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The relativistic Langevin dynamics of heavy quark diffusion from Holography

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Collaborators

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Work based on

- U. Gursoy, E. Kiritsis, L. Mazzanti and F. Nitti,
"Langevin diffusion of heavy quarks in non-conformal holographic backgrounds"
[ArXiv:1006.3261][[hep-th]].

and previous work:

- U. Gursoy, E. Kiritsis, G. Michalogiorgakis and F. Nitti,
" Thermal Transport and Drag Force in Improved Holographic QCD"
[ArXiv:0906.1890][[hep-ph]].

Previous work in the context of N=4 sYM:

- J. Casalderrey-Solana and D. Teaney, [arXiv:hep-ph/0605199].
- S. S. Gubser, [arXiv:hep-th/0612143].
- J. Casalderrey-Solana and D. Teaney, [arXiv:hep-th/0701123].
- J. de Boer, V. E. Hubeny, M. Rangamani and M. Shigemori, [ArXiv:0812.5112][hep-th].
- D. T. Son and D. Teaney, [ArXiv:0901.2338][hep-th].
- G. C. Giecold, E. Iancu and A. H. Mueller, [ArXiv:0903.1840][hep-th].

Plan of the presentation

- Introduction
- Langevin Dynamics and Brownian Motion
- Holographic computation of the Langevin diffusion.
- Relevance for RHIC and LHC
- Interesting ramifications
- Outlook

Introduction

- An important class of probes in Heavy ion collisions are heavy quarks.

- They are heavy and relatively easily identifiable from end products.

- They can provide useful "localized" information about the quark-gluon fireball in a heavy-ion collision.

- They have not been so prominent in the initial phases of RHIC due to energy availability.

- They are becoming more prominent recently and they are expected to play a leading role at LHC, complementing the collective observables.

Brownian Motion and Langevin Dynamics

• Heavy particles moving inside a thermal bath undergo Brownian motion: once in a while they collide with fluid particles and suddenly change path.

• This phenomenon has an elegant description in terms of the (local) Langevin equation which in its simplest form is

$$\frac{dp^i(t)}{dt} = -\eta_D^{ij}\, p^j(t) + \xi^i(t) \quad , \quad \langle \xi^i(t) \rangle = 0 \quad , \quad \langle \xi^i(t)\xi^j(t') \rangle = 2\kappa^{ij}\ \delta(t-t')$$

η_D^{ij} is an average "viscous" (dissipative) force

κ^{ij} is the diffusions coefficients.

• Physically both of them have a common origin: the interactions of the heavy probe with the heat-bath.

• The first describes the averaged out (smooth) motion, while the second the (stochastic) fluctuations around the average motion.

• The Langevin equation is a stochastic equation and as such makes sense only in a (time) discretized form (Itô calculus).

The Kramers Equation

• The Brownian motion induced by the Langevin equation can be remodeled as an evolution in phase space

• Let $P(x^i, p^i, t)\ d^3x d^3p$ be the probability of an ensemble of probes. The Langevin evolution translates to

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{E} \cdot \frac{\partial}{\partial \vec{x}}\right) P = \frac{\partial}{\partial p^i}\left(\eta_D^{ij} p^j + \frac{1}{2}\kappa^{ij}\frac{\partial}{\partial p^j}\right) P$$

• The equilibrium distribution in a homogeneous ensemble is expected to satisfy,

$$\frac{\partial}{\partial p^i}\left(\eta_D^{ij} p^j + \frac{1}{2}\frac{\partial}{\partial p^j}\kappa^{ij}\right) P = 0$$

• It will be a (non-relativistic) Boltzmann distribution $P \sim e^{-\frac{E}{T}}$ if the Einstein relation holds

$$\kappa^{ij} = 2MT\ \eta_D^{ij} \quad , \quad E = \frac{\vec{p}^2}{2M}$$

where T is the bath temperature.

Solution of the Langevin Equation

$$\dot{p} = -\eta p + \xi \quad , \quad \langle \xi(t)\xi(t')\rangle = \kappa\delta(t-t')$$

with solution

$$p(t) = p(0)e^{-\eta t} + \int_0^t dt' e^{\eta(t'-t)}\xi(t')$$

$$\langle p(t)\rangle = p(0)e^{-\eta t}$$

$$\langle p(t)^2\rangle - \langle p(t)\rangle^2 = \int_0^t dt' e^{\eta(t'-t)} \int_0^t dt'' e^{\eta(t''-t)} \langle \xi(t)\xi(t')\rangle = \frac{\kappa}{2\eta}\left(1 - e^{-2\eta t}\right)$$

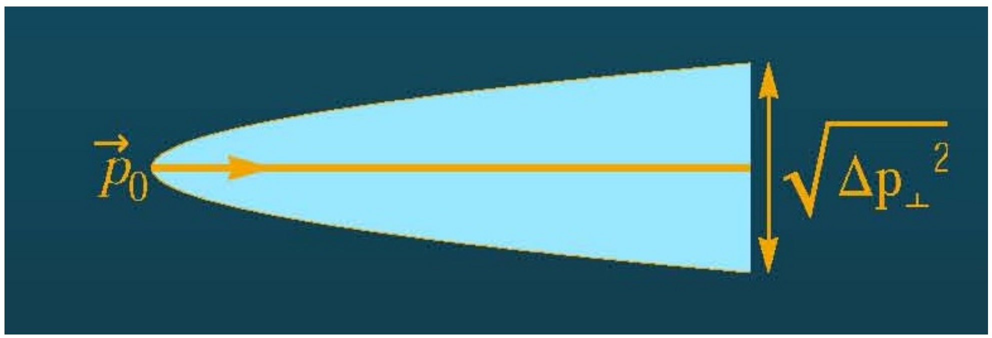
- Long times: $t \gg \frac{1}{\eta}$: $\langle p\rangle \to 0$ and $\langle \Delta p^2\rangle \to \frac{\kappa}{2\eta}$.
- Short times: $t \ll \frac{1}{\eta}$: $\langle p\rangle \simeq p(0)$ and $\langle \Delta p^2\rangle \to \kappa\ t$.

- Consider a multidimensional motion and separate

$$\vec{p} = p^{||} + p^{\perp} \quad , \quad \vec{v} \cdot p^{\perp} = 0$$

- The transverse momentum obeys a Langevin process with (by definition) $\langle p^{\perp} \rangle = 0$ but with an increasing dispersion

$$\langle (\Delta p^{\perp})^2 \rangle \to 2\kappa^{\perp} \ t$$



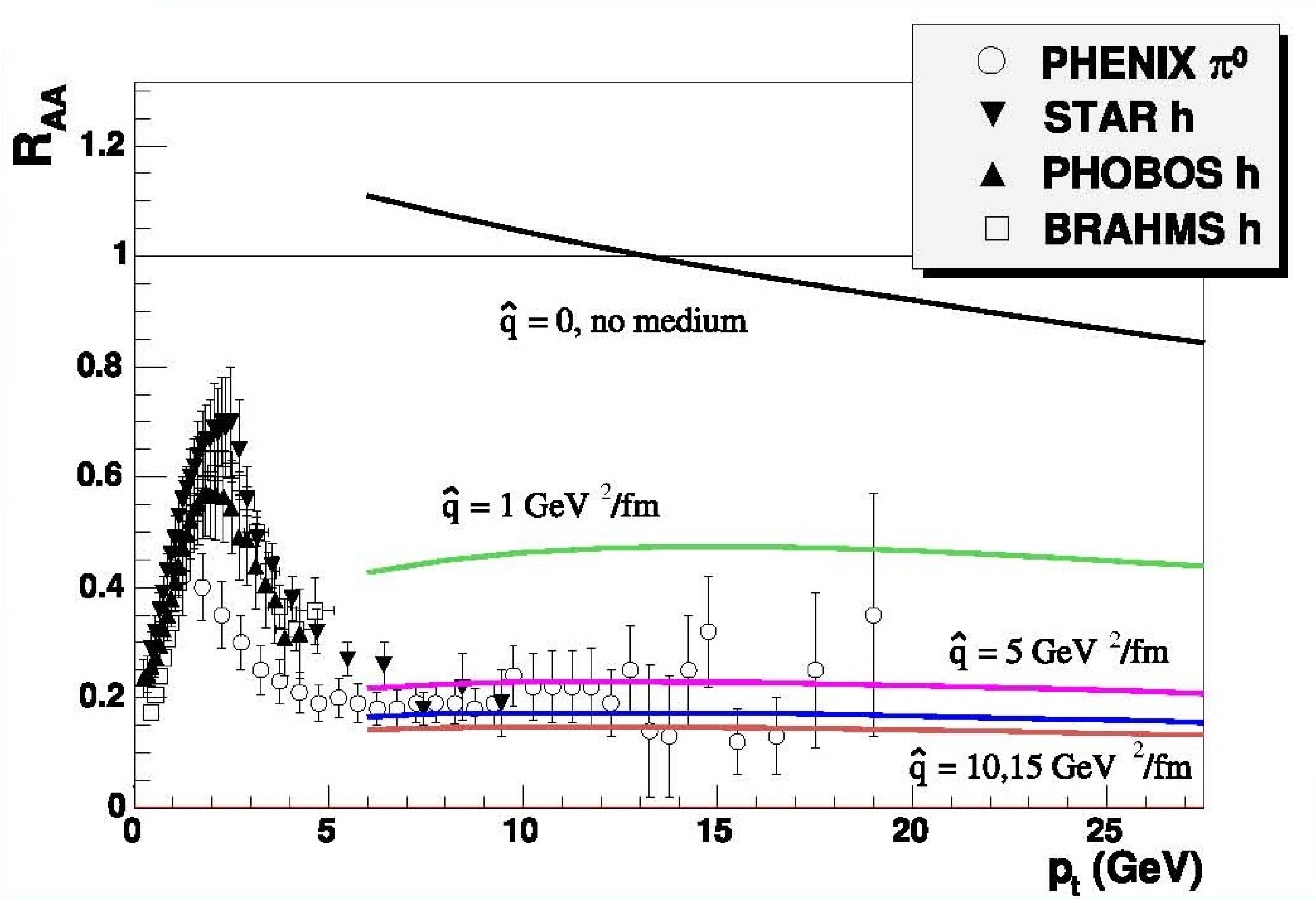
- This defines the "jet quenching parameter"

$$\hat{q} = \frac{\langle (\Delta p^{\perp})^2 \rangle}{vt} = 2\frac{\kappa^{\perp}}{v}$$

- This is a "local" (in time) transport coefficient.

Jet quenching influence

• A non-zero value for the jet quenching parameter for light quarks is essential in explaining the RHIC data. Below we show the nuclear modification factor



Eskola et al. 2005

The generalized Langevin equation

• For our purposes a more general analysis is necessary. We consider the coupling of the coordinates of the probe with the bath degrees of freedom

$$S_{int} = \int d\tau \vec{X}(\tau) \cdot \vec{\mathcal{F}}$$

where $\mathcal{F}^i$ is the force from the heat-bath.

