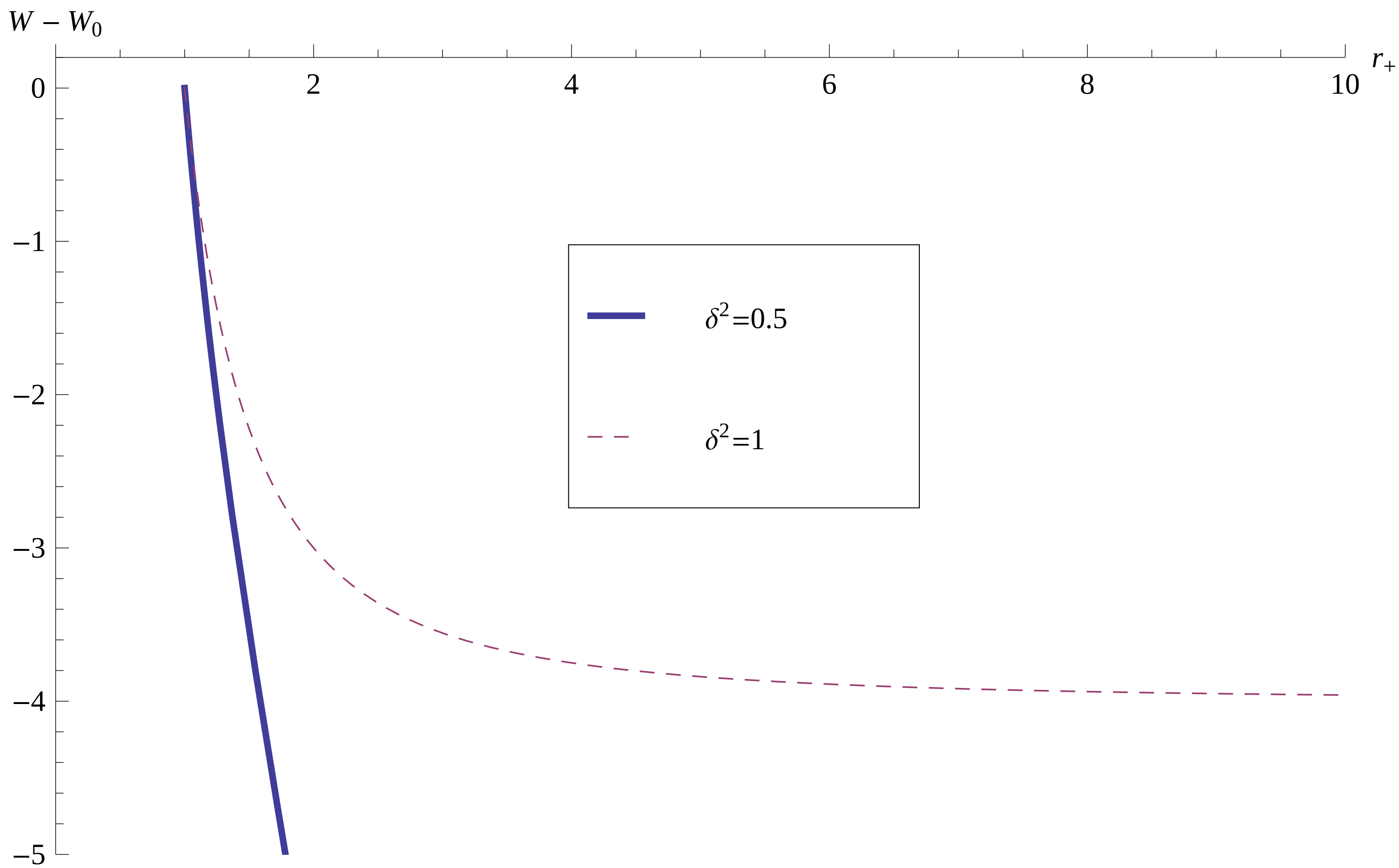
$$0 \leq \delta^2 \leq 1 \qquad\qquad 1 \leq \delta^2 \leq 3$$

