TOPICS IN CONFORMAL FIELD THEORY

Thesis by

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ABSTRACT

In this work two major topics in Conformal Field Theory are discussed. First a detailed investigation of N=2 Superconformal theories is presented. The structure of the representations of the N=2 superconformal algebras is investigated and the character formulae are calculated. The general structure of N=2 superconformal theories is elucidated and the operator algebra of the minimal models is derived. The first minimal system is discussed in more detail. Second, applications of the conformal techniques are studied in the Ashkin-Teller model. The $c = 1$ as well as the $c = \frac{1}{2}$ critical lines are discussed in detail.

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INTRODUCTION

In the past five years there was a renewed interest in string theories as candidates for a unified theory of nature, [1]. This came as no surprise in an era where the theoretical ideas of the subject seemed to be inadequate for further advancement. On the other hand string theory seemed to provide new solutions to problems we did not know how to attack before. It remains to be seen if string theory is the theory that describes Nature. But even if the answer is negative it is certainly true that now we understand Quantum Field Theory much better than we did five years ago.

The connection between string theories and two-dimensional critical models is well known. Any two-dimensional critical model with the appropriate local symmetry and central charge is a classical ground state for string theory. The fact above was responsible for the great interest in 2-d critical models in recent years and the emergence of new ideas and techniques in order to classify and solve them. In the light of string theory the classification problem is quite important. Knowing all conformal field theories (CFT) in 2-d is equivalent, as mentioned above, to knowing the classical vacua of string theories. Of course this knowledge is not the whole story since it is expected that only a subclass of them will be stable under quantum fluctuations. At the present status of string theory our knowledge of taking account of quantum fluctuations in a field theoretic way is very limited, and as far as our calculational tools are concerned we are in an even worse situation. It is conceivable though that a knowledge of all string vacua will enable us to see if string theory has anything to do with the real world. Because imagine that in our complete list of classical string vacua we find none with a spectrum that resembles the real world. Then it is very hard to see how one can reconcile the theory with basic experimental facts (spectrum). In the opposite case one will be pushed to investigate more closely vacua with properties in accord with Nature.

From the view-point of condensed matter physics the problems of 2-d critical phenomena discussed above are fundamental. There are two reasons for one being interested in the critical behaviour of 2-d models. First, there are a lot of situations in real life where the system under study is a 2-d one, (e.g. surface behaviour). Second it is well known by now that 2-d critical phenomena possess the richest structure compared with higher dimensional ones.

In the last five years there emerged a new approach to 2-d critical phenomena which proved to be very powerful and illuminating at the same time. We will refer to this approach with the name Conformal Field Theory. It was introduced by Belavin, Polyakov and Zamolodchikov, [2], in an attempt to introduce both as a principle and as a tool the group theoretic structure of conformal symmetry. The main hope which becomes more plausible as time goes by is the use of the representation theory of the conformal group and its extensions as well as elements of 2-d geometry as tools to classify and solve all universality classes of critical behaviour in 2-d.

Since its introduction, CFT has advanced considerably and has been recognized as a valuable tool both in string theory and in critical phenomena. Now more than ever it seems that a classification of 2-d CFTs is not a hopelessly difficult task. It also gave the biggest collection of exactly solvable 2-d models that we know so far.

In this thesis I will try to present my own contribution to the subject. In chapter 1 the basic principles and tools of CFT are presented in a way which (hopefully) will make the rest more intelligible to the non-expert. In chapter 2 various aspects of N=2 superconformal symmetry are discussed as well as its relevance for realistic string compactification. Chapter 3 contains some applications to condensed matter systems, in particular the Askin-Teller model. Some ideas pertaining on non-standard bosonization techniques are also presented.

This thesis is based on both published and unpublished work of the author. The published work on the subject has been presented in references [3,4,5,6,7,8].

CHAPTER 1

Introduction to Conformal Field Theory

1.1 Conformal Symmetry and Ward Identities

Conformal symmetry was introduced in quantum field theory inspired by certain scaling ideas in the theory of second order phase transitions, [9]. The basic hypothesis was based on the idea that the physics of the systems at the critical point was invariant under scalings of the system. In terms of coordinates, $\xi^a \to \lambda\xi^a$. Such a transformation is a symmetry if the stress-energy tensor is traceless,

$$T^a_a(\xi) = 0 \tag{1.1.1}$$

If the condition above is true then one can show that the system not only possesses the aforementioned scaling symmetry but it is also invariant under coordinate transformations, $\xi^a \to \eta^a(\xi)$, which have the property that the metric tensor transforms as,

$$g_{ab} \to \frac{\partial\xi^{a'}}{\partial\eta^a}\frac{\partial\xi^{b'}}{\partial\eta^b}g_{a'b'} = \rho(\xi)g_{ab} \tag{1.1.2}$$

Such transformations constitute the conformal group. The condition for an infinitesimal coordinate transformation of the form $\xi^a \to \xi^a + f^a(\xi)$ to have the property (2) turns out to be,

$$\partial_a f_b + \partial_b f_a = \frac{2}{d}\delta_{ab}\partial^c f_c \tag{1.1.3}$$

where d is the dimension of space-time.