• The generalized Langevin equation in general has memory and reads

$$\dot{P}^i + \int_0^\infty dt' \gamma^{ij}(t') \dot{X}^j(t-t') = \xi^i(t) \ , \ \dot{\gamma}^{ij}(t) = G_R^{ij}(t) \ , \ \langle \xi^i(t) \xi^j(0) \rangle = G_{sym}^{ij}(t)$$

where

$$G_R(t) = -i\theta(t)\langle[\mathcal{F}(t), \mathcal{F}(0)]\rangle \quad , \quad G_A(t) = i\theta(-t)\langle[\mathcal{F}(t), \mathcal{F}(0)]\rangle$$

$$G_{sym}(t) = -\frac{i}{2}\langle\{\mathcal{F}(t), \mathcal{F}(0)\}\rangle \quad , \quad G_{anti-sym}(t) = -\frac{i}{2}\langle[\mathcal{F}(t), \mathcal{F}(0)]\rangle$$

$$\langle T\mathcal{F}(t)\mathcal{F}(0)\rangle \equiv \theta(t)\langle \mathcal{F}(t)\mathcal{F}(0)\rangle + \theta(-t)\langle \mathcal{F}(0)\mathcal{F}(t)\rangle = G_{sym} + \frac{1}{2}(G_R + G_A)$$

• The main goal is to use holography in order to evaluate G_{sym} and G_R for the forces of interest in QGP

The local limit

- For $t \gg t_c$ the autocorrelation time of the force

$$\int_0^\infty dt'\gamma(t')\dot{X}(t-t') \to \eta\dot{X}(t) \quad , \quad \eta = \int_0^\infty dt'\gamma(t')$$

$$G_{sym}(t-t') \to \kappa\delta(t-t') \quad , \quad \kappa = \int_0^\infty dt\ G_{sym}(t)$$

$$\dot{P} + \eta_D P = \xi \quad , \quad \eta_D = \frac{\dot{X}}{P}\eta = \frac{\eta}{\gamma M}$$

- In Fourier space

$$\kappa = G_{sym}(\omega = 0) \quad , \quad \eta = -\lim_{\omega\to 0}\frac{Im\ G_R(\omega)}{\omega}$$

- The relation between G_R and G_R is ensemble-dependent. For a thermal ensemble

$$G_{sym}(\omega) = \coth\left(\frac{\omega}{2T}\right) Im\ G_R(\omega) \quad \Rightarrow \quad \kappa = 2T\eta = 2MT\eta_D$$

we recover the non-relativistic Einstein relation.

The holographic strategy

• To determine the stochastic motion of heavy quarks we must therefore calculate the force correlator in QCD as

$$e^{iS_{eff}} = \langle e^{i\int X\mathcal{F}} \rangle$$

• We will calculate them using a holographic dual.

1. We must identify the force operator $\mathcal{F}$.

2. We must solve the classical equations of motion

3. Calculate the correlators from the boundary on-shell action using the Son-Starinets prescription for the real-time correlators.

The holographic setup

• There is a 5D bulk described by a general 5D black hole with metric (in the string frame)

$$ds^2 = b^2(r)\left[\frac{dr^2}{f(r)} - f(r)dt^2 + d\vec{x}^2\right]$$

• The boundary is at

$$r \to 0 \quad , \quad f \to 1 \quad , \quad b \to \frac{\ell}{r} + \cdots$$

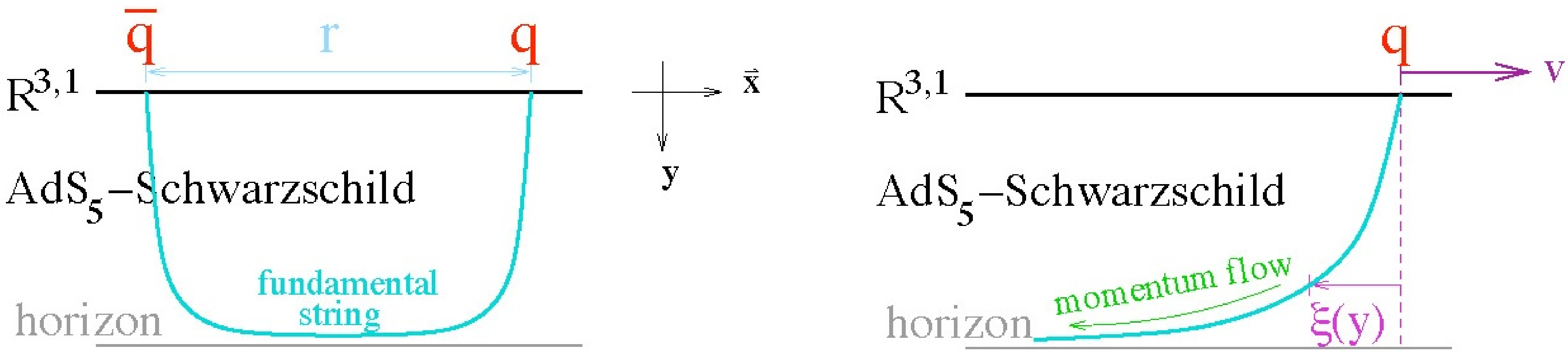
and at the BH horizon

$$r \to r_h \quad , \quad f(r_h) = 0 \quad , \quad 4\pi T = |\dot{f}(r_h)|$$

• This is the holographic description of a general strongly coupled plasma (deconfined phase) in a heat bath.

Classical Heavy quark motion

- A heavy quark is modeled by a string moving in the BH background with (constant) velocity $\vec{v}$.



Herzog+Karch+kovtun+Kozcac+Yaffe, Gubser
Casaldelrrey-Solana+Teaney, Liu+Rajagopal+Wiedeman

- The dynamics of the string is given by the Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi\ell_s^2}\int d^2\xi\sqrt{\det\ \hat{g}} \quad , \quad \hat{g}_{\alpha\beta} = g_{\mu\nu}\ \partial_\alpha X^\mu \partial_\beta X^\nu$$

The drag force

- The classical dragging string solution is

$$X^{\perp} = 0 \quad , \quad X^{||} = vt + \xi(r) \quad , \quad \xi(0) = 0$$

$$\xi'(r) = \frac{C}{f(r)} \sqrt{\frac{f(r) - v^2}{b^4(r) f(r) - C^2}} \quad , \quad f(r_s) = v^2 \quad , \quad C = b^2(r_s) f(r_s)$$

- The "drag" force is in the longitudinal direction

$$\frac{dp^{||}}{dt} = -\frac{b^2(r_s)}{2\pi \ell_s^2} \, v = -\eta_D^{class} p^{||} \quad , \quad \eta_D^{class} = \frac{1}{M\gamma} \frac{b^2(r_s)}{2\pi \ell_s^2} \quad , \quad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

Gursoy+Kiritsis+Michalogiorgakis+Nitti,2009

The world-sheet black hole

- Change coordinates to

$$t = \tau + \zeta(r) \quad , \quad \zeta' = \frac{v\xi'}{f - v^2}$$

and write the induced world-sheet metric as

$$ds^2 = b^2(r)\left[-(f(r) - v^2)d\tau^2 + \frac{b^4(r)}{b^4(r)f(r) - C^2}dr^2\right] \quad ,$$

- This has a (world-sheet) horizon at $r = r_s$.

- It is an asymptotically AdS_2, two-dimensional black-hole.

- The Hawking temperature can be calculated to be

$$T_s = \frac{1}{4\pi}\sqrt{f(r_s)f'(r_s)\left[4\frac{b'(r_s)}{b(r_s)} + \frac{f'(r_s)}{f(r_s)}\right]}$$

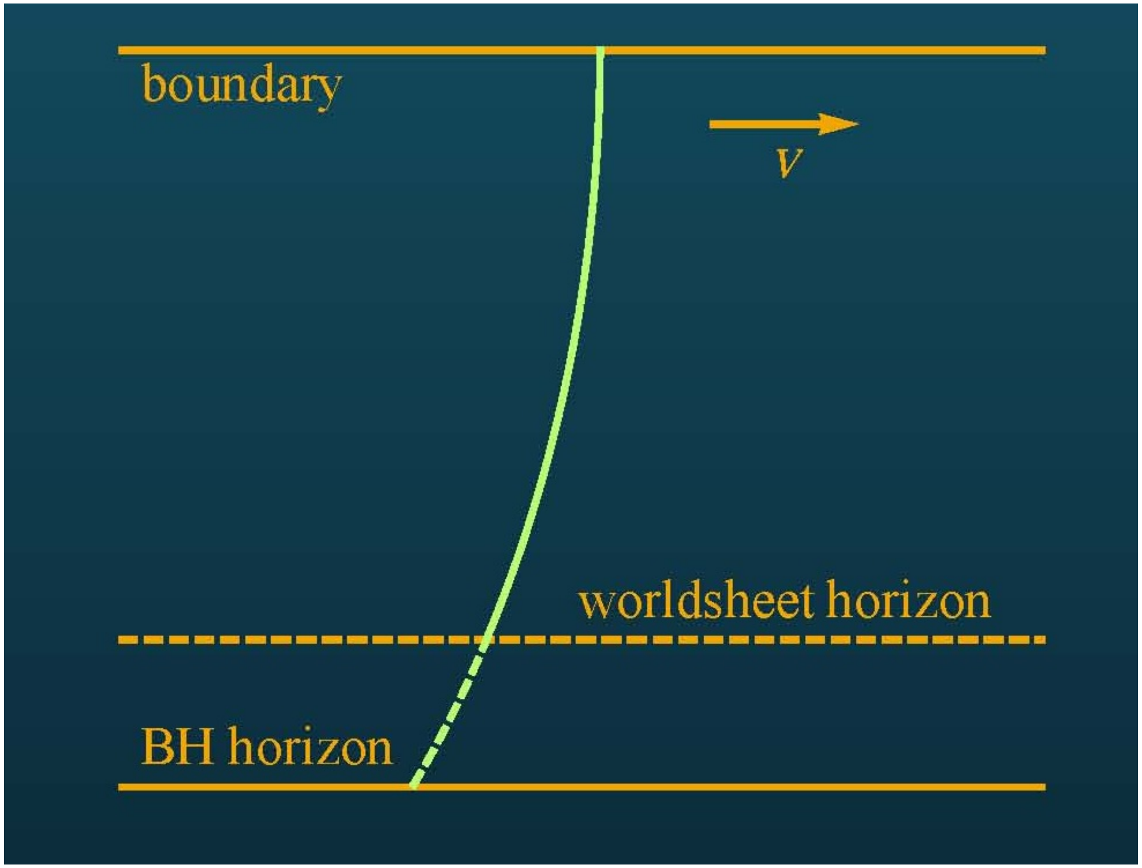
- In general T_s depends on T, Λ, v. In the conformal case, $T_s = \frac{T}{\sqrt{\gamma}}$.

Giecold+Iancu+Mueller, 2009

$$T_s \to T \quad as \quad v \to 0 \quad , \quad T_s \to \frac{T}{\sqrt{\gamma}} \quad as \quad v \to 1$$

- In all examples we analyzed, $T_s \leq T$.

- We always have $0 \leq r_s \leq r_h$. $r_s = 0$ when $v = 1$ and $r_s = r_h$ when $v = 0$.



Fluctuations of the trailing string

• So far we have calculated the average *damped* motion of the trailing string.

• To study the fluctuations we set

$$\vec{X}(r,t) = (vt + \xi(r))\frac{\vec{v}}{v} + \delta\vec{X}(r,t)$$

• From the boundary coupling

$$S_{bdr} = \int dt\ X_i(t)\ \mathcal{F}^i(t) \simeq S^0_{bdr} + \int dt\ \delta X_i(t)\ \mathcal{F}^i(t)$$

• Correlators of $\mathcal{F}$ in the dual QFT are given by holographic correlators of $\delta X_i(t)$ in the bulk string theory.

• They can be obtained according to the standard holographic prescriptions by solving the second order fluctuation equations for $\delta X_i(t)$.

• For the retarded correlator we must impose "incoming" boundary conditions at the world-sheet horizon and unit normalization at the boundary.

