International Journal of Modern Physics A, Vol. 6, No. 16 (1991) 2871–2890

Universal W-Algebras in Quantum Field Theory * †

Ioannis Bakas

Center for Theoretical Physics
Department of Physics and Astronomy
University of Maryland
College Park, MD 20742

Elias B. Kiritsis

Department of Physics, University of California
and
Theoretical Physics Group, Lawrence Berkeley Laboratory
Berkeley, CA 94720

Abstract

We outline some of the main results derived recently on the representation theory of universal W-algebras, which arise in two dimensional field theory as large N limits of W_N (extended) conformal symmetries. Certain connections with integrable systems of non-linear differential equations, hyper-Kähler geometries in four spacetime dimensions and the infinitely long chain of hermitian matrix models are also discussed.

1. Introduction and Motivations

Despite all efforts made so far, our present understanding of string theory is far from being complete and satisfactory. Some of the main problems seem to be related to the lack of non-perturbative methods that could capture quantum effects characteristic of strings and stimulate new directions in the development of the theory. Recently, considerable progress was made in the context of 2-d quantum gravity using the double scaling limit of hermitian (multi)-matrix models, [1]. The non-perturbative results obtained in this area turn out to be quite appropriate for the description of toy string models in which the embedding (spacetime)

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grants PHY-85-15857 and PHY-87-17155.

†Presented by I. Bakas

dimension is less than one. In particular the double scaling limit of the (N-1)-chain of matrix models (with N $\geq$ 2) provides a non-perturbative formulation of 2-d quantum gravity coupled to conformal matter with $c \leq 1$. The value of the central charge is zero for N=2 (corresponding to pure gravity) and tends to one as N$\rightarrow \infty$. For the purposes of string theory it is mostly important to understand the physics of infinitely long matrix chains, because in this case the embedding space is one dimensional and the toy string model becomes more realistic, [2]. Although string theory might require more sophisticated non-perturbative techniques for its solutions in spacetime dimensions $D > 1$, at this stage one hopes to gain a modest insight into this complicated problem by considering the $D = 1$ theory first.

The relations found in the context of matrix models between (N-1)-hermitian chains and the generalized KdV hierarchy of integrable non-linear differential equations of $SL(N)$-type, have suggested quite strongly an alternative description of these models using W_N-gravity, [3, 4], where W_N is Zamolodchikov's extended conformal symmetry algebra generated by chiral fields with spin $s = 2, 3, \cdots, N$, [5]. In fact, apart from the emergence of the $SL(N)$-KdV equations in these models, it has been shown, [6] that the loop equations can be written as highest weight conditions plus an eigenvalue equation using W_N generators. From this point of view, $D = 1$ string theory could be formulated in more algebraic terms using W_∞, the large N limit of the chiral operator algebras W_N. It is our purpose to present an account of some recent results obtained on the structure and representation theory of W_∞ (often called Universal W-Algebra) and discuss the relevance of this infinite dimensional algebra to the geometry of 4-d hyper-Kähler manifolds. A novel synthesis of the algebraic and geometrical ideas involved in the large N limit of W-algebras may provide the key ingredients for a non-perturbative formulation of $D = 1$ strings and eventually lead to their solution.

On the other hand, universal W-algebras (UWA) might be closely related to operator algebras of a certain class of higher dimensional quantum field theories, whose operators could be Fourier decomposed into W-generators. From an algebraic point of view this seems plausible because UWA are generated by an infinite collection of conserved (chiral) fields in two dimensions. Moreover, as we will see later, W_∞ is a deformation of the algebra of area-preserving diffeomorphisms, a symmetry which is certainly not intrinsic to the 2-d geometry of conformal field theories (CFT). The possibility to reach higher dimensions using large N limit techniques in the theory of W_N chiral operator algebras has provided one of the main driving forces in our investigation. More specifically, one would like to construct some physically interesting quantum field theories in $d > 2$ which are exactly solvable by the bootstrap (operator algebra) approach (see for instance [7, 8]). Of course, this problem is highly non-trivial because there is very little known about the representation theory of the infinite dimensional algebras involved. In fact, the situation is more subtle than that. In generic cases, equal-time commutation relations do not make sense. This problem motivated Wilson to introduce the operator product expansion (which always makes sense) as an alternative to commutation relations, [7]. The results we describe here provide a way to construct unitary highest weight representations (HWR) of W_∞. In particular, all the unitary HWR of W_∞ that we obtain are the limits of the corresponding HWR of W_N as $N \to \infty$. However, we do not maintain that the representations obtained by such limiting procedures exhaust all possible unitary HWR of W_∞. In this regard, the steps we have taken should be considered as the starting point for a more ambitious investigation in this area.

Next, having presented our main motivations and objectives for considering the large N limit of W_N-algebras, we outline the organization of this paper. In section 2 we review the notion of UWA together with their operator content and geometric interpretation. In section 3 we study unitary HWR of W_∞ (associated with the Lie algebra $sl(\infty)$), using complex free bosons in 2-d that generate representations of affine $U(1)$ current algebras. In section 4 we discuss physical applications of W_∞ in the theory of integrable non-linear differential equations, self-dual Einstein equations and the multi-matrix models describing $D = 1$ string theory. Finally, section 5 contains our conclusions.

2. The Notion of Universal W-Algebras

Let us consider first the infinite dimensional (double graded) algebra

$$[W_m^s, W_n^{s'}] = ((s'-1)m - (s-1)n)W_{m+n}^{s+s'-2} \ , \tag{2.1}$$

where both $m, n \in Z$ and s, s' are integers ≥ 2. This algebra is quite interesting from the point point of view of 2-d CFT, because it contains the Virasoro algebra

$$[W_m^2, W_n^2] = (m-n)W_{m+n}^2 \tag{2.2}$$

as a subalgebra, with central charge $c = 0$. Moreover, we have the relations

$$[W_m^2, W_n^s] = ((s-1)m - n)W_{m+n}^s \ , \tag{2.3}$$

which imply that the operator content of the algebra (2.1) can be identified as follows: W_n^s are the Fourier components (labeled by n) of primary conformal fields with weight s. Consequently, the algebra we are considering here, viewed as a module of the (centerless) Virasoro algebra, decomposes into a direct sum of 2-d higher spin fields. The restriction imposed on s, $s \geq 2$, is dictated from physical considerations. Of course, we could also include generators with $s = 1$, corresponding to $U(1)$ currents, but we postpone this possibility for later.