In the generic case, $d \neq 2$, only a finite number of solutions exist for (1.1.3). This can be easily seen by rewriting (1.1.3) in the more suggestive form,

$$(\delta_{ab}\Box + (d-2)\partial_a\partial_b)\partial^c f_c = 0 \tag{1.1.4}$$

which implies that for $d \neq 2$ f_c must be at most quadratic in ξ^a. Thus the conformal group in $d > 2$ dimensions is finite dimensional consisting of translations, rotations, dilatations and special conformal transformations. But in $d = 2$ equation (1.1.3) becomes the Cauchy-Riemann equations. Thus any meromorphic function is a solution. The conformal group in this case is infinite dimensional. It is generated by the components of the stress energy tensor.

From now on we restrict ourselves to the two-dimensional case. We will also work in Euclidean space where both statistical mechanics and quantum field theory are well defined. Along with conformal invariance one needs to assume a strong version of the Operator Product Expansion (OPE): Assume that there exists an infinite set of local fields $\phi_i(\xi)$.Then the set of operators $\phi_i(0)$ is assumed to be complete in the following sense. The set $[\phi_i]$ contains the identity operator I as well as all coordinate derivatives of local fields. The completeness of the set $[\phi_i]$ means that any state can be generated by the linear action of these operators. This is equivalent to the OPE:

$$\phi_i(\xi)\phi_j(0) = \sum_k C_{ij}^k(\xi)\phi_k(0) \tag{1.1.5}$$

The structure constants $C_{ij}^k(\xi)$ are c-number functions which are single valued. The previous relations are understood as an exact expansion of the correlation functions,

$$\langle \phi_{i_1}(\xi_1)\phi_{i_2}(\xi_2)\cdots\phi_{i_n}(\xi_n)\rangle = \sum_k C_{i_1 i_2}^k(\xi_1-\xi_2)\langle\phi_k(\xi_2)\cdots\phi_{i_n}(\xi_n)\rangle \tag{1.1.6}$$

which converges in some finite domain of ξ, dependent on the positions ξ_i. The most restrictive requirement is the associativity of the operator algebra (1.1.5). This gives an infinite number of equations for the structure functions $C_{ij}^k(\xi)$. Conformal symmetry fixes the form of the structure functions up to numerical parameters. Then these equations should determine these parameters. For $d > 2$ the system is too complicated due to the difficulty of classifying the fields participating in the algebra. In $d = 2$ the situation is tractable. The conformal group is infinite dimensional and the operators can be classified successfully by the irreducible representations of the group.

In order to describe the group we will choose complex coordinates z and $\bar{z}$ (in Minkowski space they correspond to light cone coordinates).

$$z \equiv \xi^1 + i\xi^2 \ , \quad \bar{z} \equiv \xi^1 - i\xi^2 \tag{1.1.7}$$

From now on we will restrict our attention to the flat Euclidean space. The results however can be generalized to the most general case and we will have to say more in subsequent chapters. The metric is written as, $ds^2 = dzd\bar{z}$. In these coordinates a conformal transformation becomes an analytic transformation,

$$z \rightarrow \zeta(z) \ , \quad \bar{z} \rightarrow \bar{\zeta}(\bar{z}) \tag{1.1.8}$$

where ζ, $\bar{\zeta}$ are arbitrary analytic functions. It will be useful to consider the transformations (1.1.8) as independent and thus the conformal group G will be the direct product, $G = \Gamma \otimes \bar{\Gamma}$, where Γ ($\bar{\Gamma}$) is the group of analytic (anti-analytic) transformations.

An infinitesimal transformation of the group Γ is, $z \rightarrow z + \epsilon(z)$, where $\epsilon(z)$ is an arbitrary infinitesimal meromorphic function. If we represent it in terms of its Laurent series $\epsilon(z) =$

$\sum_{-\infty}^{\infty} \epsilon_n z^{n+1}$ then the Lie algebra of Γ coincides with the algebra of differential operators $l_n = z^{n+1}\partial_z$. The commutation relations are,

$$[l_n, l_m] = (n-m)l_{m+n} \tag{1.1.9}$$

The generators $l_{-1},$, l_0, l_1 generate a subalgebra $sl(2,C)$. The corresponding subgroup consists of the projective transformations,

$$z \to \zeta = \frac{az+b}{cz+d}, \quad ad - bc = 1 \tag{1.1.10}$$

An important operator in a theory is the stress-energy tensor. It is defined as the variation of the action with respect to the metric, (whenever there exists an action), $T^{ab} \equiv \frac{\delta S}{\delta g_{ab}}$.

Let's consider an arbitrary correlation function of the form,

$$\langle X \rangle = \langle \phi_{i_1}(\xi_1) \cdots \phi_{i_n}(\xi_n) \rangle \tag{1.1.11}$$

The fields in the correlation function are local fields. Let's now perform a coordinate transformation, $\xi^a \to \xi^a + \epsilon^a(\xi)$. We can derive the appropriate Ward identity this way which reads,

$$\sum_{k=1}^{n} \langle \phi_{i_1}(\xi_1) \cdots \delta_\epsilon \phi_{i_k}(\xi_k) \cdots \phi_{i_n}(\xi_n) \rangle + \int d^2\xi \partial^a \epsilon^b(\xi) \langle T^{ab}(\xi) X \rangle = 0 \tag{1.1.12}$$

where $\delta_\epsilon \phi_i$ denotes the variation of the local field under the coordinate transformation. A corollary of (1.1.12) is the conservation of the stress-energy tensor,

$$\partial_a \langle T^{ab}(\xi) X \rangle = 0 \tag{1.1.13}$$

everywhere except at ξ_i. In a conformally invariant theory the trace of the stress-energy tensor vanishes. Combining relations (1.1.1) and (1.1.13) we obtain,

$$\partial_{\bar{z}} \langle T(\xi) X \rangle = 0 \,, \quad \partial_z \langle \bar{T}(\xi) X \rangle = 0 \tag{1.1.14}$$

where,

$$T \equiv T_{11} - T_{22} + 2iT_{12} \,, \quad \bar{T} \equiv T_{11} - T_{22} - 2iT_{12} \tag{1.1.15}$$