• $0 \leq \delta^2 \leq 1$



• Temperature as a function of horizon position

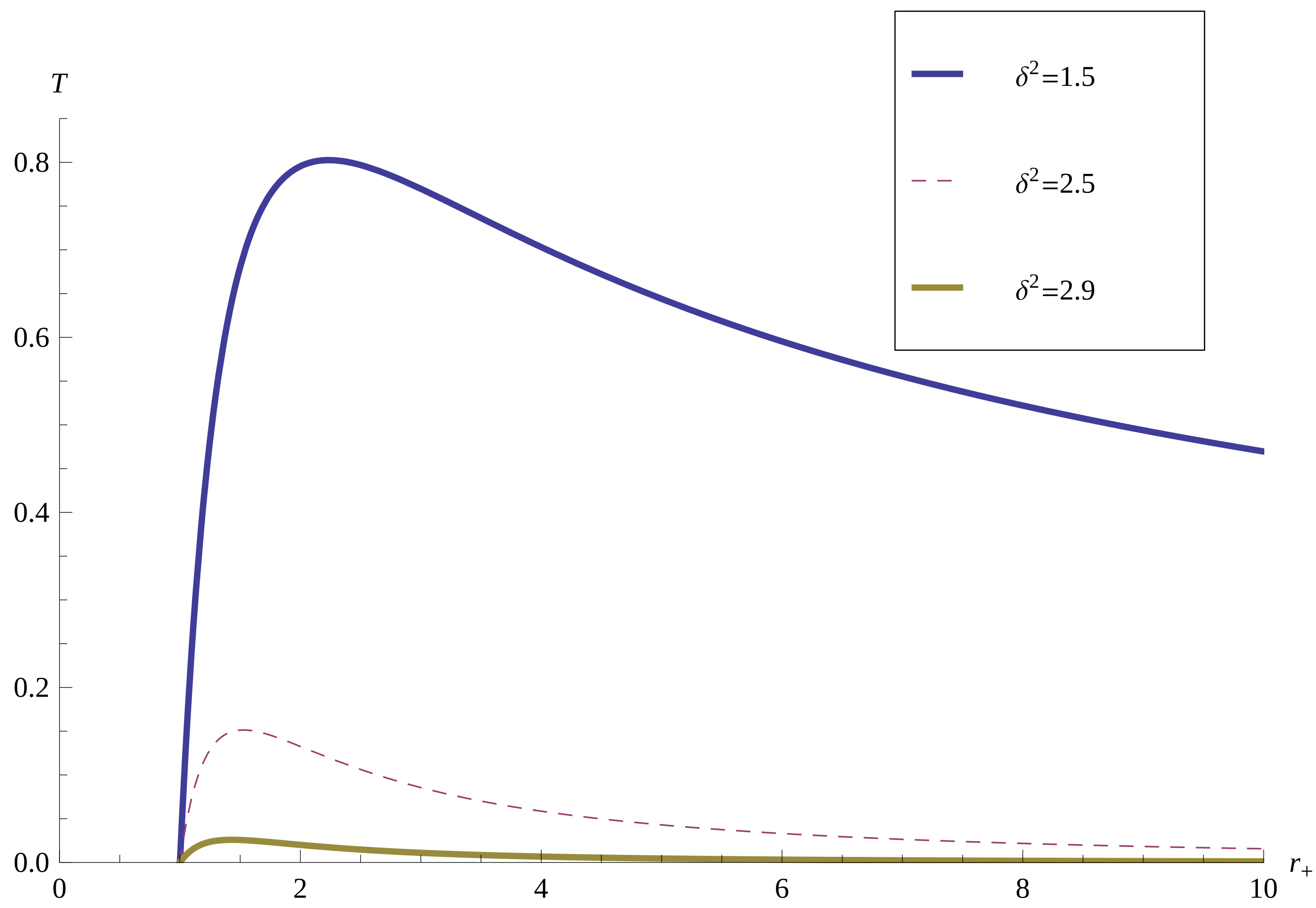
- $0 \leq \delta^2 \leq 1$



- Difference of free energies vs horizon position

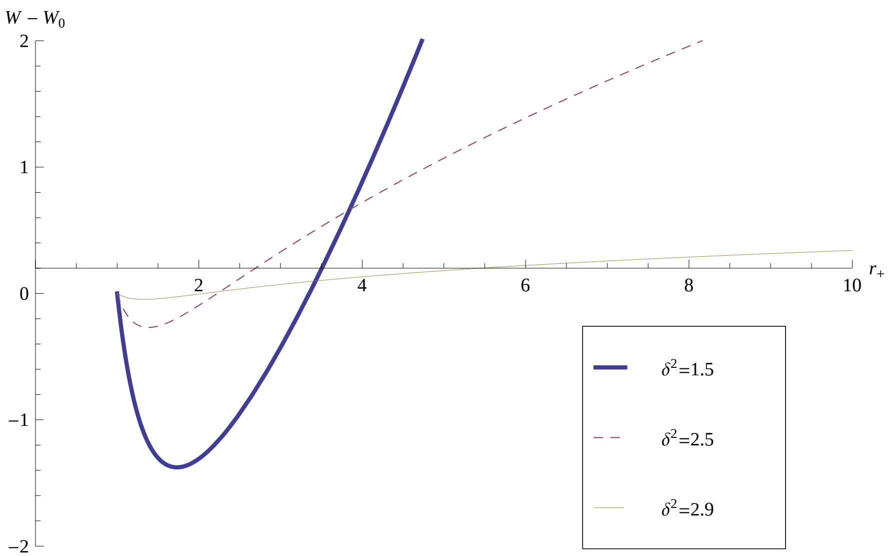
- The BH always dominates

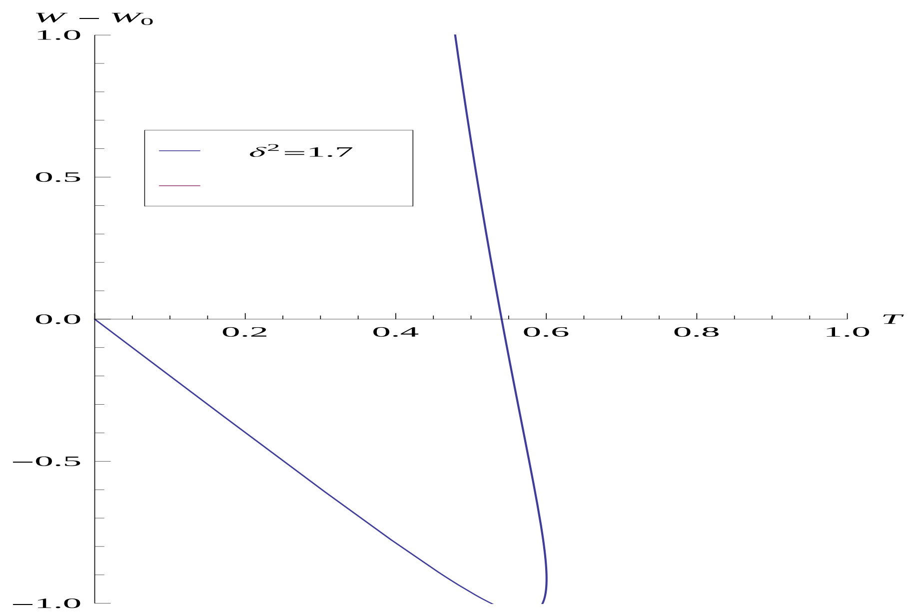