• Introducing the Fourier modes of fluctuations

$$\delta\vec{X}(r,t) = e^{i\omega\tau}\delta\vec{X}(r,\omega)$$

the second-order radial equations are of the form

$$\partial_r\left[\sqrt{(f-v^2)(b^4 f - C^2)}\;\partial_r\left(\delta X^{\perp}\right)\right] + \frac{\omega^2 b^4}{\sqrt{(f-v^2)(b^4 f - C^2)}}\,\delta X^{\perp} = 0$$

$$\partial_r\left[\frac{1}{Z^2}\sqrt{(f-v^2)(b^4 f - C^2)}\;\partial_r\left(\delta X^{\parallel}\right)\right] + \frac{\omega^2 b^4}{Z^2\sqrt{(f-v^2)(b^4 f - C^2)}}\delta X^{\parallel} = 0$$

$$Z \equiv b^2\sqrt{\frac{f-v^2}{b^4 f - C^2}}.$$

• They are different for longitudinal and transverse fluctuations.

The diffusion constants

• From direct calculation of the IR asymptics of fluctuation correlators we obtain

$$\kappa^{\perp} = \frac{b^2(r_s)}{\pi \ell_s^2} T_s \quad , \quad \kappa^{||} = \frac{b^2(r_s)}{\pi \ell_s^2} \frac{(4\pi)^2}{f'(r_s)^2} T_s^3$$

• We also obtain the relation

$$G^i_{sym}(\omega) = \coth\left(\frac{\omega}{2T_s}\right) G^i_R(\omega)$$

and therefore the temperature entering the fluctuation-dissipation relations is T_s.

• Because the diffusion and friction coefficients are generically momentum dependent there are non-trivial relations between Langevin equations for momenta and position fluctuations.

$$\dot{\vec{p}} = -\eta_D^{||}\ p^{||}\ \hat{v} - \eta_D^{\perp}\ p^{\perp} + \vec{\xi}(t)$$

In configuration space (where all of this is calculated)

$$\gamma M \delta \ddot{X}^{\perp} = -\eta^{\perp} \delta \dot{X}^{\perp} + \xi^{\perp} \quad , \quad \gamma^3 M \delta \ddot{X}^{||} = -\eta^{||} \delta \dot{X}^{||} + \xi^{||}$$

$$\eta^{\perp} = \frac{1}{\gamma M} \eta_D^{\perp} \quad , \quad \eta^{||} = \frac{1}{\gamma^3 M} \left[\eta_D^{||} + \gamma M v \frac{\partial \eta_D^{||}}{\partial p} \right]$$

- We have computed holographically

$$\eta^{||,\perp} = \frac{\kappa^{||,\perp}}{2T_s}$$

which lead to the modified Einstein relations

$$\kappa^{\perp} = 2\gamma M T_s\ \eta_D^{\perp} = 2 E T_s\ \eta_D^{\perp} \quad , \quad \kappa^{||} = 2\gamma^3 M T_s \left[\eta_D^{||} + \gamma M v \frac{\partial \eta_D^{||}}{\partial p} \right]$$

to be compared with the standard one $\kappa = 2MT\eta_D$.

• Consistency check:

$$\eta_D^{||} = \eta_D^{\perp} = \frac{b^2(r_s)}{M\gamma(2\pi\ell_s^2)}$$

satisfies both Einstein relations.

• This type of relativistic Langevin evolution is different from what has been described so far in the mathematical physics literature.

Debasch+Mallick+Ribet, 1997

• The diffusion constants satisfy the general inequality (in the deconfined phase)

$$\frac{\kappa_{||}}{\kappa_{\perp}} = \left(\frac{4\pi T_s}{f'(r_s)}\right)^2 = 1 + 4v^2 \; \frac{b'(r_s)}{f'(r_s)b(r_s)} \geq 1$$

equality is attained at $v = 0$.

• For systems similar to QCD, the WKB approximation valid for large ω seems to be valid down to very low frequencies, providing an analytical control over the Langevin correlators.

Validity of locality approximation

- The validity of the local approximation demands that

$$t \gg t_{correlation} \sim \frac{1}{T_s}$$

- For $(\Delta p^{\perp})^2$ to be characterized by $\kappa^{\perp}$ we must have

$$t \ll t_{\text{relaxation}} \sim \frac{1}{\eta_D}$$

Therefore we need

$$\frac{1}{\eta_D} \ll \frac{1}{T_s}$$

If this fails we need the full non-local (in time) Langevin evolution.

This translates into an upper bound for the momentum ultra-relativistic quarks of the form

$$p \ll \frac{1}{4}\left(\frac{\ell_s}{\ell}\right)^4 \frac{M_q^3}{T^2}\lambda_s^{-8/3}.$$

Calculations in Improved Holographic QCD

- This is Einstein-dilaton gravity with

$$S = M^3 N_c^2 \int d^5x \sqrt{g} \left[R - \frac{4}{3}\frac{\partial\lambda^2}{\lambda^2} - V(\lambda) \right]$$

- λ is approximately the QCD 't Hooft coupling

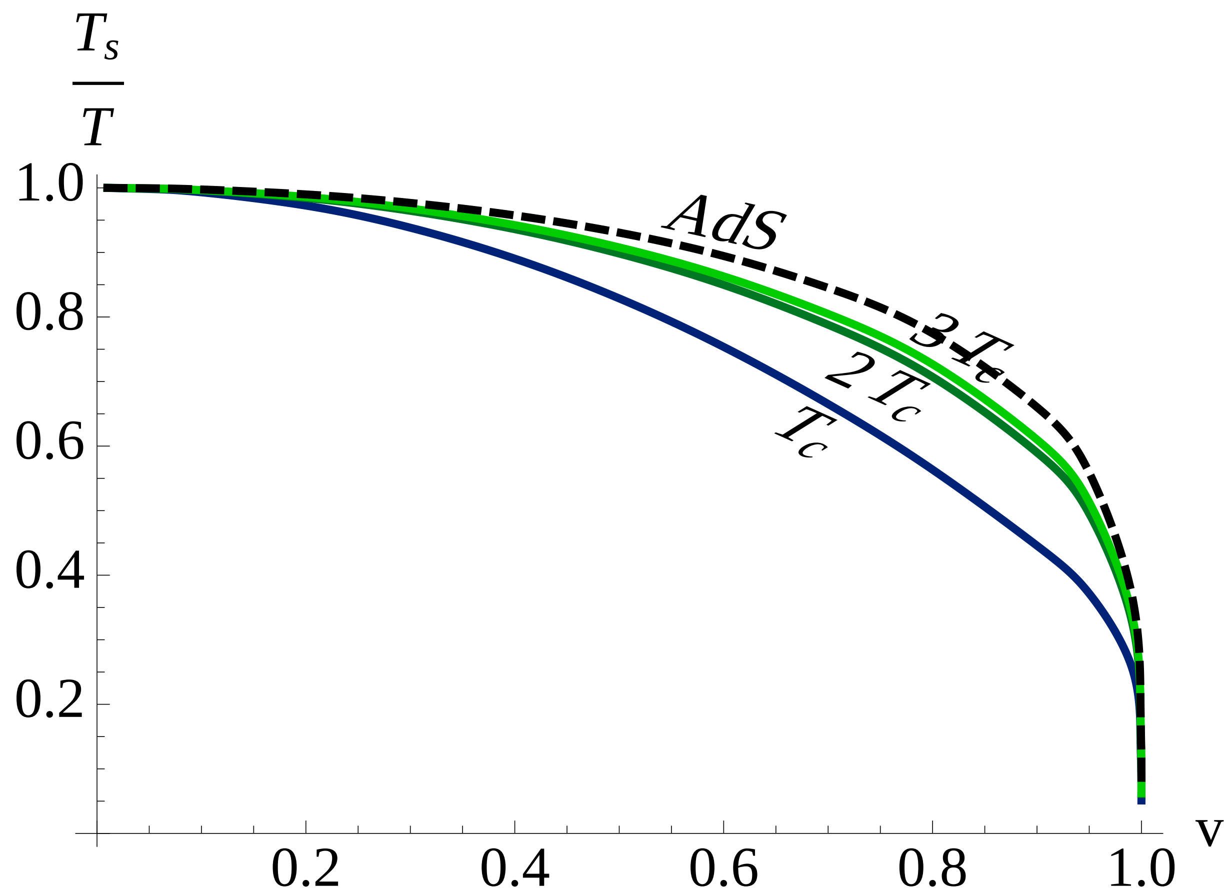
- $V(\lambda) = \frac{12}{\ell^2}\left[1 + c_1\lambda + c_2\lambda^2 + \cdots\right] \quad , \quad \lambda \to 0$

- $V(\lambda) \sim \lambda^{\frac{4}{3}}\sqrt{\log\lambda} \quad , \quad \lambda \to \infty$

- It agrees well with pure YM, both a zero and finite temperature.

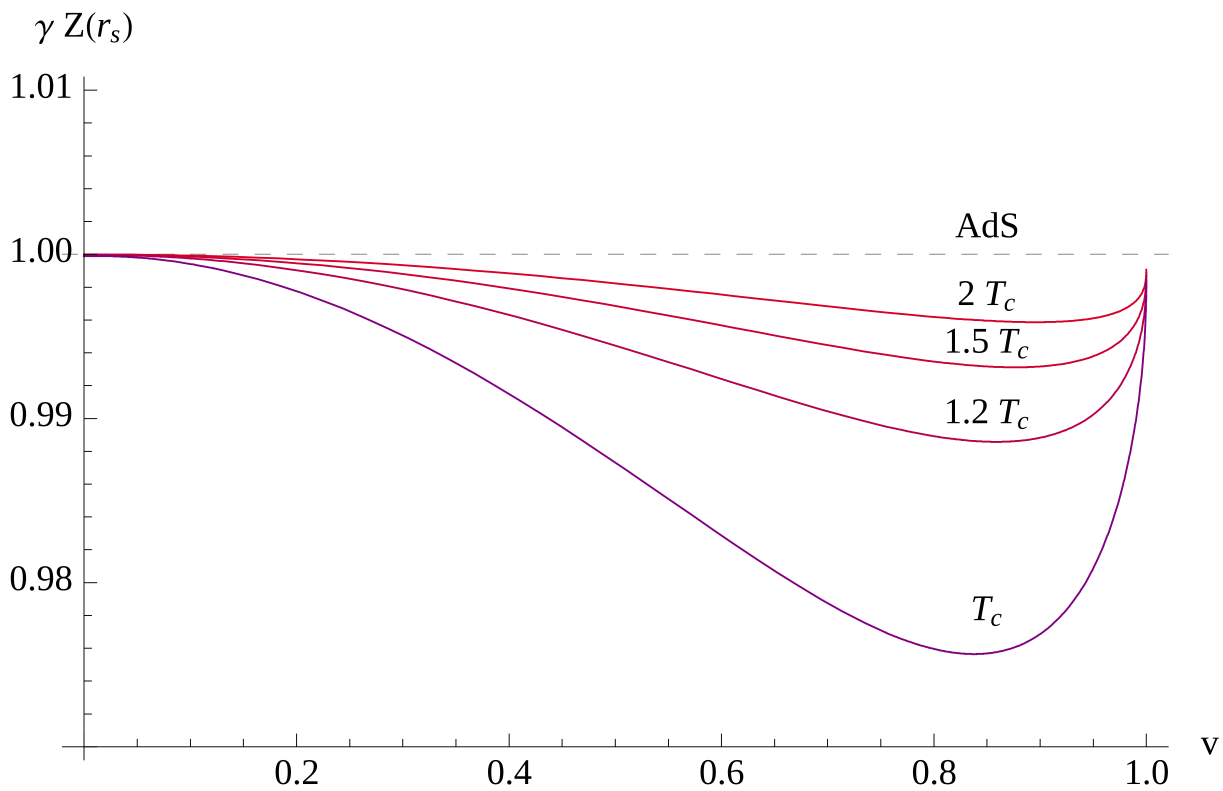
Gursoy+Kiritsis+Mazzanti+Nitti, 2007-2009