It is clear that a quantum version of (2.1) which allows for central terms in the commutation relations, qualifies as a chiral operator algebra in the context of CFT. In the realm of 2-d CFT, this algebra would be associated with models whose stress tensor $T(z)$ is identified with $W^2(z)$ and which possess an infinite collection of additional conserved currents generated by the chiral fields[‡] $W^s(z) \equiv \sum_{n \in Z} W_n^s z^{-n-s}$. For this reason it is natural to view the quantum version of (2.1) as a large N limit of W_N algebras, [9].

Recall that W-algebras are closed operator algebra extensions of the Virasoro algebra with additional higher spin generators and non-linear commutation relations in general. The latter are typically of the form

$$[W_m^s, W_n^{s'}] = \sum_{\{s_i\},\{k_i\}} C_{s_1 s_2 \cdots s_p}^{ss'}(n, m, k_1, k_2, \cdots k_p; c) W_{k_1}^{s_1} \cdots W_{k_p}^{s_p} \ , \tag{2.4}$$

where $k_1 + k_2 + \cdots + k_p = m + n$, $W_k^0 = \delta_{k,0}$ (inclusion of the identity operator that accounts for central terms) and $s_1 + s_2 + \cdots + s_p \leq s + s' - 2$. The structure constants $C_{\{s_i\}}^{ss'}$ may depend on the central charge c of the Virasoro subalgebra and they are calculable using the Jacobi identity constraint. However, explicit derivation of their form requires extremely long calculations. Despite the complicated nature of their commutation relations, W-algebras have provided a unifying conceptual framework for classifying and solving rational CFT that describe 2-d critical statistical models. Moreover, it was realized that W-algebras are closely connected with simple Lie algebras which determine their operator content (ie. the range of values for the spin of the generating fields $W^s(z)$) and the anomalous dimensions of the primary W-operators in representation theory. Also, the details of their structure as well as various field theoretic realizations of the commutation relations (2.4) (Sugawara, Feigin-Fuks, Toda) can be derived essentially from the corresponding current algebras using the method of Hamiltonian reduction. We are not going to present any details here, because some of these issues have

[‡]Since we are considering chiral algebras, no distinction will be made between weight (left dimension) and spin of conformal fields.

been discussed in this meeting by Graeme Segal, [10]. We point out only that the connection between W-algebras and affine current algebras provides the most systematic way to define W-symmetries and study their properties, without ever computing explicitly the structure constants in (2.4).

The simplest example in the class of W-type symmetries is associated with the A series in Cartan's classification of simple Lie algebras. For $A_{N-1} \simeq sl(N)$ we have $N-1$ chiral fields $\{W^s(z)\}$ (as many as the vertices of the corresponding Dynkin diagram) with integer spin $s = 2, 3, \cdots, N$, which generate $W(A_{N-1})$, Zamolodchikov's W_N-algebra. Different types of operator algebras $W(G)$ are associated with other simple Lie algebras G, but these cases (see for instance [11] and references therein) will not be considered here at all. One of the practical difficulties in deriving explicit forms for the commutation relations of W_N is that for any given pair of spins $2 \leq s, s' \leq N$, the structure constants $C^{ss'}_{\{s_i\}}$ are not universal, in the sense that many of them depend explicitly on N. However, taking a suitable limit in which $N \to \infty$, the structure of W-algebras simplifies considerably and the commutation relations of the resulting infinite dimensional symmetry (when appropriately defined) are determined only by universal constants. This justifies the use of the term "universal W-algebra" for characterizing W_∞.

It has been established that the leading (highest spin) linear contribution to the commutation relations of W_N at large N is fully described by (2.1), [9]. This is interesting because the infinite dimensional algebra (2.1), which captures the universal features of the higher spin transformations in two dimensions, has a natural geometric interpretation as area preserving diffeomorphisms. Indeed, if we consider a 2-d plane with (canonical) coordinates x, y, such that

$$\{x, y\} = 1 \ , \tag{2.5}$$

the Poisson bracket algebra of the functions $W^s_n = x^{n+s-1} y^{s-1}$ coincides with (2.1). Alternatively, in order to avoid singularities when $x = 0$, one may use the smooth functions $e^{inx} y^{s-1}$ on a cylinder $S^1 \times R$ to represent the generators W^s_n.

The complete structure of W_∞ can be described as a deformation of the symmetry algebra (2.1). In particular, the results of ref. [9] imply that for any given s and s', the commutation relations of the area preserving diffeomorphism algebra and W_∞ differ from each other by local functionals of the generating fields with spin less than $s + s' - 2$. Since both algebras satisfy the Jacobi identity (associativity) the deformation terms cannot be arbitrary; they are 2-cocycles of the algebra (2.1) with non-trivial coefficients in general. The existence of consistent gauge interactions among higher spin fields with all integer values $s \geq 2$, imposes physical restrictions on the form of the deformation terms that differentiate W_∞ from the algebra of area preserving diffeomorphisms. It is natural to expect that these terms are central or linear but not quadratic (or higher polynomial) in the W-fields. However, their direct computation is highly non-trivial and the results obtained so far using the Feigin-Fuks (free-field) realization of W-algebras, seem to depend on the way that the value of the background charge α_0 behaves for large N, [9, 12]. As a starting point, it is practically more advantageous to make an ansätz for the form of the deformation terms and then try to justify them by appealing to the representation theory of W_N algebras at large N. A two parameter deformation of the algebra (2.1) was constructed recently by Pope, Romans and Shen (PRS), which allows for central terms in the commutation relations of all higher spin fields, [13]. We adopt their result in the sequel and study (some) of the unitary representations of W_∞ from the point of view of 2-d CFT. In fact, the constructions we will present next are motivated by the behavior of the W_N series of minimal models in the limit $N \to \infty$. This way we will demonstrate that the PRS algebra is identical to W_∞ and obtain unitary HWR of it, in terms of free bosons.