In view of (1.1.14) we can write $T = T(z)$, $\bar{T} = \bar{T}(\bar{z})$. The correlation function $\langle T(z) X \rangle$ is a meromorphic function of z which is single valued and regular everywhere except at the points

z_i where it has poles. The Ward identity (1.1.12) for a holomorphic coordinate transformation becomes,

$$\langle \delta_\epsilon X \rangle = \oint_C d\zeta \langle T(\zeta) X \rangle \tag{1.1.16}$$

where the contour C encloses all singularities z_i of the correlation function. Thus we can write the following relation for the variation of a local field under a holomorphic transformation $z \to z + \epsilon(z)$,

$$\delta_\epsilon \phi_i(z, \bar{z}) = \oint_{C_i} d\zeta \epsilon(\zeta) T(\zeta) \phi_i(z, \bar{z}) \tag{1.1.17}$$

The same arguments are valid for anti-holomorphic transformations.

The transformation properties of $T(z)$ are important since they are related to the realization of the algebra of conformal transformations in the quantum theory. The following theorem is due to Mack and Lüscher*, [10]:

Theorem: In a local relativistic quantum field theory (satisfying the Wightman axioms) let the stress-energy tensor be symmetric, conserved (absence of gravitational anomalies) and traceless (scale invariance). Let also the stress-energy tensor be dilation covariant and the vacuum state be dilatation invariant,

$$U(\rho) T_{ab}(\xi) U^{-1}(\rho) = \rho^2 T_{ab}(\rho\xi)\,,\ U(\rho)|0\rangle = |0\rangle \tag{1.1.18}$$

then,

$$\delta_\epsilon T(z) = \epsilon(z)\partial_z T(z) + 2\partial_z \epsilon(z) T(z) + \frac{c}{12}\partial_z^3 \epsilon(z) \tag{1.1.19}$$

One defines a quantum field theory in 2-d space (σ,τ) and imposes periodicity requirements in the space direction σ. Then we can go from the cylinder to the complex plane by means of the transformation $z = exp(\tau + i\sigma)$. The correlation functions in the (σ, τ) space can be defined through time ordering in the "time" τ. In the operator formalism the variations $\delta_\epsilon \phi_i$ can be expressed in terms of equal time commutators,

$$\delta_\epsilon \phi_i(\sigma, \tau) = [T_\epsilon, \phi_i(\sigma, \tau]\ ,\ T_\epsilon \equiv \oint_{log|z|=\tau} \epsilon(z) T(z) dz \tag{1.1.20}$$

Then relation (1.1.19) becomes,

$$[T_\epsilon, T(z)] = \epsilon(z) T'(z) + 2\epsilon'(z) T(z) + \frac{c}{12}\epsilon'''(z) \tag{1.1.21a}$$

⋆ A more general theorem is in fact true. In any 2-d theory that satisfies the Mack-Lüscher assumptions and also has a continuous global symmetry or global supersymmetry then this symmetry is automatically local, [10] or in OPE form,

$$T(z)T(w) = \frac{1}{2}\frac{c}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots \qquad (1.1.21b)$$

It is convenient to expand $T(z)$ in a Laurent series,

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad (1.1.22)$$

Then using (1.1.21) one can derive the commutation relations of the operators L_n which are the generators of the holomorphic part of the conformal group.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \qquad (1.1.23)$$

The same commutation relations are valid for the antiholomorphic generators $\bar{L}_n$. The algebra of equation (1.1.23) is known as the Virasoro algebra. It contains $sl(2,C)$ as a subalgebra. The operators L_{-1}, $\bar{L}_{-1}$ generate translations on the complex plane whereas L_0, $\bar{L}_0$ generate dilatations of z, $\bar{z}$. In the (σ,τ) coordinates $L_0+\bar{L}_0$ generates time translations and consequently it is the Hamiltonian. The infinite past ($\tau = -\infty$) and the infinite future ($\tau = \infty$) correspond to the points $z = 0$ and $z = \infty$ on the complex plane.

The Cartan subalgebra of the Virasoro algebra is generated by L_0. Its eigenvalues Δ (holomorphic critical dimensions) classify the irreducible representations. The raising operators are L_n, $n = 1, 2, 3, \ldots$. The unique vacuum state of the theory corresponds to the identity operator and has zero L_0 eigenvalue. It is also a highest weight vector (hwv) of the algebra, that is it is annihilated by the raising operators,

$$L_n|0>= 0\,, \quad n = 0, 1, 2, 3, \ldots \qquad (1.1.24)$$

Using (1.1.23) and (1.1.24) we can show that the vacuum state is also annihilated by L_{-1}. This the maximal subset of the conformal group generators that can annihilate the vacuum. The fact that we cannot impose a bigger set of generators to annihilate the vacuum is due to the non-zero central element c in the algebra (1.1.23). This fact can be cast in conventional field theoretic terms as "the full conformal invariance of the theory is spontaneously broken". The statement above reflects the fact that the vacuum state is not invariant under the full conformal group but the conformal Ward identities are still valid. However this interpretation should be used with care. As already explained above the $sl(2,C)$ symmetry is still manifest.