World-Sheet Hawking temperature



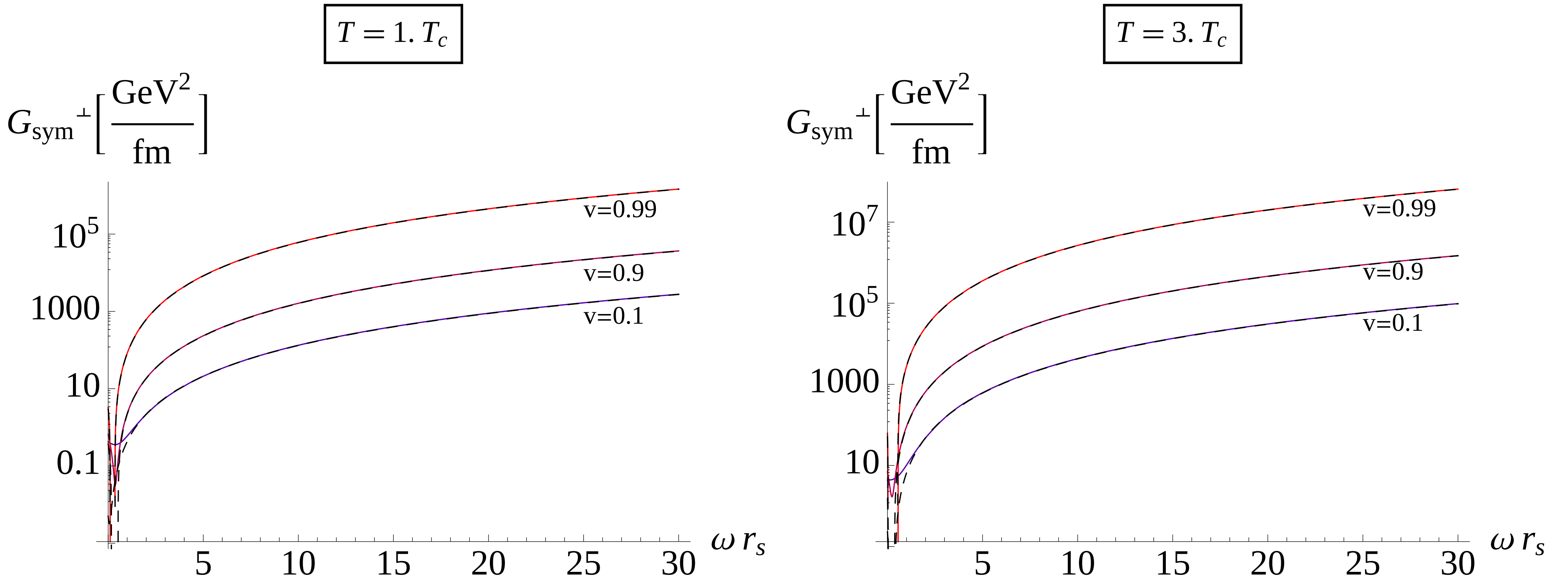
The ratio of the world-sheet temperature to the bulk black hole temperature, as a function of velocity, for different values of the bulk temperature. The dashed line indicates the AdS-Schwarzschild curve, $T_s = T/\sqrt{\gamma}$.

Asymmetry factor (Z)



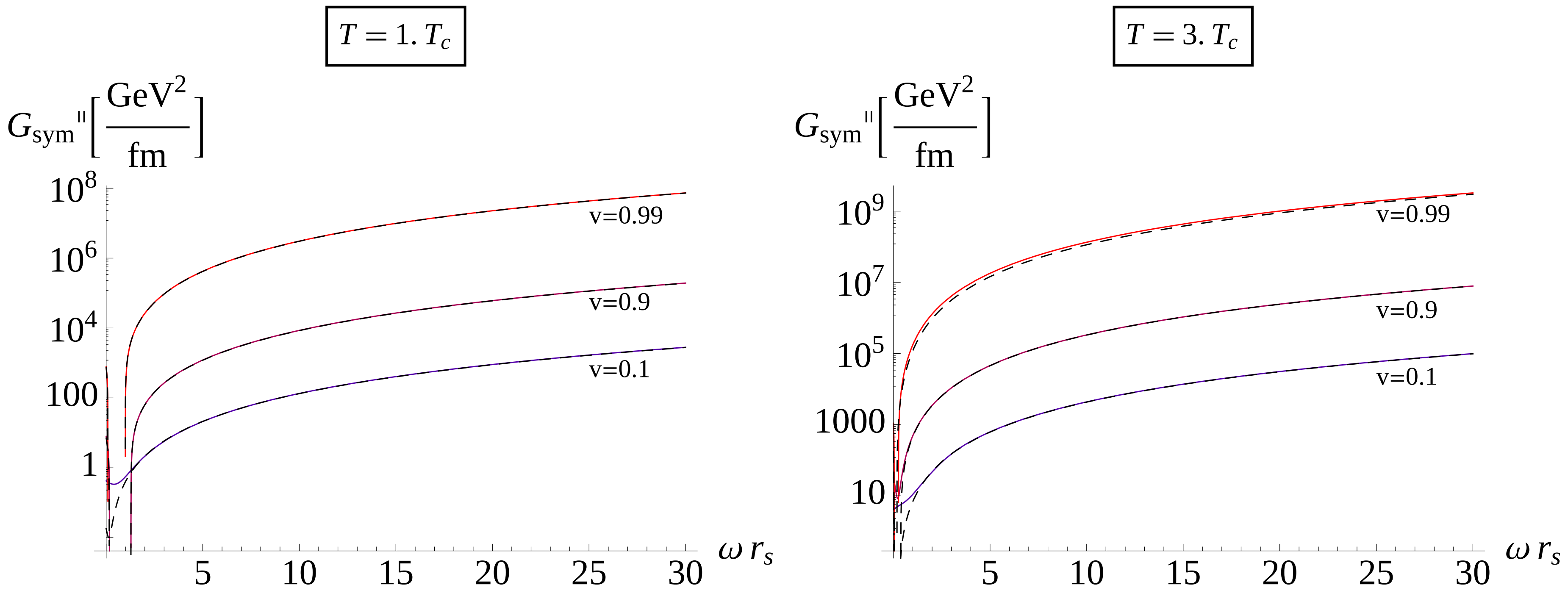
The function $\gamma Z(r_s)$ as a function of velocity, ($\gamma \equiv 1/\sqrt{1-v^2}$), computed numerically varying the velocity, at different temperatures. The dashed line represents the conformal limit, in which $\gamma Z = 1$ exactly.

Symmetric Transverse, Langevin Correlator

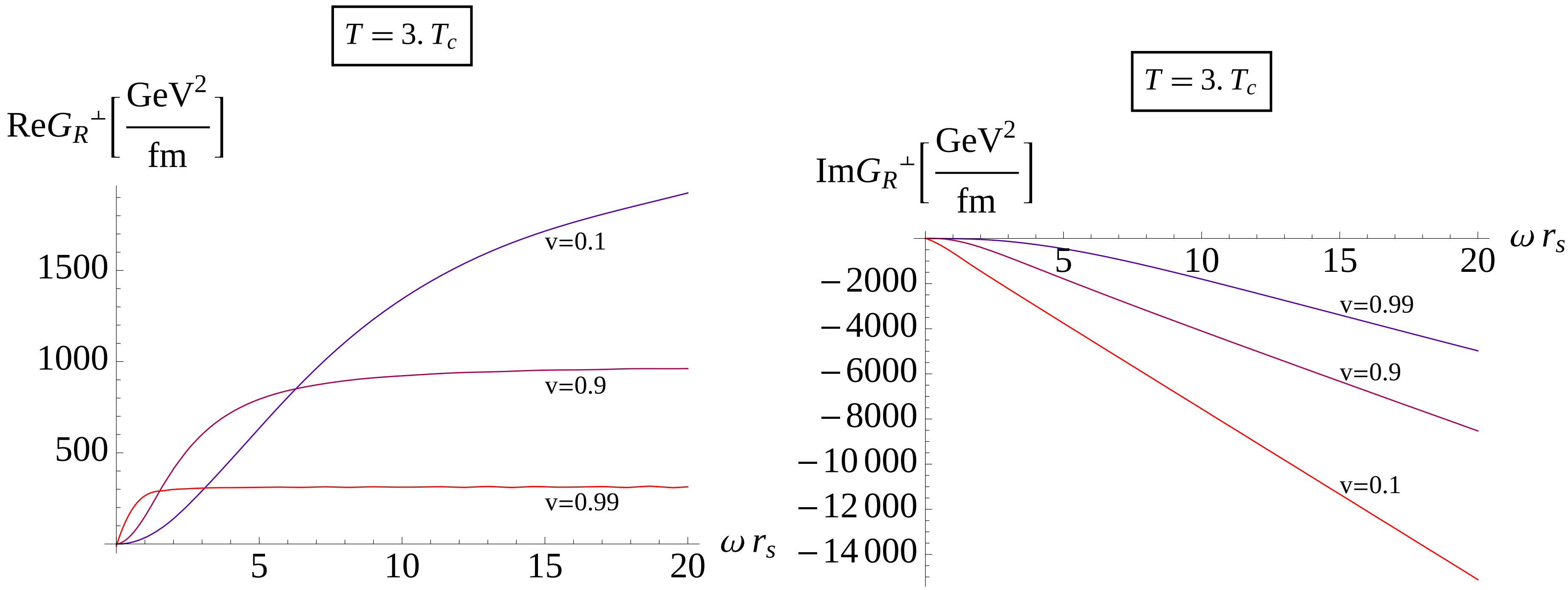


The symmetric correlator of the $\perp$ modes by the numerical evaluation (solid line) in the $M_q \to \infty$ limit. We show in each plot the curves corresponding to the velocities $v = 0.1, 0.9, 0.99$ and different plots for the temperatures $T = T_c, 3T_c$.