Finally, we point out that it is possible to extend W_∞ to $W_{1+\infty}$ by including a $U(1)$ field (with $s = 1$) in the algebra (2.1), [14]. Using W_∞ and $W_{1+\infty}$ an N=2 extended supersymmetric analog of W_∞ has been constructed in ref. [15]. It might be also interesting to investigate the large N limit of W-algebras associated with other series of classical Lie algebras. However, we reserve the discussion of these investigations to a future occasion.

3. Realization and Representations of W_∞

Next we introduce the linear PRS deformation of the commutation relations (2.1). For this purpose we need the following combinatorial expressions

$$g_{2r}^{ss'}(m,n) = \frac{1}{2(2r+1)!}\varphi_{2r}^{ss'} N_{2r}^{ss'}(m,n) \ , \tag{3.1}$$

where

$$\varphi_{2r}^{ss'} = \sum_{k=0}^{r} \frac{(-\frac{1}{2})_k(\frac{3}{2})_k(-r-\frac{1}{2})_k(-r)_k}{k!(-s+\frac{3}{2})_k(-s'+\frac{3}{2})_k(s+s'-2r-\frac{3}{2})_k} \ , \tag{3.2}$$

$$N_{2r}^{ss'}(m,n) = \sum_{k=0}^{2r+1}(-1)^k\binom{2r+1}{k}(2s-2r-2)_k[2s'-k-2]_{2r+1-k}\cdot$$
$$\cdot[s-1+m]_{2r+1-k}[s'-1+n]_k \tag{3.3}$$

and

$$(a)_k \equiv a(a+1)(a+2)\cdots(a+k-1) \ , \qquad (3.4a)$$

$$[a]_k \equiv a(a-1)(a-2)\cdots(a-k+1) \ . \qquad (3.4b)$$

We also set $(a)_0=[a]_0=1$ for all values of a. Then, the commutation relations of the PRS algebra are

$$[W^s_m, W^{s'}_n] = ((s'-1)m-(s-1)n)W^{s+s'-2}_{m+n} + q^{2(s-2)}c_s(m)\delta_{s,s'}\delta_{m+n,0}+$$
$$+q^2 g_2^{ss'}(m,n)W^{s+s'-4}_{m+n} + q^4 g_4^{ss'}(m,n)W^{s+s'-6}_{m+n} + \cdots \ , \qquad (3.5)$$

where the coefficients of the central terms are

$$c_s(m) = \frac{c}{2}m(m^2-1)(m^2-4)\cdots(m^2-(s-1)^2)\frac{2^{2(s-3)}s!(s-2)!}{(2s-1)!!(2s-3)!!} \qquad (3.6)$$

and the sequence of $\cdots$ terms in (3.5) terminates with W^2_{m+n} for $s+s'$ even and with W^3_{m+n} for $s+s'$ odd. The value of c depends on the underlying theory, while the second deformation parameter q can be normalized to 1 by rescaling the generators W^s_n by q^{s-2}. From now on we choose to work with $q=1$ without loss of generality.

We are in a position now to present our results on the representation theory of the algebra (3.5). Our exposition follows closely that of ref. [16]. Let us consider first a free complex scalar field in 2-d with two-point function given by,

$$\langle\phi(z)\phi(w)\rangle = \langle\bar{\phi}(z)\bar{\phi}(w)\rangle = 0 \ , \ \langle\phi(z)\bar{\phi}(w)\rangle = -\log(z-w) \ . \qquad (3.7)$$

The standard stress tensor of the theory

$$W^2(z) \equiv T(z) = -\partial_z\phi\partial_z\bar{\phi} \qquad (3.8)$$

has the operator product expansion (OPE)

$$T(z)T(w) = \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \ , \qquad (3.9)$$

which leads to the Virasoro algebra with central charge $c=2$. It is possible to extend this representation to the full PRS algebra by introducing the ansätz

$$W^s(z) = B(s)\sum_{k=1}^{s-1}(-1)^k A^s_k \partial^k_z\phi\partial^{s-k}_z\bar{\phi} \ , \qquad (3.10)$$

for all $s \geq 2$. Normal orderings are implicitly assumed throughout this paper. The prescription for choosing appropriate numerical coefficients $B(s)$ and A^s_k will be given shortly.

We point out that the ansätz (3.10) is motivated by the linear structure of the commutation relations (3.5), which seem to require only bilinear expressions in the $U(1)\otimes U(1)$ currents $\partial\phi$, $\partial\bar{\phi}$ (and their derivatives) for a bosonic realization of all higher spin generators. A proper field theoretic motivation can be obtained by viewing W_N (and its large N limit, W_∞) as the symmetry algebra of the parafermion theory. The dimension of the $\psi_{\pm 1}$ parafermions is 1 for $N=\infty$ and they can be identified as $\psi_1 = i\partial\phi$ and $\psi_{-1} = i\partial\bar{\phi}$. Since we know that all W-generators appear in the operator product of ψ_1 with ψ_{-1}, it is not difficult to see that only terms of the form (3.10) can appear in this OPE. Thus, if we manage to find a basis which reproduces the PRS algebra, we will prove automatically that the PRS algebra and W_∞ are identical.

With these explanations in mind we proceed to calculate the coefficients A^s_k. Notice that there are no centrals terms in eq. (3.5) for $s \neq s'$, which means that W^s are quasiprimary operators. Since we assume that the W-symmetry is unbroken, ie. $\langle W^s(z)\rangle = 0$ for all s, we have

$$\langle W^s(z)W^{s'}(w)\rangle \sim \frac{\delta_{s,s'}}{(z-w)^{s+s'}} \ . \tag{3.11}$$

This equation alone is sufficient to determine the form of A^s_k uniquely up to an overall normalization constant, which we denote by $B(s)$ in (3.10). The solution is

$$A^s_k = \frac{1}{s-1}\binom{s-1}{k}\binom{s-1}{s-k} \tag{3.12}$$

and enjoys the property $A^s_k = A^s_{s-k}$. We also reach agreement with the standard normalization of the central terms in (3.6) provided that

$$B(s) = \frac{2^{s-3}s!}{(2s-3)!!} \ . \tag{3.13}$$

We point out that the ansätz (3.10) with the choices (3.12) and (3.13) for its numerical coefficients, yields an infinite tower of quasiprimary conformal fields with integer spin $s \geq 2$. These fields are not necessarily primary; in fact, only $W^3(z)$ is primary, as can be readily verified.[§] Introducing Fourier modes, it is possible to show (with the help of symbolic manipulation) that the OPE of the operators $W^s(z)$ given by (3.10), reproduces not only the central terms but the entire PRS algebra (3.5). For further details we refer the reader to ref. [16].