- $1 \leq \delta^2 \leq 3$



- Temperature vs horizon position

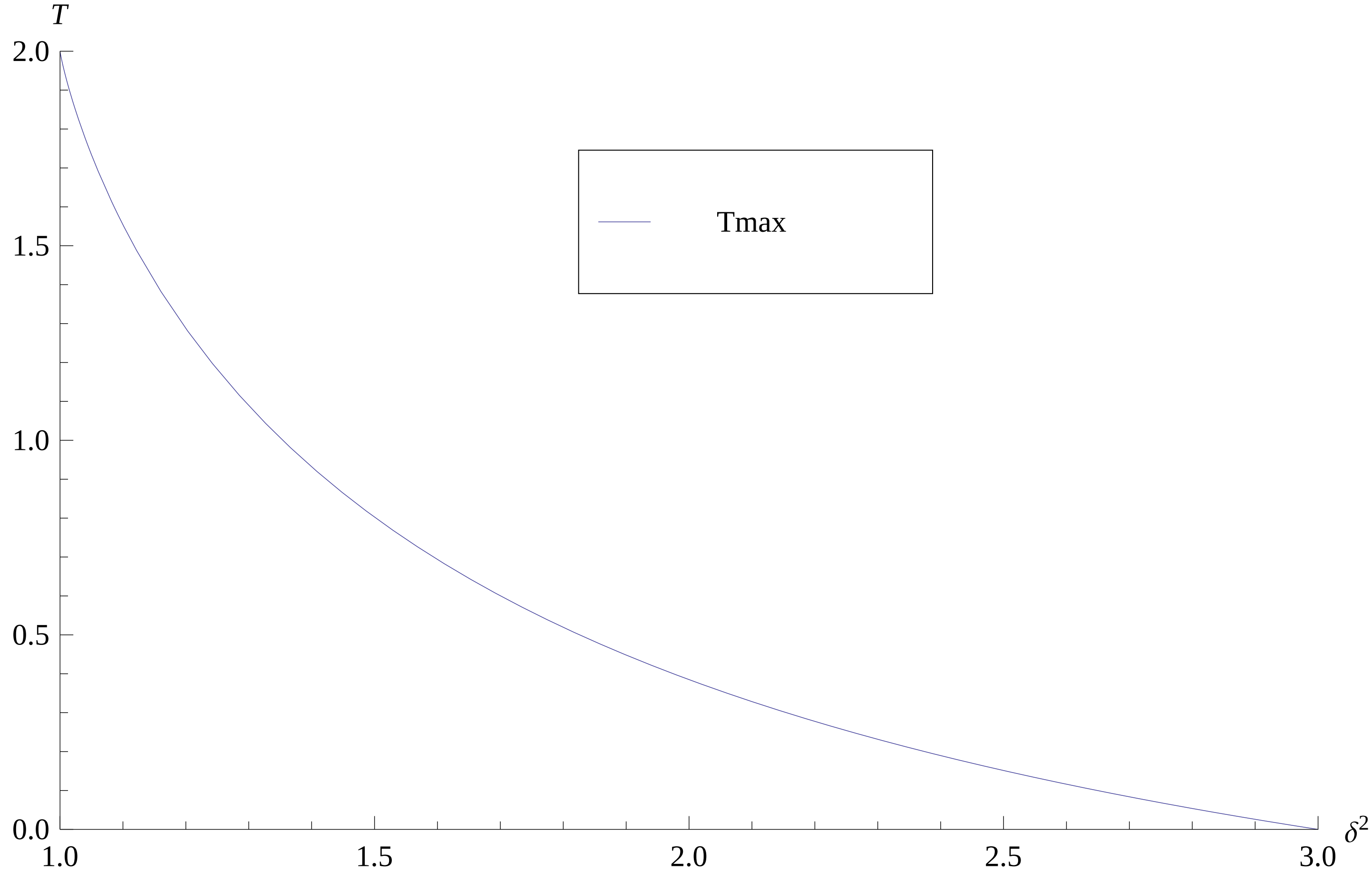
• $1 \leq \delta^2 \leq 3$





• Difference of free energies as a function of horizon position and temperature.

• The BH dominates at low temperatures up to the phase transition

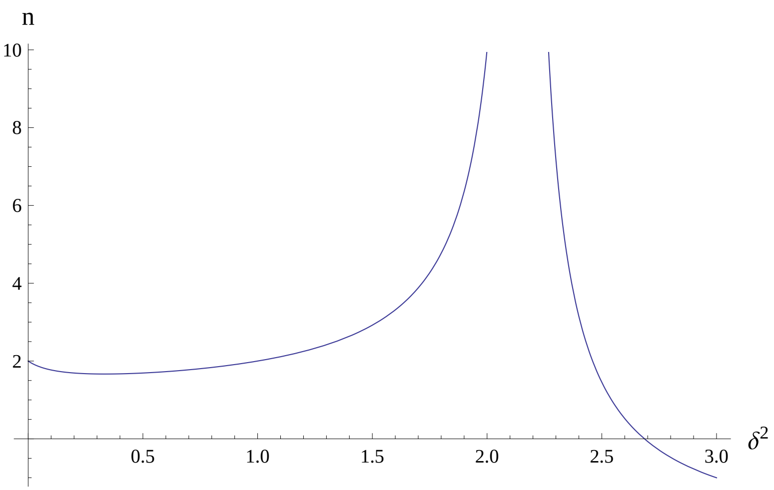


• The maximum temperature as a function of δ^2.

Conductivity of the $\gamma\delta = 1$ solutions

In the first two regimes $0 \leq \delta^2 \leq 1 + \frac{2}{\sqrt{3}}$ the AC conductivity is

$$\sigma(\omega) \simeq \omega^n , \qquad n = \frac{(3-\delta^2)(5\delta^2+1)}{|3\delta^4 - 6\delta^2 - 1|} - 1 \, .$$