Symmetric Longitudinal Langevin Correlator

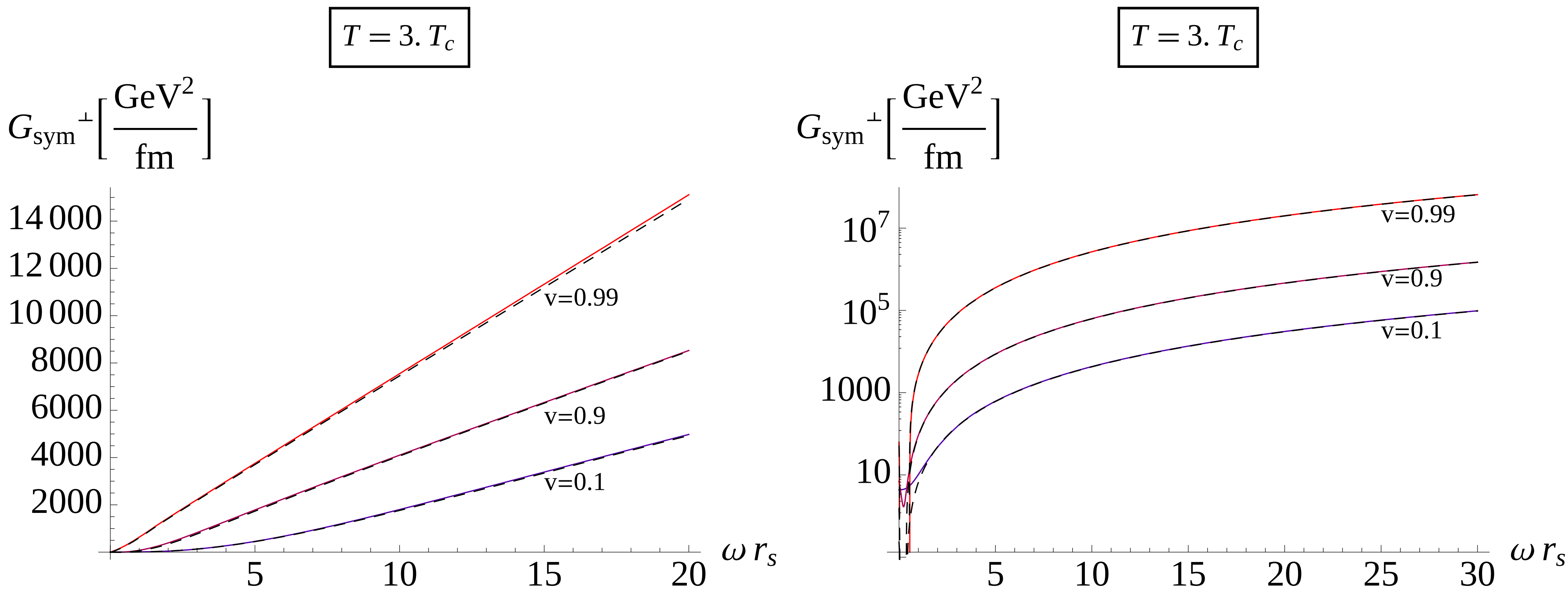


Retarded correlators for finite mass quarks

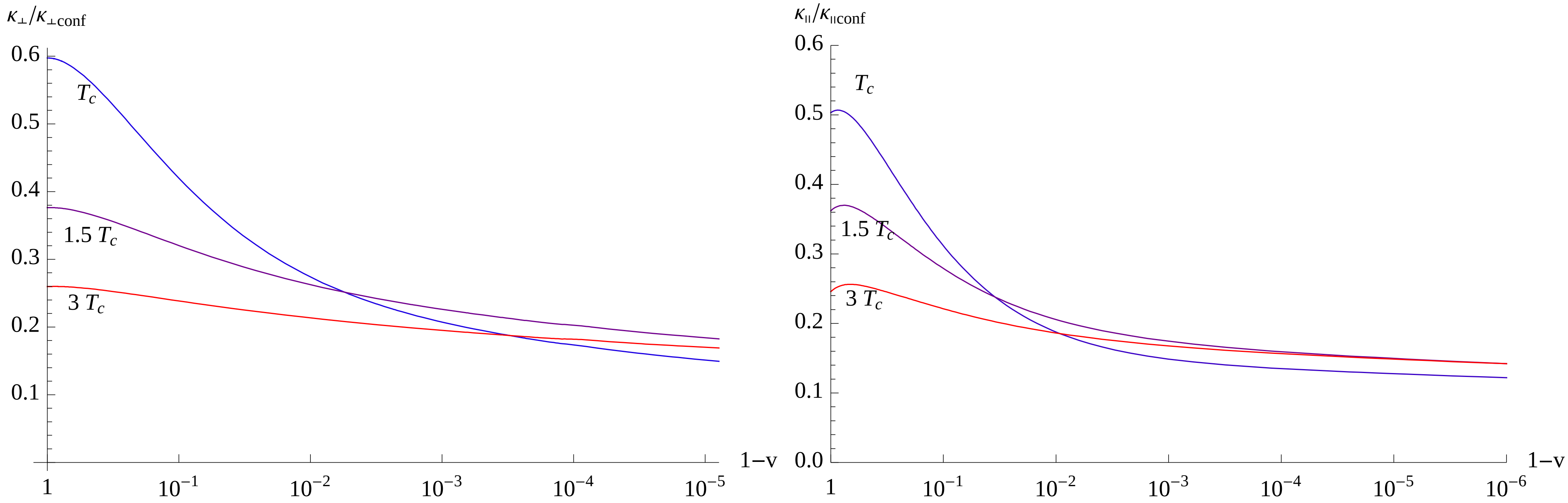


The retarded correlator real and imaginary part for finite but large quark mass, calculated numerically. The mass is chosen equal to that of charm.

Symmetric correlator for finite mass quarks



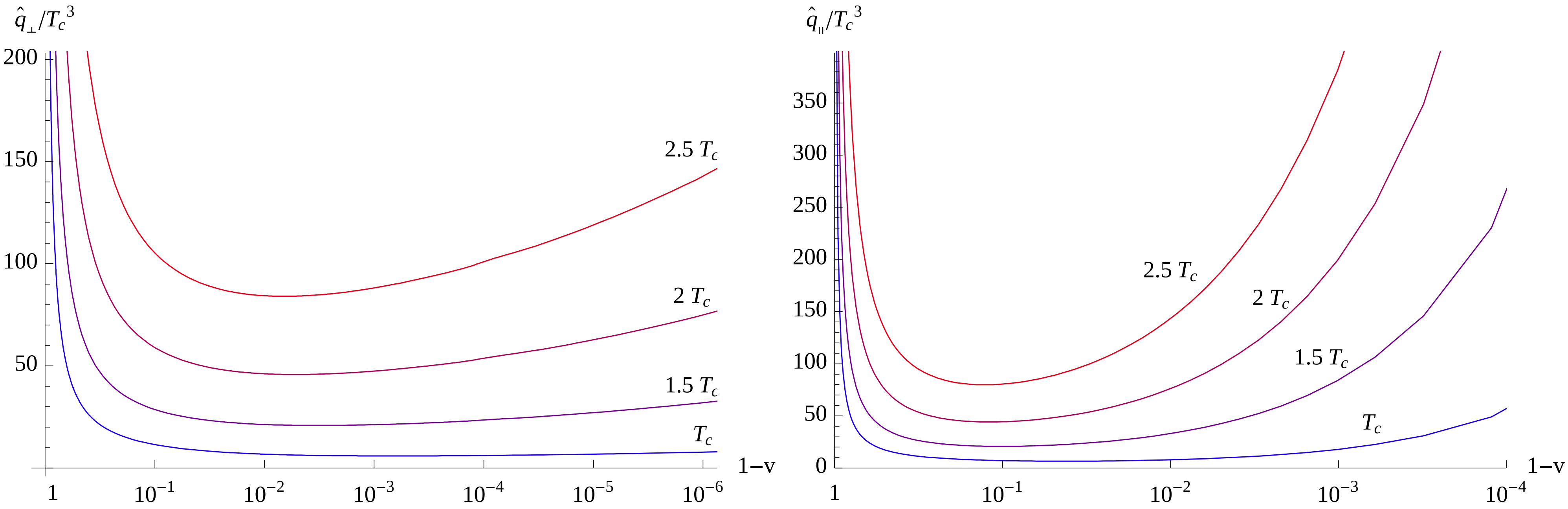
Comparison with N=4



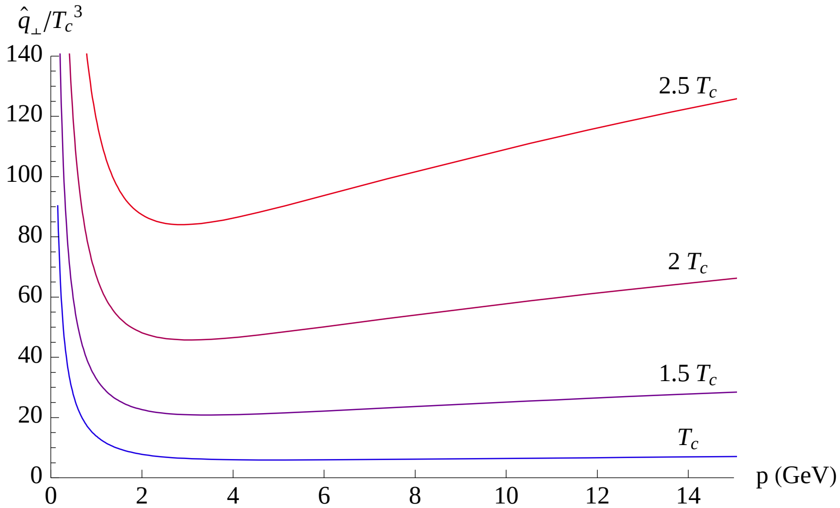
The ratio of the diffusion coefficients $\kappa_\perp$ and $\kappa_{||}$ to the corresponding value in the holographic conformal $\mathcal{N} = 4$ theory (with $\lambda_{\mathcal{N}=4} = 5.5$) are plotted as a function of the velocity v (in logarithmic horizontal scale) The results are evaluated at different temperatures $T = T_c, 1.5T_c, 3T_c$ in the deconfined phase of the non-conformal model.

• If we choose $\lambda = 0.5$ instead of $\lambda = 5.5$ in the conformal case then our result agrees with the conformal result within the 10% level, in the range $v > 0.6$ and for $T > 1.5T_c$.

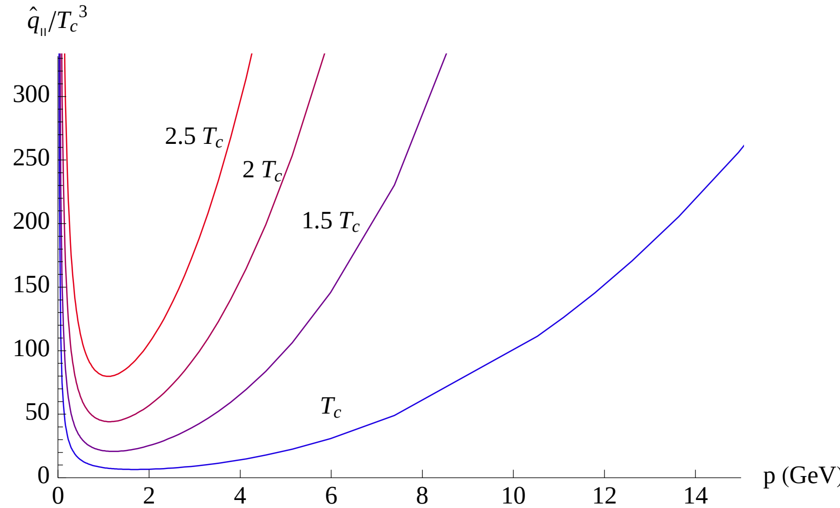
Jet quenching parameters



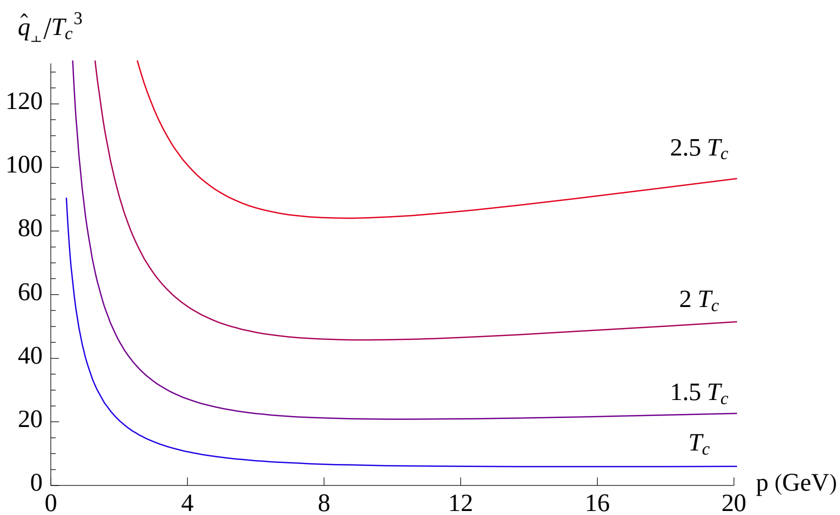
The jet-quenching parameters $\hat{q}_{\perp}$ and $\hat{q}_{||}$ obtained from the diffusion constants $\kappa_{\perp}$ and $\kappa_{||}$, normalized to the critical temperature T_c, are plotted as a function of the velocity v (in a logarithmic horizontal scale). The results are evaluated at different temperatures.