[§]Of course we can define primary fields for all $s > 3$ at the cost of the algebra becoming non-linear.

It is obvious from the preceding that W_∞ is realized as a subalgebra in the enveloping algebra of the $U(1) \otimes U(1)$ current algebra generated by $i\partial\phi$ and $i\partial\bar{\phi}$. Consequently, we can use highest weight (hw) representations of the latter to construct representations of the W_∞ algebra with $c = 2$. A state $|Q\rangle$ will be a hw state of W_∞, if the following conditions are satisfied:

$$W^s_{n>0}|Q\rangle = 0 \ , \ W^s_0|Q\rangle = Q_s|Q\rangle \ . \tag{3.14}$$

In the quasiprimary basis (3.10), the standard hermiticity condition of the $U(1)\otimes U(1)$ current algebra translates into $(W^s_n)^\dagger = W^s_{-n}$. Thus, any representation that we obtain from decomposing a unitary hw representation of the $U(1)\otimes U(1)$ current algebra is automatically a unitary representation of the W_∞ algebra. To be more precise, let us consider a hw representation of the $U(1) \otimes U(1)$ current algebra generated from the $SL(2, C)$ invariant vacuum, $|0\rangle$ by the vertex operator

$$V_{a,\bar{a}} = exp(ia\phi + i\bar{a}\bar{\phi}) \ . \tag{3.15}$$

It is straightforward to show that $V_{a,\bar{a}}(0)|0\rangle \equiv |Q\rangle$ is a hw state of the W_∞ algebra, with charges

$$Q^{a,\bar{a}}_s = |a|^2[1 + (-1)^s]\frac{2^{s-3}(s-1)!(s-2)!}{(2s-3)!!} \ . \tag{3.16}$$

The rest of the representation is generated from $|Q\rangle$ by acting with the W-lowering operators. $V_{a,\bar{a}}$ gives rise to a unitary hw representation iff $a^* = \bar{a}$.

The (reduced) character of W-representations with $c = 2$ can be computed by taking a suitable limit of the parafermionic characters. The answer turns out to be, [16]

$$\chi^{\bar{a}}_a(q) \equiv Tr[q^{L_0 - \frac{c}{24}}] = \frac{q^{|a|^2 - \frac{1}{12}}}{\prod_{n=1}^{\infty}(1-q^n)^2} \tag{3.17}$$

and coincides with the $U(1) \otimes U(1)$ character. This proves directly that any $U(1) \otimes U(1)$ representation decomposes into a single W_∞ representation.¶ In fact, these are all the unitary irreducible HWR of the W_∞ algebra with $c = 2$, which arise as large N limits of CFT representations.

A few remarks are in order concerning the theory of Z_N parafermions in the limit $N \to \infty$. Recall that the series of minimal models associated with the W_N algebra have central charges given by, [5]

$$c^N_p = (N-1)[1 - \frac{N(N+1)}{(N+p)(N+p+1)}] \ ; \ p = 1, 2, \cdots \ . \tag{3.18}$$

¶From (3.16) it is also obvious that these W_∞ representations are highly degenerate, as expected.

These theories can be identified with the coset models

$$\frac{SU(N)_1 \otimes SU(N)_p}{SU(N)_{p+1}} \ , \tag{3.19}$$

or equivalently with the Grassmannian coset models,

$$G_N(p) = \frac{SU(p+1)_N}{SU(p)_N \otimes U(1)} \ . \tag{3.20}$$

We adopt the second picture here in order to avoid unnecessary complications dealing with the large N limit of $SU(N)$. The Z_N parafermions, $\psi_k(z)$, occur in the simplest model of the series (3.18), p=1 and have central charge $c_1^N = 2\frac{N-1}{N+2}$, [17]. As such they generalize ordinary Weyl fermions in 2-d for $N \geq 2$. However, the spin of the parafermions ψ_k is fractional in general, given by $\Delta_k = k(N-k)/N$ for $0 \leq k \leq N-1$. Local fields are obtained only for $N = 2$ or $N \to \infty$. In the first case the chiral algebra is generated by W_2, which is the Virasoro algebra and the corresponding CFT is the Ising model. In the second case we have W_∞ and an infinite collection of parafermion fields ψ_k (and their conjugates $\psi_k^\dagger$) with integer spin, $\Delta_k = k \in Z^+$. The fields ψ_1 and $\psi_1^\dagger$ are identified with the $U(1)$ currents $i\partial\phi$ and $i\partial\bar{\phi}$ respectively, which parametrize the coset $G_N(1) = SU(2)_N/U(1)$ in the limit $N \to \infty$. It is trivial to see that in that limit the $SU(2)_N$ current algebra "flattens" becoming an abelian $U(1)^3$ algebra. Going to the coset, one of the $U(1)$ currents is factored out and we are left with two $U(1)$ currents only. The rest of the Z_∞ parafermions are given by $\psi_k \sim (\partial\phi)^k$, $\psi_k^\dagger \sim (\partial\bar{\phi})^k$ and in this case, the parafermion algebra coincides with the enveloping algebra of the $U(1) \otimes U(1)$ current algebra. This argument provides a solid justification for the bosonic realization of W_∞ we presented earlier.