In order to define Hermitian conjugation we have to remind ourselves that the matrix elements are evaluated between the "in" ($z = 0$) and the "out" ($z = \infty$) states. These are related by $z \to \frac{1}{z}$. From this and the reality of $T(z)$ it can be inferred that,

$$L_n^\dagger = L_{-n} \tag{1.1.25}$$

Using the commutation relations (1.1.23) along with (1.1.25) one can evaluate any correlation function involving stress-energy tensors only. For example,

$$\langle T(z_1)T(z_2)\rangle = \frac{c}{2}\frac{1}{(z_1-z_2)^4} \tag{1.1.26}$$

Reflection positivity (unitarity in Minkowski space) and (1.1.26) imply that $c \geq 0$.

The states in the Hilbert space of the theory are generated by local operators acting on the vacuum state,

$$|\phi_i\rangle \equiv \phi_i(0)|0\rangle \tag{1.1.27}$$

The representations of the conformal group are generated by hwvs (primary fields). The whole representation is generated by the action of the lowering operators of the algebra on the hwvs. Consider a hwv $|\Delta\rangle$. It satisfies the usual hwv conditions,

$$L_n|\Delta\rangle = 0\,,\ \ n > 0\,,\ \ L_0|\Delta\rangle = \Delta|\Delta\rangle \tag{1.1.28}$$

These are equivalent to the commutation relations ($|\Delta\rangle \equiv \phi_\Delta(0)|0\rangle$),

$$[L_m, \phi_\Delta(z)] = z^{m+1}\partial_z\phi_\Delta(z) + \Delta(m+1)z^m\phi_\Delta(z) \tag{1.1.29}$$

or to the OPE,

$$T(z)\phi_\Delta(w) = \Delta\frac{\phi_\Delta(w)}{(z-w)^2} + \frac{\partial_w\phi_\Delta(w)}{(z-w)} + \cdots \tag{1.1.30}$$

where the dots in (1.1.30) denote terms regular as $z \to w$. The operators appearing in the regular terms are the descendants of the hwv under the action of the lowering operators. In the Hilbert space language an arbitrary state in the representation generated by $|\Delta\rangle$ is of the form,

$$|\Delta, (k_i)\rangle \equiv (L_{-1})^{k_1}(L_{-2})^{k_2}(L_{-3})^{k_3}\cdots|\Delta\rangle \tag{1.1.31}$$

These states constitute a basis in the representation. They are not orthogonal in general but they are linearly independent (modulo a subtlety which will be discussed later).

So far we neglected the existence of the anti-holomorphic Virasoro operators $\bar{L}_n$. But it is quite easy to take them into account due to the fact that the conformal group is simply a direct product of the holomorphic and anti-holomorphic factors. Thus a hwv is characterized by the two eigenvalues Δ and $\bar{\Delta}$ of L_0 and $\bar{L}_0$. Then the whole representation is generated by the action on the hwv of the lowering operators L_{-n} and $\bar{L}_{-n}$. In a few words the representations of the full conformal group are tensor products of representations of its left and right components. The physical dimension of an operator is given by $\Delta + \bar{\Delta}$ and its "spin" by $\Delta - \bar{\Delta}$.

Using the information above we can derive the conformal Ward identities. An important ingredient is the fact that a meromorphic function on the Riemann sphere is determined by its singularities and the corresponding residues. Thus let's consider correlation functions of primary fields with an insertion of the stress-energy tensor, $T(z)$. We will view it as meromorphic function of z. Then we know its singularities and residues from (1.1.30) so that,

$$\langle T(z)\phi_1(z_1)\cdots\phi_n(z_n)\rangle = \sum_{i=1}^{n}\left(\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{(z-z_i}\frac{\partial}{\partial z_i}\right)\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle \tag{1.1.32}$$

The Ward identity (1.1.32) is important to determine the correlation functions of the descendants of the primary fields. From (1.1.31) they are defined by modes of $T(z)$ acting on the primary fields. Thus we can use (1.1.32) in order to determine their correlation functions. We can also derive the projective Ward identities which illustrate the fact that $sl(2,C)$ is an exact symmetry of the theory. From our previous discussion it become obvious that the generators L_{-1}, L_0, L_1 annihilate both the "in" and the "out" vacuum. We can isolate their action on the correlation functions by taking appropriate contour integrals in (1.1.32). This results in the following projective Ward identities.

$$\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\langle\phi_1(z_1)...\phi_n(z_n)\rangle = 0 \tag{1.1.33a}$$

$$\sum_{i=1}^{n}\left(z_i\frac{\partial}{\partial z_i} + \Delta_i\right)\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0 \tag{1.1.33b}$$

$$\sum_{i=1}^{n}\left(z_i^2\frac{\partial}{\partial z_i} + 2z_i\Delta_i\right)\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = 0 \tag{1.1.33c}$$

where all the fields in the correlation function are primary.