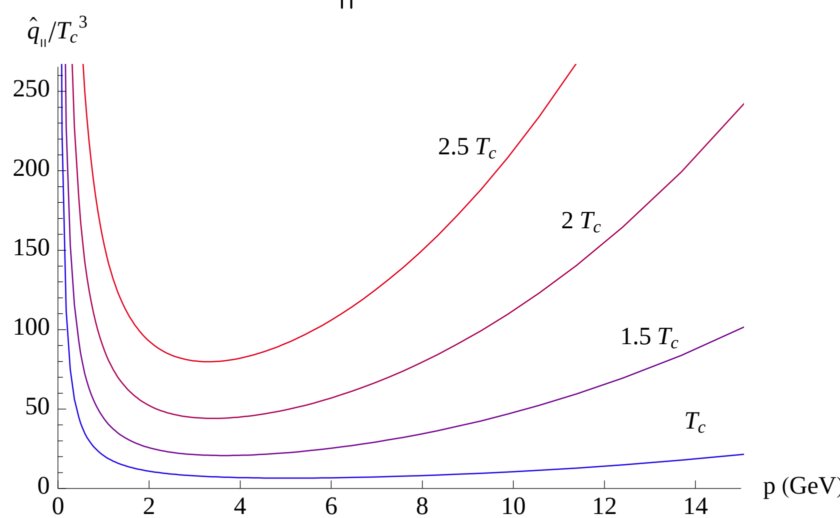
$\hat{q}_\perp$ charm



$\hat{q}_{||}$ charm

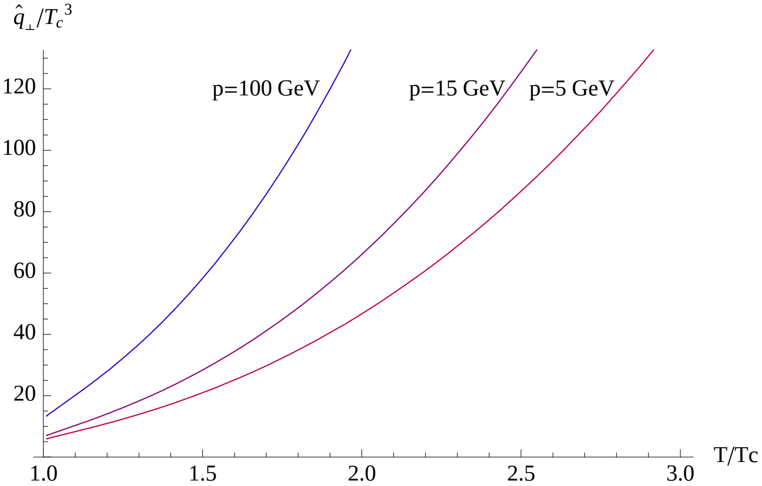


$\hat{q}_\perp$ bottom

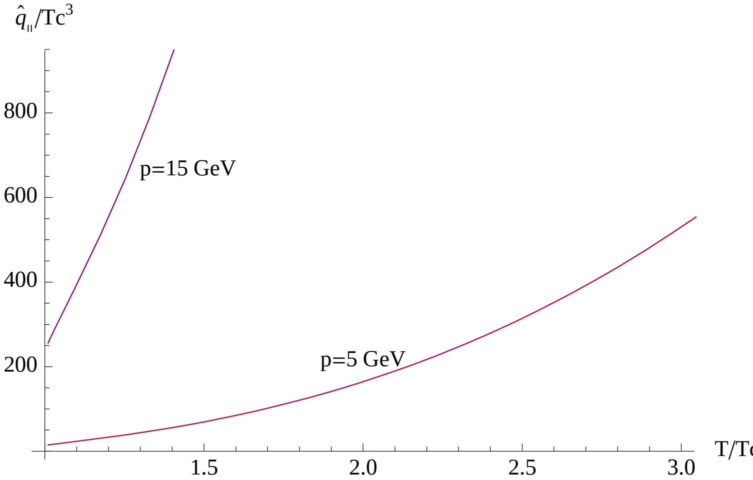


$\hat{q}_{||}$ bottom

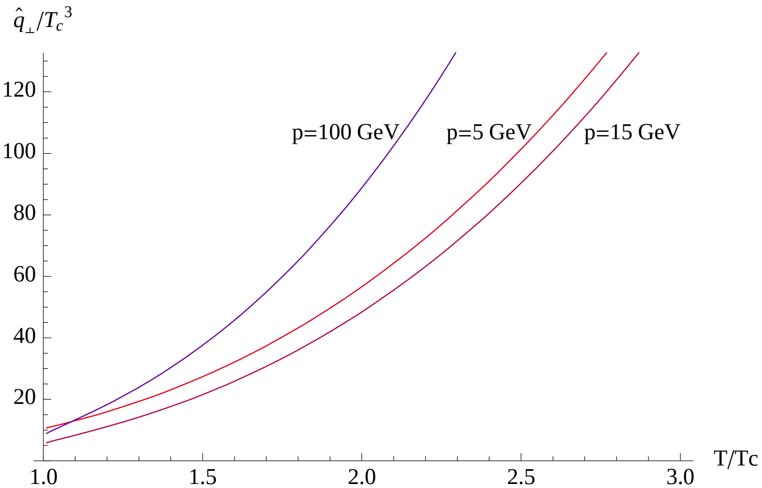
The quantities $\hat{q}_\perp/T_c^3$ and $\hat{q}_{||}/T_c^3$ plotted as a function of the quark momentum p. The plots for the charm and the bottom quark differ by a scaling of the horizontal direction.



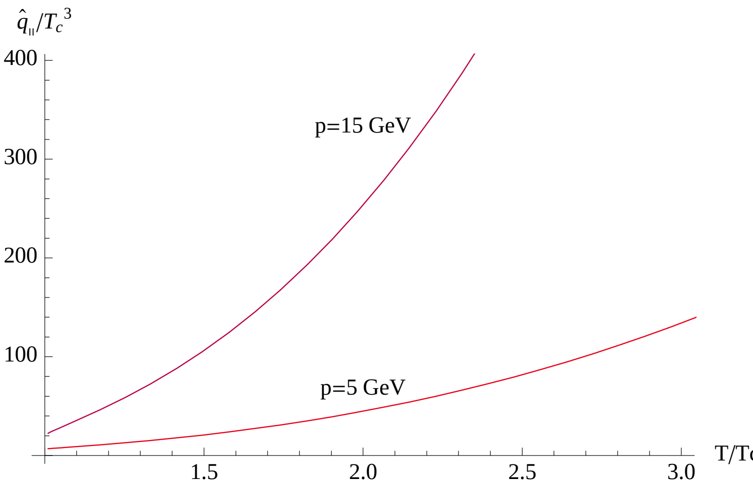
$\widehat{q}_\perp$ charm



$\widehat{q}_{||}$ charm



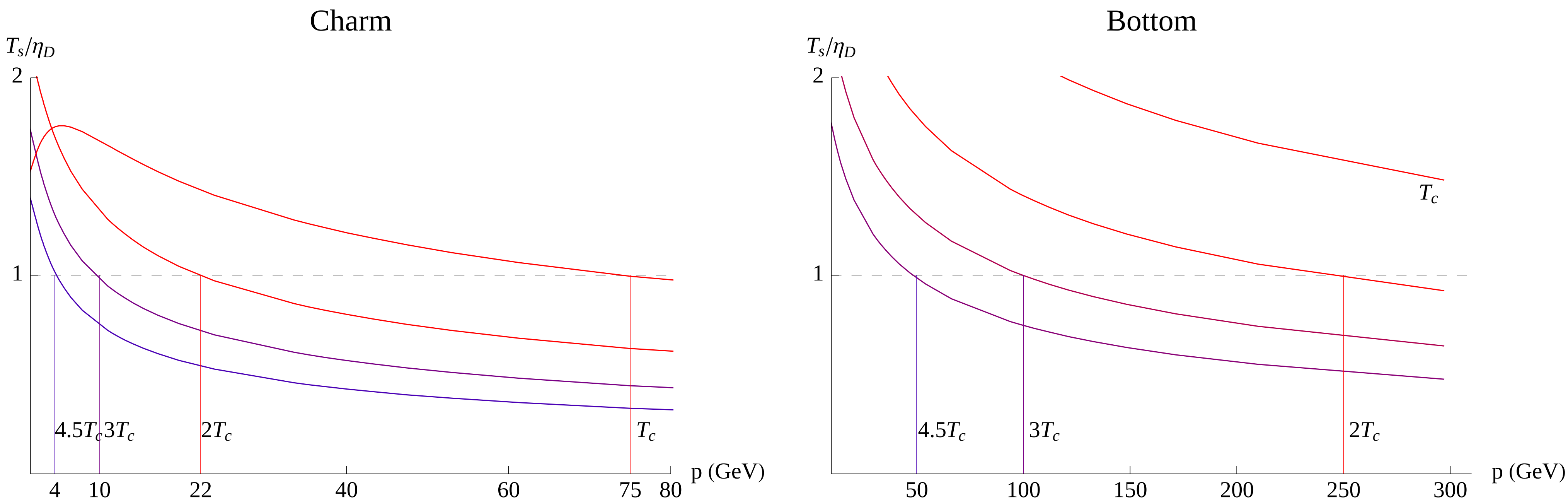
$\widehat{q}_\perp$ bottom



$\widehat{q}_{||}$ bottom

The jet-quenching parameters $\widehat{q}_\perp$ and $\widehat{q}_{||}$ plotted as a function of T/T_c, for different quark momenta.

Locality of Langevin evolution



(a) (b)

The quantity T_s/η_D is plotted against quark momentum, for different bulk temperatures. Figures refer to the charm and bottom quark, respectively. For each temperature, the validity of the local Langevin equation constrains p to the left of the corresponding vertical line, which marks the transition of T_s/η_D across unity.

Systematic uncertainties and approximations

- Large-N_c limit
- Lack of a first principles string theory dual for YM
- Not included light quark degrees of freedom in plasma
- Finite mass corrections (may be relevant for charm)
- Extra (quantum) fermionic degrees of freedom.

General considerations

• It has been observed in some systems with complicated dynamics, that when they are gently stirred in contact with a heat bath, they reach equilibrium at a temperature $T_s > T$.

Cugliandolo+Kurchan+Peliti, 1997

• This is what we have shown to happen to all strongly-coupled holographic systems, with the difference that always here $T_s < T$.

• The following question is a hundred years-old and unsettled: "What are the Lorentz transformation properties of temperature?"

• Our analysis suggest that

1. A heavy quark probe moving in a plasma acts as a moving thermometer.

2. It measures temperature via the fluctuation-dissipation relation.

3. The temperature it measures is $T_s(v, T, ...)$. Its dependence on temperature and velocity is simple in conformal systems ($T_s = T/\sqrt{\gamma}$) but more complicated in non-conformal systems, and depends in particular on the dynamical mass scales.

Outlook

- The Langevin diffusion of heavy quarks in the QGP may be an interesting observable that will provide extra clues for the dynamics in the deconfined phase.

- The relativistic Langevin dynamics expected from QCD is providing a novel paradigm for asymmetric evolution that is captured by holographic techniques.

- The jet quenching transport coefficients may be calculated and provide important input for the evolution of heavy quarks.

- They thermalize at a temperature dictated by a world-sheet black hole and is distinct from the plasma temperature.

- The local Langevin evolution breaks down for lighter quarks rather early and full correlators are needed. These are captured by a WKB analysis.

- A detailed simulation done recently (Akamatsu+Hatsuda+Hirano) used a different Langevin evolution that is not in accord with the one derived via holography

- A new simulation seems necessary in order to test (qualitatively at least the holographic templates and predictions.

- The holographic calculations have room for improvement, most importantly by including the fundamental degrees of freedom in the plasma.