More generally, unitary HWR of W_∞ with central charge $c = 2p = 2, 4, \cdots$ will be generated from the Grassmannian coset models $G_N(p)$, as $N \to \infty$. As before, and for all values of p, the $SU(p)_N$ current algebra abelianizes at $N = \infty$, becoming a $U(1)^{p^2-1}$ current algebra. Therefore, the Grassmannian coset models $G_\infty(p)$ are parametrized by $U(1)^{2p}$ affine currents which can be identified with a collection of p independent free complex scalar fields, $\partial\phi^i$ and $\partial\bar{\phi}^i$, $(i = 1, 2, \cdots, p)$. Then, W_∞ is a subalgebra of the parafermionic algebra of these models, or equivalently it is a subalgebra of the enveloping algebra of the $U(1)^{2p}$ current algebra. It is immediate and obvious (thanks to the linear structure of the commutation relations of the PRS algebra) that the tower of quasiprimary fields

$$W^s(z) = B(s) \sum_{i=1}^{p} \sum_{k=1}^{s-1} (-1)^k A_k^s \partial_z^k \phi^i \partial_z^{s-k} \bar{\phi}^i \tag{3.21}$$

with $s \geq 2$ and $B(s)$, A_k^s given by (3.12) and (3.13), provides a realization of W_∞ with $c = 2p$. In analogy with the simplest case $p = 1$, $U(1)^{2p}$ hw unitary irreducible representations decompose into representations of the W_∞ algebra with $c = 2p$. In particular, hw states generated by vertex operators are also W_∞ hw states. However, for $p > 1$ a $U(1)^{2p}$ hw representation decomposes into more than one W_∞ representations. The precise decomposition law, although straightforward to derive in principle, is rather involved and we do not have a concrete answer at the moment.

4. Diverse Applications or a Synthesis of Ideas?

The need to investigate the structure of W_N algebras as $N \to \infty$ arises not only in problems of 2-d CFT, but also in other areas of mathematical physics. In particular, the KP hierarchy of integrable non-linear differential equations, the continual Toda field equations, the theory of stationary gravitational instantons and the infinitely long chain of hermitian matrix models, all seem to be related (in one way or another) in this limit. Although many questions raised in our discussion will remain unanswered, the sole purpose of this section is to show that the concept of UWA could be useful for bringing together some (otherwise unrelated) ideas in the formulation of $D = 1$ string theory.

4.1 **Non-linear differential equations**: Let us consider first the KP hierarchy of integrable non-linear differential equations (see for instance [18] and references therein) described by the evolution

$$\frac{\partial Q}{\partial t_r} = [(Q^r)_+, Q] \ , \tag{4.1}$$

where Q is a formal pseudo-differential operator

$$Q = \partial_z + q_1(z, t_i)\partial_z^{-1} + q_2(z, t_i)\partial_z^{-2} + \cdots \tag{4.2}$$

and $(Q^r)_+$ denotes the purely differential part of Q^r. Then, for every fixed positive integer r, we obtain a system of infinitely many coupled non-linear equations of the form

$$\frac{\partial q_i}{\partial t_r} = F_i^{(r)}(q, q', q'', \cdots) \ ; \ i \in Z^+ \ , \tag{4.3}$$

where $F_i^{(r)}$ are certain polynomials in the q-variables and their derivatives determined by eq. (4.1). It is known that all KdV systems of $SL(N)$-type hierarchies (whose members are parametrized by r) are special cases of (4.1). For each value of r, their embedding in the KP hierarchy is described simply by the requirement that Q is the (unique) N-th root of a differential operator L_N,

$$Q^N \equiv L_N = \partial_z^N + u_{N-2}(z)\partial_z^{N-2} + \cdots + u_0(z) \ . \tag{4.4}$$

Then, the equations (4.3) reduce to a system of N-1 independent differential equations and the rest are functionally related to them.

At this point recall that the Hamiltonian structure of the KdV systems can be formulated using the commutation relations of the Gelfand-Dickey algebra $GD(SL(N))$. For $N = 2$, this reduces to the standard description of the KdV equations in terms of the Virasoro algebra. On the other hand, it has been established that Gelfand-Dickey algebras provide a classical Hamiltonian framework for all W_N symmetry algebras with $N \geq 2$ (for details see Segal's talk, [10] and references therein). Therefore, it is natural to view the full unrestricted KP hierarchy as a universal KdV system associated with the large N limit of W_N algebras. This observation alone suggests an interesting reformulation of the problem (4.1) and its integrability properties. In fact, we suspect that there is a deeper connection between the (classical and/or quantum) KP hierarchy and the algebra of area preserving diffeomorphisms, but the precise details need to be worked out.

Another interesting system of integrable non-linear differential equations is provided by the Toda theory, (see [19, 20] and references therein). In this case we have a collection of scalar fields $\{\phi_i(z,\bar{z}) \ ; i = 1,2,\cdots,N-1\}$, which are coupled through the Cartan matrix K of a simple Lie algebra (here it is taken to be $sl(N)$) and satisfy the following system of equations

$$\partial_z\partial_{\bar{z}}\phi_i = exp\left(\sum_{j=1}^{N-1} K_{ij}\phi_j\right) \ . \tag{4.5}$$

It is known that the Toda theory (4.5) possesses a number of (mutually independent) conserved currents $\{W^s(z) \ ; \ s = 2,3,\cdots,N\}$, which generate the chiral operator algebra W_N, [20], provided that the fields $\phi_i(z,\bar{z})$ satisfy the equal $\bar{z}$ commutation relations

$$[\partial\phi_i(z,\bar{z}), \partial\phi_j(w,\bar{z})] = (K^{-1})_{ij}\delta'(z-w) \ . \tag{4.6}$$

In the limit $N \to \infty$, the Cartan matrix K of $sl(N)$ becomes infinite dimensional and it is appropriate to replace it with a distribution $K(t-t') = -\delta''(t-t')$, where t is a continuous variable which labels the roots of $sl(\infty)$. Introducing a "master" field $\Phi(z,\bar{z},t)$, it is straightforward to show that in this limit the associated system of Toda field equations assumes the form of a non-linear differential equation in three dimensions $(z,\bar{z},t)$. Differentiating twice with respect to t and letting $\Psi \equiv -\partial_t^2\Phi$, we obtain

$$\partial_z\partial_{\bar{z}}\Psi(z,\bar{z},t)+\partial_t^2 e^{\Psi(z,\bar{z},t)}=0 \ . \tag{4.7}$$