The constraints that (1.1.33) put on the correlation function are the following. The 2-point functions are fixed:

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{(z_1 - z_2)^{2\Delta_1}} \tag{1.1.34}$$

The 3-point functions are also fixed up to an overall constant:

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = C_{\Delta_1,\Delta_2,\Delta_3} \prod_{i<j}^{3} (z_i - z_j)^{-\Delta_{ij}} \tag{1.1.35}$$

where $\Delta_{12} = \Delta_1 - \Delta_2 - \Delta_3$ and so on. The general n-point function is constrained to be of the form,

$$\langle\phi_1(z_1)\cdots\phi_n(z_n)\rangle = \prod_{i<j}^{n} (z_i - z_j)^{\gamma_{ij}} G(x_{ij}^{kl}) \tag{1.1.36}$$

where the γ_{ij} are any solutions of $\sum_{j\neq i}\gamma_{ij} = 2\Delta_i$ and G is an arbitrary function of the $n-3$ anharmonic quotients, x_{ij}^{kl},

$$x_{ij}^{kl} = \frac{(z_i - z_j)(z_k - z_l)}{(z_i - z_l)(z_k - z_j)} \tag{1.1.37}$$

Thus all the non-trivial information of the theory is in the spectrum of critical dimensions Δ_i of the primary fields and the OPE coefficients C_{ij}^k.

1.2 Minimal Theories and Unitarity

In this section we will be discussing a special set of CFTs that contain representations of the conformal group which are "unusual". Such theories have a finite number of primary fields and are exactly solvable.

There are certain cases where the representations of the conformal group (Verma modules) as constructed above in (1.1.31), are not irreducible. This happens when one of the descendant states $|\Delta,(k_i)\rangle$ happens to have the properties of a hwv. Then one can show that such a state $|\chi\rangle$ is null, ($\langle\chi|\chi\rangle = 0$), and orthogonal to all the other states of the representation. Such a state generates another representation which is embedded in the previous one. Thus the true irreducible representation is obtained after discarding all such states and their descendants. Since in a unitary theory the Hilbert space is positive definite such a state is identically zero. This means in particular that any correlation function, where such a state is participating in, is zero. To give a concrete example consider a descendant state at level two,

$$|\chi\rangle = \left(L_{-2} + \kappa L_{-1}^2\right)|\Delta\rangle \tag{1.1.38}$$

In order for this to satisfy the hwv conditions (1.1.28) we must have,

$$\kappa = -\frac{3}{2(2\Delta+1)}, \quad 4\Delta + \frac{c}{2} + \frac{9\Delta}{2\Delta+1} = 0 \tag{1.1.39}$$

Assume that $c = \frac{1}{2}$, then Δ can take only two values satisfying (1.1.39), $\Delta = \frac{1}{2}$ or $\Delta = \frac{1}{16}$. Let's take $\Delta = \frac{1}{2}$ for concreteness. We would like to show that the existence of a null state implies extra constraints on the correlation functions of the theory. In particular, if in a theory all the primary fields are of this kind (that is their representations contain null vectors) then these constraints are enough to determine all the correlation functions. Such theories will be referred to as "minimal" and the corresponding representations as "degenerate". Let's now show how the null state $|\chi\rangle = \left(L_{-2} - \frac{3}{4}L_{-1}^2\right)|\frac{1}{2}\rangle$ implies constraints in the correlation functions. As it was argued before,

$$\langle 0|\phi_1(z_1)\cdots\phi_n(z_n)|\chi\rangle = \langle 0|\phi_1(z_1)\cdots\phi_n(z_n)\left(L_{-2} - \frac{3}{4}L_{-1}^2\right)\phi_{\frac{1}{2}}(0)|0\rangle = 0 \qquad (1.1.40)$$

On the other hand we can use the Ward identities (1.1.32) to move the Virasoro operators to the left, picking up on the way various terms and eventually annihilating the "out" vacuum. Thus we end up with a differential equation for the correlation function,

$$\left(\frac{3}{4}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\frac{\Delta_i}{(z-z_i)^2} - \sum_{i=1}^{n}\frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\right)\langle\phi_{\frac{1}{2}}(z)\phi_1(z_1)\cdots\phi_{n-1}(z_n)\rangle = 0 \qquad (1.1.41)$$

One can use the projective Ward identities to substitute the derivatives with respect to z_i with derivatives with respect to z so that (1.1.41) becomes an ordinary differential equation.

Another important issue is unitarity (positivity). This is the statement that the Hilbert space of the theory is positive definite . An important concept in the discussion of unitarity is the Kač determinant. This is an object that can be defined for every representation of the conformal group. As we mentioned before the representation is built by the action of the lowering operators on a hwv. We will define the level of a descendant state $|i>$ as the eigenvalue of $L_0 - \Delta$ on that state, (Δ is the dimension of the hwv). Then it is easy to show that states at different levels are orthogonal. Now consider the space of states at a given level n. Choose a basis in this space, for example the basis in (1.1.31) will do. Now consider the matrix, M_n, of all the inner products between states in this space, (such a matrix is known as the Shapovalov matrix in the mathematics literature). The determinant of M_n is the Kač determinant. It is a polynomial in two variables, the dimension of the hwv, Δ, and the central charge, c. Then the statement of unitarity becomes the statement that the Kač determinant has positive eigenvalues.