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Thank you for your Patience

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The large-N_c expansion in QCD

• The generalization of QCD to N_c colors, has an extra parameter: the theory simplifies in a sense when $N_c \to \infty$.

t 'Hooft 1974

• It has the structure of a string theory, with $g_s \sim \frac{1}{N_c}$. When $N_c = \infty$ the theory contains an infinite number of particles with finite masses and no interactions. The "string" is the "flux tube" of confined color flux that binds quarks and glue together.

• Therefore, at $N_c = \infty$ the theory is "free".

• The particles are color singlets (glueballs, mesons and baryons).

• It is therefore a good starting point for a perturbative expansion in $\frac{1}{N_c}$.

• There is always the usual coupling constant: $\lambda \equiv a_s N_c$.

- it turns out that $N_c = 3$ is not that far from $N_c = \infty$

Alas, even the leading order in QCD (classical at large N_c) is not easy to compute.

- If $\lambda << 1$ we compute in perturbation theory

- This is not the case in QCD at low energy.

AdS/CFT correspondence and holography

• A new twist to the large-N_c expansion was added from standard string theory.

Maldacena 1997

• It involved a cousin theory to QCD: $\mathcal{N}$=4 sYM theory. This is a scale invariant theory: the t 'Hooft coupling λ does not run.

• It is claimed to be equivalent to a ten-dimensional (IIB) string theory propagating on a curved space $AdS_5 \times S^5$

♠ At strong coupling $\lambda \to \infty$ the string is stiff, therefore we can approximate it with a point-particle, $\to$ (super)-gravity approximation.

• we obtain a duality: (a) at $\lambda \to 0$ perturbative description in terms of gauge theory (b) at $\lambda \to \infty$ perturbative description in terms of supergravity

Holography in Anti-de-Sitter space

- AdS_5= maximally symmetric, with negative curvature

- A space with a "radial" direction, where each slice $r = constant$ is a Minkowski$_4$ space.

- The radial direction can be thought of as an RG scale ($r \sim \frac{1}{E}$): r=0 (boundary) is the UV, while $r = \infty$ is the IR.

- It has a single boundary at $r = 0$.

- The gravity fields are "dual" to sYM operators:$g_{\mu\nu} \sim T_{\mu\nu}$, $\phi \sim Tr[F^2]$ etc. One can think of them as "composites".

- The string theory effective action is capturing the dynamics of such "composites"

- Closed strings generate the glueballs. Open strings the mesons. Baryons are more complicated (solitons).

• There have been many non-trivial tests of AdS/CFT correspondence

• The gravity approximation is a (important) bonus because we cannot solve (yet) such string theories.

• But $\mathcal{N} = 4$ sYM is not QCD. How can we describe QCD?

• The problem is the weak coupling in the UV

♠ One can add a "phenomenological twist": write a (gravity) theory that has the features of QCD and is motivated from holography/string theory.

♠ The simplest model is known as AdS/QCD: its AdS space with an IR cutoff: its advantage is that is simple. The flip-side is that it has no real dynamics and the coupling does not run.

Polchinski+Strassler 2001, Erlich+Katz+Son+Stephanov 2005, DaRold+Pomarol 2005

• The state of the art: Improved Holographic QCD

Gursoy+Kiritsis+Nitti 2007

Schwinger-Keldysh derivation

Consider a system with degrees of freedom $\{ Q \}$ and density matrix $\rho(Q, Q', t)$. that evolves as

$$\rho(Q_f, Q'_f, t) = U(Q_f, Q_0, t, t_0)\rho(Q_0, Q'_0, t_0)U^\dagger(Q'_f, Q'_0, t, t_0)$$

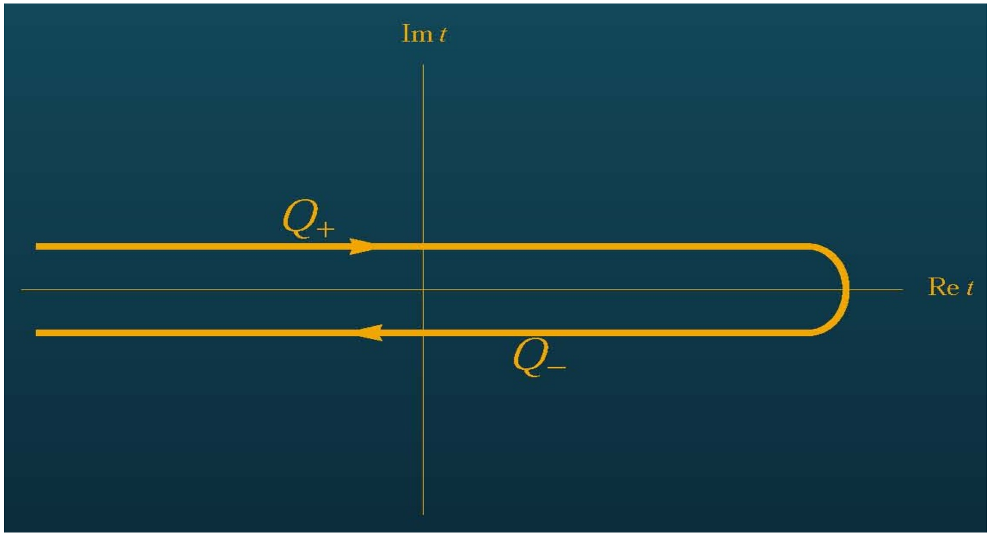
where the evolution operator is given by the path integral

$$U(Q_f, Q_0, t, t_0) = \int DQ\ e^{iS(Q)} = \int DQ\ e^{i\int_{t_0}^{t}\ L(Q,\dot{Q})} \quad , \quad Q(t_0) = Q_0 \quad , \quad Q(t) = Q_f$$

Therefore the density matrix is a double path integral

$$\rho(Q_f, Q'_f, t) = \int DQ \int DQ'\ e^{i(S(Q)-S(Q'))}\rho(Q_0, Q'_0, t_0)$$

It is natural to double the fields, call $Q = Q_+$, $Q' = Q_-$, and consider $Q_\pm$ the values of Q on the double (Keldysh) contour



We now consider a single particle described by $X(t)$, and a statistical ensemble, described by a QFT with degrees of freedom $\Phi(x,t)$. We assume a linear interaction between X and some functional $\mathcal{F}(t)$ of the QFT fields Φ.

$$S = S_0(X) + S_{QFT}(\Phi) + S_{int}(X,\Phi) \quad , \quad S_{int}(X,\Phi) = \int dt \; X(t)\mathcal{F}(t)$$

We assume that the particle starts at $X = x_i$ at $t_i = -\infty$

$$\rho_i = \delta(X - x_i)\delta(X' - x_i)\rho_i(\Phi,\Phi')$$

We would like to compute the reduced density matrix at time t:

$$\rho(X,X',t) = Tr_\Phi \rho(X,X',\Phi,\Phi',t)$$

That we can now write as a path integral using a doubled set of fields

$$\rho(X,X',t) = \int DX_+ \int DX_- \; e^{iS_0(X_+)-iS_0(X_-)} \int D\Phi_+ D\Phi_- \; e^{iS_+(X_+,\Phi_+)-iS_-(X_-,\Phi_-)}\rho_i(\Phi_+,\Phi_-)$$

where the the trace in the QFT path integral is obtained by setting $(\Phi_+)_f = (\Phi_-)_f$ and

$$S_\pm = S_{QFT} + \int X\mathcal{F}$$

Therefore the effective density matrix evolves according to the effective action

$$S_{eff}(X_+,X_-) = S_0(X_+) - S_0(X_-) + S_{IF}(X_+,X_-)$$

$$e^{iS_{IF}} = \langle e^{i\int X_+\mathcal{F}_+ - i\int X_-\mathcal{F}_-}\rangle_{\text{QFT ensemble}}$$

Feynman+Vernon, 1963

We expand the exponential to quadratic order

$$\langle e^{i\int X_+\mathcal{F}_+ - i\int X_-\mathcal{F}_-}\rangle_{\text{QFT ensemble}} \simeq 1 + i\int dt\ \langle\mathcal{F}(t)\rangle(X_+ - X_-) -$$

$$-\frac{i}{2}\int dt\int dt'\left[-X_+(t)\ i\langle\mathcal{F}_+(t)\mathcal{F}_+(t')\rangle X_+(t') + X_-(t)\ i\langle\mathcal{F}_-(t)\mathcal{F}_+(t')\rangle X_+(t') +\right.$$

$$\left.+ X_+(t)\ i\langle\mathcal{F}_+(t)\mathcal{F}_-(t')\rangle X_-(t') - X_-(t)\ i\langle\mathcal{F}_-(t)\mathcal{F}_-(t')\rangle X_-(t')\right]$$

$$\simeq \exp\left[i\int dt\ \langle\mathcal{F}(t)\rangle(X_+ - X_-) - \frac{i}{2}\int X_a(t)\ G_{ab}(t,t')X_b(t')\right]$$

with

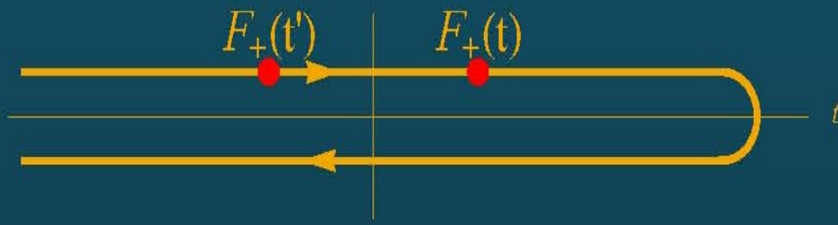
$$G_{ab}(t,t') \equiv i\langle\mathcal{P}\mathcal{F}_a(t)\mathcal{F}_b(t')\rangle$$

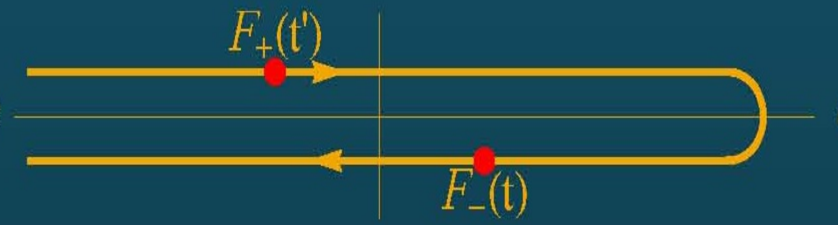
with $\mathcal{P}$ being path ordering along the keldysh contour:

- $+$ operators are time-ordered, $-$ operators are anti-time-ordered
- $-$ operators are always in the future of $+$ operators.

$$G_{++}(t,t') = -i\left\langle T\,\mathcal{F}_+(t)\mathcal{F}_+(t')\right\rangle$$

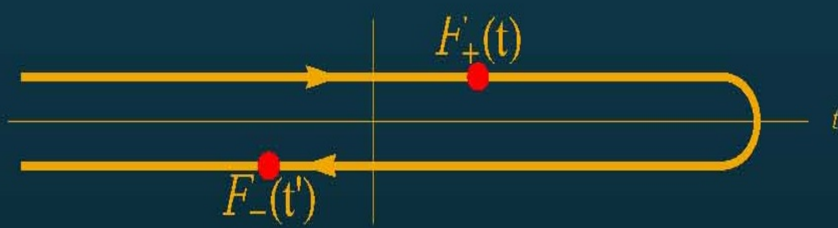
$$G_{-+}(t,t') = -i\left\langle \mathcal{F}_-(t)\mathcal{F}_+(t')\right\rangle$$