In this approach the large N limit of $sl(N)$ algebras is treated as a continual algebra, in the sense that the corresponding set of roots becomes continuous, [21], rather than an infinitely long discrete chain of vertices, [22]. It is also quite amusing that $sl(\infty)$ can be identified with the algebra of area preserving diffeomorphisms of the torus, [21, 23], which a priori has nothing to do with the relation of W_∞ to the algebra of area preserving diffeomorphisms (2.1) we have been discussing so far. It is natural to expect that (at least formally) there is an infinite collection of chiral currents associated with the continual Toda field equations (4.7), which quantum mechanically satisfy the commutation relations of W_∞. Although further work is required to establish and understand the meaning of such results, the 3-d interpretation of $sl(N)$ Toda field equations at large N is quite curious and suggestive. Also, it is interesting to note that like in the theory of Z_N parafermions the values $N=2$ and ∞ are singled out uniquely by the requirement of locality, $sl(N)$ Toda field theories have a natural geometric interpretation only for $N=2$ or ∞. The first case corresponds to the Liouville equation, which is the condition satisfied by the conformal factor $\phi(z,\bar{z})$ of a 2-d metric $ds^2=e^\phi dzd\bar{z}$ with constant negative curvature. The importance of the second case was appreciated only more recently in the context of hyper-Kähler geometry in four dimensions [9, 21, 24, 25].

4.2. **Self-dual Einstein equations**: Consider a metric in 4-d Euclidean space with local coordinates (x,y,t,τ) which is stationary with respect to τ and depends on a single scalar function $\Psi(x,y,t)$ in the following fashion:

$$ds^2=\Psi_{,t}[e^\Psi(dx^2+dy^2)+dt^2]+\frac{1}{\Psi_{,t}}[\epsilon(\Psi_{,x}dy-\Psi_{,y}dx)+d\tau]^2 . \tag{4.8}$$

For convenience we introduce the variables $z=(x+iy)/2$, $\bar{z}=(x-iy)/2$ and let $\epsilon=\pm1$. Then, the self-duality condition for the Riemann curvature tensor,

$$R_{\kappa\lambda\mu\nu}=\frac{\epsilon}{2}\sqrt{g}\varepsilon_{\mu\nu\rho\sigma}R_{\kappa\lambda}{}^{\rho\sigma} \ , \tag{4.9}$$

is equivalent to the continual Toda field equation (4.7). This result follows by straightforward calculation and establishes a link between the large N limit of

Toda theories and hyper-Kähler geometry in 4-d. Of course, the regularity condition for the metric (4.8), $detg \equiv (\Psi_{,t}e^{\Psi})^2 \neq 0$, is assumed implicitly.

Self-dual metrics in four dimensions are quite important because they imply Ricci flatness and hyper-Kähler structure. They also play a role in Euclidean quantum gravity, because they describe gravitational instanton solutions to the vacuum Einstein equations (see [26] for a review). The most obvious solution of eq. (4.7), which is non-trivial and physically interesting for the problem (4.8), is given by

$$\Psi(z,\bar{z},t) = \log \frac{t^2 - a^2}{2(1+\frac{z\bar{z}}{2})^2} \ , \ \ t^2 \geq a^2 \ . \tag{4.10}$$

It can be recognized almost immediately that this solution describes the two-center Eguchi-Hanson instanton, which is topologically the cotangent bundle of the sphere, T^*S^2. It would be very interesting to find other regular solutions of eq. (4.7) and identify the resulting 4-d Euclidean geometries. We point out that the multi-center generalization of the Eguchi-Hanson instanton due to Gibbons and Hawking, is of a different nature and does not fit our purposes. In their work, the metric also depends on a scalar function $V(\vec{x}) = V(x_1, x_2, x_3)$, but the self-duality condition (4.9) is equivalent to the Laplace equation in $\vec{x}$-space. However, it is interesting to note that there exist 4-d Kähler metrics which are Ricci scalar flat (but not necessarily Ricci flat) and interpolate between the hyper-Kähler class (4.8) and that of Gibbons and Hawking, [27]. We think that a better understanding of these issues will clarify further the role of $sl(\infty)$-Toda theory in geometry.

Despite their limitations, the results we have presented so far are quite suggestive. Since W_∞ is a quantum deformation of the algebra (2.1), it is reasonable to expect that area preserving diffeomorphisms are intimately related to self-dual Einstein equations in four dimensions. Such relations might appear to be somewhat limited to the continual Toda equation (4.7), because the metrics we are considering here are rather special and admit a Killing symmetry. However, our observation has more general value and fits nicely with Penrose's theory of non-linear gravitons, [28]. The twistor construction of self-dual Einstein spaces shows quite explicitly that area preserving diffeomorphisms are fundamental in the general theory of hyper-Kähler 4-manifolds. Further details can be found in Penrose's original work and in ref. [25, 29]. Recently, a link has also been found between curved twistor theory and N=2 string theory in 4-d with space-time signature (2,2), [30]. In fact, this particular string theory defines a consistent quantum theory whose classical analog is self-dual gravity. Clearly, it would be very interesting to study the quantum analog of classical symmetries associated with self-dual Einstein equations. It remains to be seen whether the deformed structure described by W_∞ is a vital symmetry in the quantum version of Euclidean (or (2,2)) gravity.

4.3 **Multi-matrix models**: In the remaining of this section we will adopt some ideas which were introduced recently in connection with the double scaling limit of $(N-1)$-matrix models and W_N algebras. Witten has conjectured that topological W_N gravity is equivalent to the $(N-1)$-matrix model. W_N-gravity is defined in the context of topological field theories, using the moduli space of flat $SL(N,R)$ connections

$$\mathcal{M}_N = Hom(\pi_1(\Sigma), SL(N,R))/SL(N,R) \tag{4.11}$$

over a Riemann surface Σ, [4]. Part of his motivation was provided by the existing relations between hermitian (multi)-matrix models and the generalized KdV hierarchies of non-linear differential equations of $sl(N)$-type. Subsequently, it has been shown that the loop equations of multi-matrix models admit a group theoretic interpretation as highest weight conditions for W_N, [6]. In this framework, the large N limit of W-algebras, W_∞, arises when the embedding space of the associated string model becomes one-dimensional.