Null states can also be seen from the Kač determinant. If at least one of the eigenvalues of M_n is zero we can show that there is a null state at level n. The eigenvector of M_n, corresponding to the zero eigenvalue, is the null state. The corresponding Kač determinant vanishes at level n. Thus zeros of the Kač determinant signal the presence of null states. The Kač determinant can be evaluated. For example the Kač determinant of the conformal group was conjectured by V. Kač, [11], and proven by Feigin and Fuks, [12]. It is the following,

$$det(M_n) = \prod_{i=1}^{n}\left(\prod_{\substack{rs=i\\ r\leq s}} f_{r,s}(\Delta, c)\right)^{P(n-i)} \qquad (1.1.42)$$

where,

$$f_{r,s}(\Delta, c) = (\Delta - A^{+}_{r,s})(\Delta - A^{-}_{r,s})\,, \quad r, s \in N\,, \; r \neq s \tag{1.1.43}$$

$$f_{r,r}(\Delta, c) = \Delta + \frac{1}{24}(r^2 - 1)(c - 1) \tag{1.1.44}$$

$$A^{\pm}_{r,s} = \frac{1}{48}\left[(13 - c)(r^2 + s^2) \pm \sqrt{c^2 - 26c + 25}(r^2 - s^2) - 24rs - 2 + 2c\right] \tag{1.1.45}$$

and $P(n)$ is defined through,

$$\prod_{n=1}^{\infty} \frac{1}{(1 - z^n)} = \sum_{n=0}^{\infty} P(n) z^n \tag{1.1.46}$$

An analysis of unitarity in conformally invariant theories using the Kač determinant was performed by Friedan, Qiu and Shenker, [13]. They found that for $c \geq 1$ no constraint comes from the unitarity analysis. But for $c < 1$ unitary models exist only for special values of the central charge c,

$$c = 1 - \frac{6}{m(m+1)}\,, \quad m = 2, 3, 4, ... \tag{1.1.47}$$

The spectrum of critical dimensions can be found.

$$\Delta_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}\,, \; 1 \leq p \leq m - 1\,, \; 1 \leq q \leq p \tag{1.1.48}$$

In these models there is a finite number of primary fields all of which are degenerate. According to our previous discussion these models are exactly solvable since their correlation functions satisfy linear ordinary differential equations. The above constitutes a first step towards the classification of 2-d CFTs since it classifies all unitary CFTs with $c < 1$.

But for $c \geq 1$ there are no null states in the conformal algebra. Moreover other constraints on the theories (e.g. modular invariance) imply that the set of primary fields must be infinite, [14]. To circumvent such difficulties one has to introduce new ideas, in particular enlarging the conformal algebra. Imagine that the symmetry algebra of the theory is a bigger local algebra that contains the conformal algebra as a subalgebra. Then one expects that irreducible representations of that algebra will be (infinitely in general) decomposable in representations of the conformal algebra. Several examples of such larger algebras are known, local gauge algebras (Kač-Moody algebras), supersymmetric algebras, parafermionic algebras etc. Thus theories with an infinite number of conformal representations can have a finite number of the extended algebra representations.

It is probably not true that extended algebras are enough to put order in the vast space of CFTs. There are more ambitious ideas on how to attack this problem, but since they are currently under study we will refrain from saying anything more.

So far we discussed CFT on the Riemann sphere. It is natural to ask how much of this machinery carries over to CFT defined on more complicated 2-d surfaces. After all condensed matter systems are usually defined on a parallelogram with periodic boundary conditions and this is topologically a torus. In string theory perturbation theory à la Polyakov is defined as dealing with CFTs on Riemann surfaces with an arbitrary number of handles.

In a theory with conformal invariance details associated with the metric of the surface are redundant. Thus one is led to consider the surfaces modulo diffeomorphisms and conformal transformations. Compact 2-d surfaces are classified topologically by their number of handles (genus). For a given genus the surfaces are parametrized by a finite dimensional space called moduli space.

There are elements of what we said so far that will not change when we go to more complicated surfaces. In particular all the local properties of a theory will remain the same. Short distance singularities, the spectrum of critical exponents and the OPE coefficients will not change. But the Ward identities for example will change since we crucially used the fact that we were working on the sphere. Correlation functions will also change since they also carry global information. We do know though how to generalize the formalism of CFT to surface of arbitrary genus. Viewing a CFT as an object that can be defined on various surfaces seems to be a promising approach towards such goals as classification and solution of 2-d critical phenomena and/or string theory.

CHAPTER 2

The Structure of N=2 Superconformal Field Theories

2.1 Introduction

In this chapter we are going to discuss various aspects of $N = 2$ superconformal field theories.

As already mentioned in the first chapter if there are extra global (super)symmetries in a conformally invariant theory then the theory is invariant under a bigger local algebra that includes the conformal algebra as a subalgebra.

An interesting class of such global symmetries are supersymmetries. A supersymmetry is a symmetry that relates bosons and fermions. Various kinds of supersymmetry in 2-d are classified by the number of supercurrents. In 2-d we can define left and right supersymmetries separately. A model invariant under m left and n right supersymmetries will be called of the type (m,n). From now on our discussion will be focused on the left (holomorphic) part of a theory to avoid repetition. When eventually we have to make a model we will have to tensor appropriately the left and right parts in a way consistent with various constraints that we will discuss later.