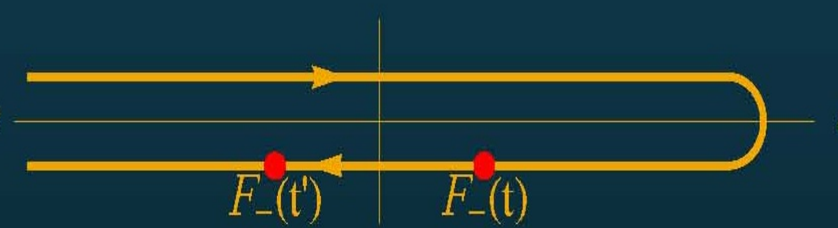




$$G_{+-}(t,t') = -i\left\langle \mathcal{F}_-(t')\mathcal{F}_+(t)\right\rangle$$

$$G_{--}(t,t') = -i\left\langle \overline{T}\,\mathcal{F}_-(t)\mathcal{F}_-(t')\right\rangle$$





The Keldysh propagators can be written in terms of the standard ones:

$$G_R(t) = -i\theta(t)\langle[\mathcal{F}(t),\mathcal{F}(0)]\rangle \quad , \quad G_A(t) = i\theta(-t)\langle[\mathcal{F}(t),\mathcal{F}(0)]\rangle$$

$$G_{sym}(t) = -\frac{i}{2}\langle\{\mathcal{F}(t),\mathcal{F}(0)\}\rangle \quad , \quad G_{anti-sym}(t) = -\frac{i}{2}\langle[\mathcal{F}(t),\mathcal{F}(0)]\rangle$$

$$\langle T\mathcal{F}(t)\mathcal{F}(0)\rangle \equiv \theta(t)\langle\mathcal{F}(t)\mathcal{F}(0)\rangle + \theta(-t)\langle\mathcal{F}(0)\mathcal{F}(t)\rangle = G_{sym} + \frac{1}{2}(G_R + G_A)$$

$$G_{++} = G_{sym} + \frac{1}{2}(G_R + G_A) \quad , \quad G_{--} = G_{sym} - \frac{1}{2}(G_R + G_A)$$

$$G_{+-} = G_{sym} + \frac{1}{2}(-G_R + G_A) \quad , \quad G_{-+} = G_{sym} + \frac{1}{2}(G_R - G_A)$$

$$G_{++} + G_{--} - G_{+-} - G_{-+} = 0$$

Using this we can rewrite the effective action as

$$S_{eff} = S_0(X_+) - S_0(X_-) + \int (X_+ - X_-)G_R(X_+ + X_-) + \frac{1}{2}(X_+ - X_-)G_{\text{sym}}(X_+ - X_-)\int$$

We now define

$$X_{\text{class}} = \frac{1}{2}(X_+ + X_-) \quad , \quad y = X_+ - X_-$$

In the semiclassical limit $y \ll X_{\text{class}}$ and we can expand

$$S_0(X_+) - S_0(X_-) \simeq \int dt\, \frac{\delta S_0}{\delta X_{\text{class}}}\, y + \mathcal{O}(y^3)$$

to obtain

$$S_{eff} = \int dt\; y(t) \left[\frac{\delta S_0}{\delta X_{\text{class}}(t)} + \int dt' G_R(t,t') X_{\text{class}}(t') \right] + \frac{1}{2} \int dt \int dt' y(t) G_{\text{sym}}(t,t') y(t')$$

Therefore the X path integral becomes

$$Z = \int DX_{\text{classs}} \int Dy\; e^{i \int dt\; y(t) \left[\frac{\delta S_0}{\delta X_{\text{class}}(t)} + \int dt' G_R(t,t') X_{\text{class}}(t') \right] + \frac{1}{2} \int dt \int dt' y(t) G_{\text{sym}}(t,t') y(t')}$$

We integrate-in a gaussian variable $\xi(t)$ with variance G_{sym}. This will linearize the y integration

$$Z = \int D\xi \int DX_{\text{class}} \int Dy \exp \left[i \int dt\; y \left(\frac{\delta S_0}{\delta X_{\text{class}}} + G_R X_{\text{class}} - \xi \right) - \frac{1}{2} \xi G_{sym} \xi \right]$$

Integrating over y we obtain a δ functional,

$$Z = \int D\xi \int DX_{\text{class}}\; \delta \left(\frac{\delta S_0}{\delta X_{\text{class}}} + G_R X_{\text{class}} - \xi \right)\; e^{-\frac{1}{2} \xi G_{sym} \xi}$$

Therefore the path integral is localized in a solution of the generalized Langevin equation

$$\frac{\delta S_0}{\delta X_{\text{class}}(t)} + \int_{-\infty}^{t} dt'\; G_R(t,t') X_{\text{class}}(t') = \xi(t) \quad , \quad \langle \xi(t) \xi(t') \rangle = G_{\text{sym}}(t,t')$$

String fluctuations and force correlators

In the diagonal wold-sheet frame

$$S_{NG}^{(2)} = -\frac{1}{2\pi\ell_s^2}\int d\tau\ dr\ \frac{H^{\alpha\beta}}{2}\left[\frac{\partial_\alpha X^{||}\partial_\beta X^{||}}{Z^2} + \sum_{i=1}^{2}\partial_\alpha X_i^{\perp}\partial_\beta X_i^{\perp}\right]$$

and the fluctuation equations are

$$\partial_\alpha(H^{\alpha\beta}\partial_\beta)X^{\perp} = 0 \quad , \quad \partial_\alpha\left(\frac{H^{\alpha\beta}}{Z^2}\partial_\beta\right)X^{||} = 0$$

$$H^{\alpha\beta} = \begin{pmatrix} -\frac{b^4}{\sqrt{(f-v^2)(b^4f-C^2)}} & 0 \\ 0 & \sqrt{(f-v^2)(b^4f-C^2)} \end{pmatrix} \quad , \quad Z \equiv b^2\sqrt{\frac{f-v^2}{b^4f-C^2}}.$$

We look for harmonic solutions $\delta X(r,t) = e^{i\omega\tau}\ \delta X(r,\omega)$

$$\partial_r\left[\sqrt{(f-v^2)(b^4f-C^2)}\ \partial_r\left(\delta X^{\perp}\right)\right] + \frac{\omega^2 b^4}{\sqrt{(f-v^2)(b^4f-C^2)}}\delta X^{\perp} = 0$$

$$\partial_r\left[\frac{1}{Z^2}\sqrt{(f-v^2)(b^4f-C^2)}\ \partial_r\left(\delta X^{||}\right)\right] + \frac{\omega^2 b^4}{Z^2\sqrt{(f-v^2)(b^4f-C^2)}}\delta X^{||} = 0$$

- Near the boundary the equations is symmetric

$$\Psi'' - \frac{2}{r}\Psi' + \gamma^2\omega^2\Psi = 0 \quad , \quad \Psi(r,\omega) \sim C_s(\omega) + C_v(\omega)r^3 + \cdots$$

• Near the world-sheet horizon, $r \to r_s$

$$\Psi'' + \frac{1}{r_s - r}\Psi' + \left(\frac{\omega}{4\pi T_s(r_s - r)}\right)^2 \Psi = 0 \quad , \quad \Psi(r,\omega) \sim C_{out}(\omega)\,(r_s - r)^{\frac{i\omega}{4\pi T_s}} + C_{in}(\omega)\,(r_s - r)^{-\frac{i\omega}{4\pi T_s}}$$

• To calculate the retarded correlator we have

$$S = \int dr d\tau \ \mathcal{H}^{\alpha\beta} \ \partial_\alpha \Psi \partial_\beta \Psi \quad , \quad \mathcal{H}^{\alpha\beta} = \begin{cases} \dfrac{H^{\alpha\beta}}{2\pi\ell_s^2} \ , \ \perp, \\ \\ \dfrac{H^{\alpha\beta}}{2\pi\ell_s^2 \ Z^2} \ , \ \|, \end{cases}$$

• For the retarded correlator

$$G_R(\omega) = \mathcal{H}^{r\alpha}(r)\Psi^*(r,\omega)\partial_\alpha\Psi(r,\omega)\Big|_{\text{boundary}} \quad , \quad \Psi(0,\omega) = 1 \quad , \quad \Psi(r \to r_s,\omega) \sim (r_s - r)^{-\frac{i\omega}{4\pi T_s}}$$

• The metric entering the wave equations for fluctuations is 2d BH metric, with temperature T_s. Using the Schwinger-Keldysh formalism we can show that

$$G^i_{sym}(\omega) = \coth\left(\frac{\omega}{2T_s}\right) G^i_R(\omega)$$

and therefore the temperature entering the fluctuation-dissipation relations is T_s.

• This is NOT the thermal equilibrium relation of the plasma.

Langevin diffusion constants

$$\kappa = G_{sym}(\omega = 0) = -2T_s \frac{G_R(\omega)}{\omega}\Big|_{\omega=0}$$

$$ImG_r(r,t) = \frac{\mathcal{H}^{rr}}{2i}\Psi^*\overleftrightarrow{\partial}\Psi = J^r(r,t) \quad , \quad \partial_r J^r = 0$$

We can compute $ImG_R(\omega)$, anywhere, and the easiest is at the horizon, $r = r_s$:

$$\Psi = C_h\,(r_s - r)^{-\frac{i\omega}{4\pi T_s}} + \cdots \quad , \quad ImG_R = \frac{\mathcal{H}^{rr}}{4\pi T_s(r_s - r)}\Big|_{r_s} |C_h|^2\omega$$

- Ψ can be computed exactly as $\omega \to 0$

$$\Psi \simeq 1 + \omega \int_0^r \mathcal{H}^{rr}(r')\,dr' \quad \Rightarrow \quad C_h = 1 \quad \Rightarrow \quad \kappa = \frac{\mathcal{H}^{rr}}{2\pi(r_s - r)}\Big|_{r_s} = \frac{1}{\pi\ell_s^2}\begin{cases} b^2(r_s)T_s \quad , \quad \perp, \\ \\ (4\pi)^2\dfrac{b^2(r_s)}{f'(r_s)^2}T_s^3 \quad , \quad ||, \end{cases}$$

Langevin friction terms

We have

$$\dot{\vec{p}} = -\eta_D^{||}\, p^{||}\, \widehat{v} - \eta_D^{\perp}\, p^{\perp} + \vec{\xi}(t)$$

To connect to the holographic equations we must rewrite them as equations for δX

$$\dot{\vec{X}} = \vec{v} + \delta\dot{\vec{X}} \quad , \quad \vec{p} = \frac{M\dot{\vec{X}}}{\sqrt{1 - \dot{\vec{X}} \cdot \dot{\vec{X}}}} = \gamma M \vec{v} + \delta\vec{p}$$

We expand to first order to obtain the equations for the position fluctuations

$$\gamma M \delta\ddot{X}^{\perp} = -\eta^{\perp}\delta\dot{X}^{\perp} + \xi^{\perp} \quad , \quad \gamma^3 M \delta\ddot{X}^{||} = -\eta^{||}\delta\dot{X}^{||} + \xi^{||}$$

$$\eta^{\perp} = \frac{1}{\gamma M}\eta_D^{\perp} \quad , \quad \eta^{||} = \frac{1}{\gamma^3 M}\left[\eta_D^{||} + \gamma M v \frac{\partial \eta_D^{||}}{\partial p}\right]$$

We have computed holographically

$$\eta^{||,\perp} = \frac{\kappa^{||,\perp}}{2T_s}$$

which lead to the modified Einstein relations

$$\kappa^{\perp} = 2\gamma M T_s \ \eta_D^{\perp} = 2 E T_s \ \eta_D^{\perp} \quad , \quad \kappa^{||} = 2\gamma^3 M T_s \left[\eta_D^{||} + \gamma M v \frac{\partial \eta_D^{||}}{\partial p}\right]$$

to be compared with the standard one $\kappa = 2MT\eta_D$.

- Consistency check

$$\eta_D^{||} = \eta_D^{\perp} = \frac{b^2(r_s)}{M\gamma(2\pi\ell_s^2)}$$

satisfies both Einstein relations.

Detailed plan of the presentation