The complex dimension of the moduli space $\mathcal{M}_N$ is $(g-1) dim\, sl(N)$ (for genus $g \geq 2$) and has a definite meaning in the BRST formulation of W_N-symmetries, [31]. Recall that for each algebra generator $W^s(z)$ (with $s = 2, 3, \cdots, N$), we introduce a ghost c_{1-s} and its conjugate field b^s with scaling dimensions $1-s$ and s respectively. Then, the Riemann-Roch theorem states that on a genus g Riemann surface,

$$(\#\text{ of zero modes of } b^s) - (\#\text{ of zero modes of } c_{1-s}) = (g-1)(2s-1) \ . \tag{4.12}$$

Summing up the contribution in (4.12) from all spins, we find that the net number of b, c zero modes (ie. moduli) for the W_N algebra is $(g-1)(N^2-1)$ which is precisely the dimension of $\mathcal{M}_N$. This counting provides a field theoretic justification for associating the moduli spaces (4.11) with W-symmetries.

The geometric interpretation of the moduli spaces $\mathcal{M}_N$ is rather obscure for $N \geq 3$, because higher order differentials (cubic, etc.) are not related to any known deformations in the classical theory of Riemann surfaces. Only for $N = 2$ we obtain the Teichmüller space which describes the inequivalent complex structures on Σ. However, the situation becomes rather interesting as $N \to \infty$.

Witten and Hitchin have argued independently, [4, 32] that $\mathcal{M}_\infty$ can be regarded as the moduli space of complex structures on $T^*\Sigma$. It is intriguing that the same spaces have been conjectured to provide a consistent background for the propagation of $N = 2$ strings, [30]. Once again it is interesting to notice that the values $N = 2$ and $N = \infty$, which correspond to the Virasoro and W_∞ algebras respectively, have a special meaning in geometry. This should be compared with our previous result in Toda field theory, which also suggests that a W_∞ structure in two dimensions has a natural geometric interpretation in four dimensions. We think that there is a direct link between the symplectic structure of the moduli spaces (4.11), [33] and $sl(N)$-Toda theory for all $N \geq 2$, which is party responsible for these analogies. Work in this direction is in progress with D. J. Smit.

In any case, the topological field theory description of multi-matrix models in terms of W_N-gravity, provides an alternative framework for studying (toy) string theories in less than one dimensions non-perturbatively. For finite N, the embedding space is discrete and its points correspond to the vertices of the Dynkin diagram of $sl(N)$. However, as $N \to \infty$, the model becomes a $D = 1$ string theory and for this purpose it is more convenient to think of $sl(\infty)$ as a continual algebra, whose root system is a smooth 1-d manifold. In this limit, the KP hierarchy provides (at least formally) a universal KdV system of non-linear differential equations for the model. However, the string equation is ill defined for an infinitely long chain of matrices and further work is required before this program is brought to completion. In view of Witten's suggestion, we think that the geometric interpretation of W_∞ structures should be taken into account seriously in search of a solution.

5. Conclusions

The complete structure of W_∞ is described as a linear deformation of the area preserving diffeomorphism algebra, which also admits central terms in the (quantum) commutation relations of all higher spin fields. Using the theory of Z_∞ parafermions, we have shown that the structure suggested by Pope, Romans and Shen is in fact identical to W_∞. As a byproduct, we derived all unitary irreducible hw representations of this algebra, which arise as smooth limits of CFT representations at large N. However, we do not know whether the list is complete. It would very interesting to find field theoretic representations of W_∞ which are naturally related to physical models in higher dimensions. In this case, W_∞ would have a direct interpretation in terms of operator algebras of (possibly conformal) quantum field theories in $d > 2$.

In ref. [16] we also introduced W_∞^p as a $U(p)$-matrix generalization of the universal algebra W_∞. Its commutation relations were obtained in closed form for all values of p and W_∞ was identified with the $U(1)$ (trace) part of W_∞^p. It turns out that the large p limit of W_∞^p is associated with the algebra of symplectic diffeomorphisms in four dimensions. For this reason, W_∞^∞ deserves a thorough study.

Finally, the action of W_∞-gravity, as formulated in [34], provides a master theory which yields by truncation all W_N-gravity theories. We think that more work on this subject will clarify further the role of universal W-algebras in quantum field theory. Also, the relations between W_∞ and the theory of integrable non-linear differential equations, in particular self-dual Einstein equations in 4-d, suggest that there are some very interesting geometrical structures underlying the non-perturbative solution of $D = 1$ string theory. These problems pose challenging questions which we hope to address elsewhere.

References

[1] E. Brezin and V. Kazakov, Phys. Lett. B236 (1990) 144; M. R. Douglas and S. H. Shenker, Nucl. Phys. B335 (1990) 635; D. J. Gross, A. A. Migdal, Phys. Rev. Lett. 64 (1990) 127; M. R. Douglas, Phys. Lett. B238 (1990) 176.

[2] E. Brezin, V. Kazakov and Al. B. Zamolodchikov, Nucl. Phys. B338 (1990) 673; G. Parisi, Phys. Lett. B238 (1990) 209, 213; G. Parisi, Europhys. Lett. 11 (1990) 595; D. J. Gross and N. Miljković, Phys. Lett. B238 (1990) 217; P. Ginsparg and J. Zinn-Justin, "*2-d Gravity + 1-d Matter*", Harvard preprint, HUTP-90/A004 (January 1990).

[3] M. F. Awada and Z. Qiu, "*Non-Perturbative Higher Spin Gravity in Two Dimensions*", U. of Florida preprint, UFIFT-90-2, (1990); "*Non-Perturbative 2-d Quantum Gravity and the Generalized KdV Hierarchy*", U. of Florida preprint, UFIFT-90-4, (1990).

[4] E. Witten, "*Surprises with Topological Field Theories*", Princeton preprint, IASSNS-HEP-90/37, (1990).

[5] A. B. Zamolodchikov, Theor. Math. Phys. 65 (1985) 1205; A. B. Zamolodchikov and V. A. Fateev, Nucl. Phys. B280[FS18] (1987) 644; V. A. Fateev and S. L. Lykyanov, Int. J. Mod. Phys. A3 (1988) 507.