The possible superconformal algebras in 2-d have been classified by Ramond and Schwarz⋆, [15]. The possibilities are:

$N = 1$ *Superconformal Algebra*. It is generated by the stress-energy tensor $T(z)$ and a dimension $\frac{3}{2}$ fermionic operator, the supercurrent $G(z)$. The algebra is given by the following OPEs:

$$T(z)T(w) = \frac{3}{4}\frac{\hat{c}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots \qquad (2.1.1a)$$

$$T(z)G(w) = \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{(z-w)} + \cdots \qquad (2.1.1b)$$

$$G(z)G(w) = \frac{\hat{c}}{(z-w)^3} + \frac{2T(z)}{(z-w)} + \cdots \qquad (2.1.1c)$$

where $\hat{c}$ is related to c in (1.1.23) by $\hat{c} = \frac{2}{3}c$ and from now on the $\cdots$ in OPEs will represent the non-singular terms as $z \to w$ (which do not contribute to the (anti)-commutation relations).

⋆ One may consider superconformal algebras for any N, [16]. The difference is that they do not, strictly speaking form an algebra. The commutation relations give expressions that are not linear in the algebra operators. However it is possible to use them in order to define CFTs.

$N = 2$ *Superconformal Algebra.* It is generated by the stress-energy tensor $T(z)$, a U(1) current $J(z)$ and two supercurrents $G(z)$, $\bar{G}(z)$.

$$T(z)T(w) = \frac{3}{2}\frac{\tilde{c}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots \tag{2.1.2a}$$

$$T(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + \cdots \quad , \quad J(z)J(w) = \frac{\tilde{c}}{(z-w)^2} + \cdots \tag{2.1.2b}$$

$$T(z)G(w) = \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{(z-w)} + \cdots \tag{2.1.2.c}$$

$$T(z)\bar{G}(w) = \frac{3}{2}\frac{\bar{G}(w)}{(z-w)^2} + \frac{\partial_w \bar{G}(w)}{(z-w)} + \cdots \tag{2.1.2d}$$

$$J(z)G(w) = \frac{G(w)}{(z-w)} + \cdots \quad , \quad J(z)\bar{G}(w) = -\frac{\bar{G}(w)}{(z-w)} + \cdots \tag{2.1.2e}$$

$$G(z)G(w) = 0 + \cdots \quad , \quad \bar{G}(z)\bar{G}(w) = 0 + \cdots \tag{2.1.2f}$$

$$G(z)\bar{G}(w) = \frac{2\tilde{c}}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} + \frac{2T(w)}{(z-w)} + \cdots \tag{2.1.2g}$$

where $c = 3\tilde{c}$.

$N = 3$ *Superconformal Algebra.* It is generated by the stress-energy tensor $T(z)$, three SU(2) currents $J^a(z)$, three supersymmetry generators $G^a(z)$ in the adjoint of SU(2) and an SU(2) singlet fermion field $\psi(z)$.

$$T(z)T(w) = \frac{3}{2}\frac{\kappa}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots \tag{2.1.3a}$$

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{(z-w)} + \cdots \tag{2.1.3b}$$

$$T(z)\psi(w) = \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial_w \psi(w)}{(z-w)} + \cdots \tag{2.1.3c}$$

$$T(z)G^a(w) = \frac{3}{2}\frac{G^a(w)}{(z-w)^2} + \frac{\partial_w G^a(w)}{(z-w)} + \cdots \quad , \quad J^a(z)\psi(w) = 0 + \cdots \tag{2.1.3d}$$

$$J^a(z)G^b(w) = i\epsilon^{abc}\frac{G^c(w)}{(z-w)} + \delta^{ab}\frac{\partial_w\psi(w)}{(z-w)} + \cdots \ , \ G^a(z)\psi(w) = \frac{J^a(w)}{(z-w)} + \cdots \tag{2.1.3e}$$

$$J^a(z)J^b(w) = i\epsilon^{abc}\frac{J^c(w)}{(z-w)} + \frac{\kappa\delta^{ab}}{(z-w)^2} + \cdots \ , \ \psi(z)\psi(w) = \frac{\kappa}{(z-w)} + \cdots \tag{2.1.3f}$$

$$G^a(z)G^b(w) = \frac{2\kappa\delta^{ab}}{(z-w)^3} + i\epsilon^{abc}\left(\frac{2J^c(w)}{(z-w)^2} + \frac{\partial_w J^c(w)}{(z-w)}\right) + 2\delta^{ab}\frac{T(w)}{(z-w)} + \cdots \tag{2.1.3g}$$

where $c = 3\kappa$ and κ takes the values $\kappa = \frac{n}{2}$, $n = 1, 2, 3, \cdots$

N = 4 Superconformal Algebra. It is generated by the stress-energy tensor $T(z)$, three SU(2) currents $J^a(z)$ and two SU(2) doublets of suppercurrents, $G^i(z)$ and $\bar{G}^i(z)$.

$$T(z)T(w) = \frac{6\kappa}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots \tag{2.1.4a}$$

$$T(z)G^i(w) = \frac{3}{2}\frac{G^i(w)}{(z-w)^2} + \frac{\partial_w G^i(w)}{(z-w)} + \cdots \tag{2.1.4b}$$

$$T(z)\bar{G}^i(w) = \frac{3}{2}\frac{\bar{G}^i(w)}{(z-w)^2} + \frac{\partial_w \bar{G}^i(w)}{(z-w)} + \cdots \tag{2.1.4c}$$