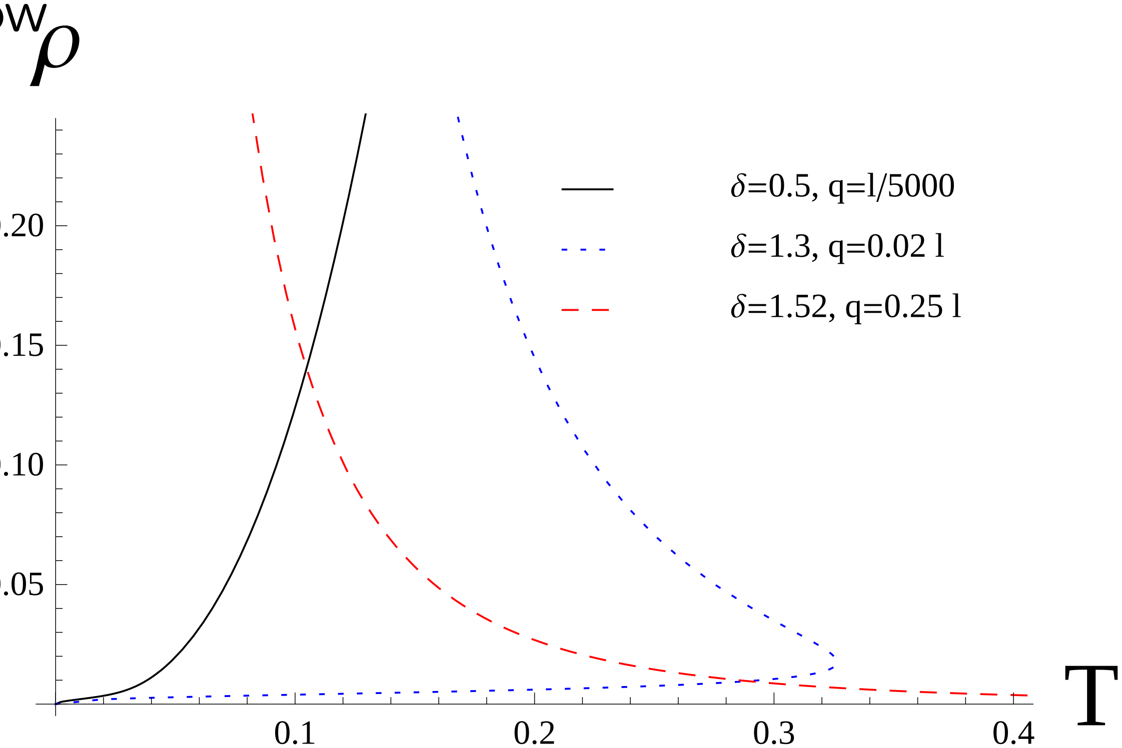


- The exponent is always larger than 5/3 in the region, $0 \leq \delta^2 < 1 + \frac{2}{\sqrt{3}}$ and diverges at $\delta^2 = 1 + \frac{2}{\sqrt{3}}$.

- The system behaves as a conductor.

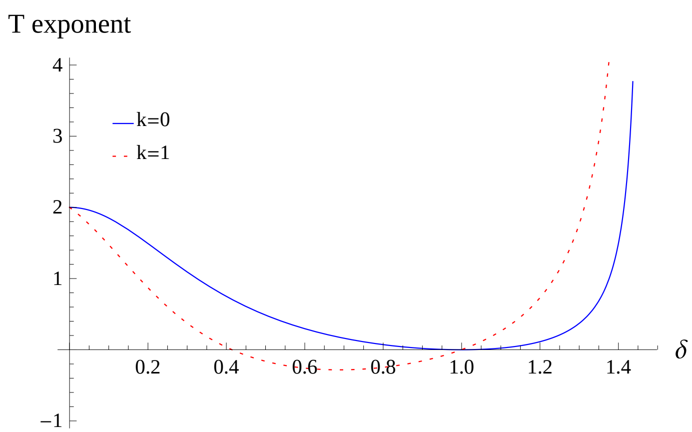
- The system is again conducting for $\frac{1}{4}(5+\sqrt{33}) < \delta^2 < 3$.

The DC resistivity is plotted below
.



The leading behavior at low temperature is

$$\rho_{\text{leading}} \sim \frac{T_f}{J^t}\left(\frac{q}{\ell}\right)^{\frac{2\delta(\delta(3-\delta^2)+(1+\delta^2)k)}{1+6\delta^2-3\delta^4}}\quad(\ell T)^{\frac{2(\delta^2-1)(\delta^2-1+2k\delta)}{1+6\delta^2-3\delta^4}}$$



- It is one at $\delta^2 = 1 + \frac{2}{\sqrt{5}}$.

Detailed plan of the presentation

- Title page 0 minutes
- Bibliography 1 minutes
- The plan 2 minutes
- Introduction 4 minutes
- Effective Holographic Theories 6 minutes
- Einstein-scalar-U(1) theory 8 minutes
- Naked singularities 11 minutes
- Solutions at Zero Charge Density 17 minutes
- The hidden scale invariance 20 minutes
- Charged near extremal solutions 24 minutes
- Hidden scaling at finite density 25 minutes

- Mott-like spectra 26 minutes
- Exact Charged solutions 27 minutes
- Solutions with $\gamma\delta = 1$ 30 minutes
- QC systems with Schröndiger symmetry 35 minutes
- Resistivity at non-zero magnetic field 42 minutes
- Outlook 43 minutes