[6] M. Fukuma, H. Kawai and R. Nakayama, "*Continuum Schwinger-Dyson Equations and Universal Structures in Two-Dimensional Quantum Gravity*", KEK preprint, KEK-TH-251, (1990); R. Dijkgraaf, H. Verlinde and E. Verlinde, "*Loop Equations and Virasoro Constraints in Non-Perturbative 2-d Quantum Gravity*", Princeton preprint, IASSNS-HEP-90/48, (1990).

[7] K. Wilson, Phys. Rev. 179 (1969) 1499.

[8] E. S. Fradkin, M. Ya. Palchik, Phys. Rep. 44 (1978) 249; G. Mack in "*Non-Perturbative Quantum Field Theory*", edited by G. t'Hooft et al., Plenum, New York (1988).

[9] I. Bakas, Phys. Lett. B228 (1989) 57; I. Bakas, "*Area Preserving Diffeomorphisms and Higher Spin Fields in Two Dimensions*", to appear in the proceedings of the Trieste Conference on "Supermembranes and Physics in 2+1 dimensions", edited by M. Duff et al., World Scientific (1990); I. Bakas, "*The Structure of the W_∞ Algebra*", to appear in Comm. Math. Phys.

[10] G. Segal, in these proceedings.

[11] V. A. Fateev and S.L. Lykyanov, Kiev preprints, ITP-88-74,75,76 (1988).

[12] A. Bilal, Phys. Lett. B227 (1989) 406; A. Morozov, "*On the Concept of Universal W-Algebra*", ITEP preprint, ITEP/148-89, (1989).

[13] C. Pope, L. Romans and X. Shen, Phys. Lett. B236 (1990) 173; Nucl. Phys. B339 (1990) 191.

[14] C. Pope, L. Romans and X. Shen, Phys. Lett. B242 (1990) 401.

[15] E. Bergshoeff, C. Pope, L. Romans, E. Sezgin and X. Shen, "*The Super-W_∞ Algebra*", CERN preprint, CERN-TH.5730/90, (1990).

[16] I. Bakas and E. Kiritsis, "*Bosonic Realization of a Universal W-Algebra and Z_∞ Parafermions*", to appear in Nucl. Phys. B; "*Grassmannian Coset Models and Unitary Representations of W_∞*", to appear in Mod. Phys. Lett. A.

[17] A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. JETP 62 (1985) 215; D. Gepner and Z. Qiu, Nucl. Phys. B285[FS19] (1987) 423.

[18] G. Segal and G. Wilson, Publ. Math. IHES 61 (1985) 5.

[19] A. Leznov and M. Saveliev, Comm. Math. Phys. 74 (1980) 111; P. Mansfield, Nucl. Phys. B208 (1982) 277.

[20] A. Bilal and J.-L. Gervais, Phys. Lett. B206 (1988) 412.

[21] M. Saveliev, Comm. Math. Phys. 121 (1989) 283; M. Saveliev and A. Vershik, Comm. Math. Phys. 126 (1989) 387; Phys. Lett. A143 (1990) 121; R. Kashaev, M. Saveliev, S. Savelieva and A. Vershik, "*On Non-Linear Equations Associated with Lie Algebras of Diffeomorphism Groups of 2-d Manifolds*", Serpukhov preprint, IHEP-90-1, (1990).

[22] V. G. Kač, "*Infinite Dimensional Lie Algebras*", Cambridge University Press, Cambridge, (1985).

[23] J. Hoppe, Phys. Lett. B215 (1988) 706; E. Floratos, J. Iliopoulos and G. Tiktopoulos, Phys. Lett. B217 (1989) 285; D. Fairlie, P. Fletcher and C. Zachos, Phys. Lett. B218 (1989) 203; D. Fairlie and C. Zachos, Phys. Lett. B224 (1989) 101; C. Pope and K. Stelle, Phys. Lett. B226 (1989) 257.

[24] C. Boyer and J. Finley, J. Math. Phys. 23 (1982) 1126; J. Gegenberg and A. Das, Gen. Rel. Grav. 16 (1984) 817; R. S. Ward, Class. Quant. Grav. 7 (1990) L95.

[25] Q-Han Park, Phys. Lett. B236 (1990) 429; Phys. Lett. B238 (1990) 287.

[26] T. Eguchi, P. Gilkey and A. Hanson, Phys. Rep. 66 (1980) 213.

[27] C. LeBrun, "*Explicit Self-Dual Metrics on* $CP_2\#\cdots\#CP_2$", Princeton preprint, (1989).

[28] R. Penrose, Gen. Rel. Grav. 7 (1976) 31.

[29] C. Boyer and J. Plebanski, J. Math. Phys. 26 (1985) 229; K. Takasaki, J. Math. Phys. 30 (1989) 1515; ibid. 31 (1990) 1877.

[30] H. Ooguri and C. Vafa, Mod. Phys. Lett. A5 (1990) 1389.

[31] J. Thierry-Mieg, Phys. Lett. B197 (1987) 368; K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Comm. Math. Phys. 124 (1989) 87.

[32] N. Hitchin, talk delivered at the Yukawa International Seminar "Common Trends in Mathematics and Quantum Field Theories", Kyoto, (1990).

[33] P. G. Zograf and L. A. Takhtadzhyan, Math. USSR Sbornik 60 (1988) 143; W. Goldman, Adv. Math. 54 (1984) 200.

[34] C. Hull, Phys. Lett. B240 (1990) 110; E. Bergshoeff, C. Pope, L. Romans, E. Sezgin, X. Shen and K. Stelle, Phys. Lett. B243 (1990) 350; E. Bergshoeff, C. Pope, L. Romans, E. Sezgin and X. Shen, "*W_∞ Gravity and Super-W_∞ Gravity*", CERN preprint, CERN-TH.5743/90, (1990); K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, "*Covariant Formulation of Classical W-Gravity*", Stony Brook preprint, ITP-SB-90-26, (1990); "*Covariant W_∞ Gravity and its Reduction to W_N Gravity*", Stony Brook preprint, ITP-SB-90-39, (1990).