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{(z-w)} + \cdots \tag{2.4.d}$$

$$J^a(z)J^b(w) = i\epsilon^{abc}\frac{J^c(w)}{(z-w)} + \frac{\kappa\delta^{ab}}{(z-w)^2} + \cdots \tag{2.1.4e}$$

$$J^a(z)G^i(w) = \frac{1}{2}\sigma^a_{ij}\frac{G^j(w)}{(z-w)} + \cdots \ , \ J^a(z)\bar{G}^i(w) = -\frac{1}{2}\sigma^a_{ji}\frac{\bar{G}^j(w)}{(z-w)} + \cdots \tag{2.1.4f}$$

$$G^i(z)G^j(w) = 0 + \cdots \ , \ \bar{G}^i(z)\bar{G}^j(w) = 0 + \cdots \tag{2.1.4g}$$

$$G^i(z)\bar{G}^j(w) = \frac{8\kappa\delta^{ij}}{(z-w)^3} + 2\delta^{ij}\frac{T(w)}{(z-w)} + 2\sigma^a_{ji}\left(\frac{2J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{(z-w)}\right) + \cdots \tag{2.1.4h}$$

where $c = 12\kappa$, $\kappa = \frac{n}{2}$ $n = 1, 2, 3, \cdots$ and σ^a_{ij} are the standard Pauli matrices.

There are various sectors in the theories invariant under the superconformal algebras above. Their existence is linked to the possibility on imposing various boundary conditions on the algebra operators that respect the algebra structure. The stress-energy tensor will have to be always periodic otherwise conformal invariance will be broken. The algebras have in general a global automorphism group G which is represented on the operators $O_i(z)$ of the algebra by matrices $M_{ij}(g)$, $g \in G$. Then the different algebras are obtained by imposing the periodicity conditions,

$$O_i(z) = M_{ij}(g) O_j(e^{2\pi i} z) \tag{2.1.5}$$

Elements of G in the same conjugacy class give equivalent algebras. The algebras also contain local automorphisms due to the local gauge symmetries present (U(1) for N=2, SU(2) for N=3,4). Thus some of the boundary conditions introduced through twists of the global automorphisms can be removed by a local gauge transformation. So the truly independent algebras are generated by global automorphisms that are not contained in the local automorphisms. Such automorphisms are known in the mathematics literature as "outer" automorphisms. In the N=1 algebra there are no local automorphisms whereas the group of global automorphisms is isomorphic to Z_2. This is a fancy way of saying that there are two possible boundary conditions for the supercurrent, anti-periodic (NS algebra) and periodic (R algebra). These two algebras are inequivalent.

In the N=2 algebra the group of global automorphisms is O(2). The local automorphism group is SO(2) and the outer automorphism group is $O(2)/SO(2) = Z_2$. Thus there are two inequivalent N=2 algebras, [17]. The one which corresponds to the non-trivial element of Z_2 is "twisted" (the U(1) invariance is broken). The untwisted algebra is really a infinite set of locally equivalent algebras which are defined through the various boundary conditions of the supercurrents, (The U(1) current is periodic).

$$G(z) = e^{2\pi i\alpha} G(e^{2\pi i} z) \ , \ \bar{G}(z) = e^{-2\pi i\alpha} \bar{G}(e^{2\pi i} z) \tag{2.1.6}$$

where $0 \leq \alpha \leq 1$. The various algebras are specified by the value of α. They can be mapped onto each other through a local U(1) transformation as follows:

$$T_\alpha(z) = T_0(z) - iz\frac{df}{dz}T_0(z) - \frac{1}{2}\tilde{c}\left(z\frac{df}{dz}\right)^2 \tag{2.1.7a}$$

$$J_\alpha(z) = J_0(z) - i\tilde{c}z\frac{df}{dz} \tag{2.1.7b}$$

$$G_\alpha(z) = e^{if(z)} G_0(z) \ , \ \bar{G}_\alpha(z) = e^{-if(z)} \bar{G}_0(z) \tag{2.1.7c}$$

where $f(z) = i\alpha log(z)$.

For the N=3 algebra the global automorphism group is O(3) whereas the local automorphism group is SO(3). Thus again the outer automorphism group is Z_2 and there only two inequivalent N=3 algebras R and NS. Finally the global automorphism group of the N=4 algebra is SO(4) (the algebra is not invariant under parity). The local automorphism group is SU(2), thus the outer automorphism group is $SO(4)/SU(2) = SU(2)$ and there is an infinite number of inequivalent N=4 superconformal algebras.

Superconformal invariance is important in superstring models. It seems to be indispensable if one hopes to get space-time fermions. There are two kinds of closed superstring models, the type II models which have (1,1) local supersymmetry (gauged in the sense that there are ghosts associated with it) and the heterotic models which have (1,0) local supersymmetry. There is also the U(1) string which has (2,2) local gauged supersymmetry but for phenomenological reasons it is uninteresting since it makes sense if space time is two-dimensional.

To construct four-dimensional string theories one has to proceed as follows. The part describing 4-d Minkowski space is constructed out of free bosons (and fermions) in the conventional way. Then the theory has to be supplemented by a conformal field theory describing the internal degrees of freedom. Such a theory has to have the appropriate value for the central charge which is $c = 22$ for the N=0 case and $c = 9$ for the N=1 case. Such a CFT may have a bigger local invariance that the gauged one. We are usually interested in a theory which has unbroken 4-d N=1 supersymmetry at the Planck scale. The reason for supersymmetry is to solve problems associated with hierarchies. We need N=1 instead of an extended supersymmetry because in 4-d only N=1 supersymmetry can accommodate chiral fermions. It can be shown that the statement that the theory has N=1 space-time supersymmetry is equivalent to the statement that the CFT describing the internal degrees of freedom has an N=2 superconformal invariance. Thus N=1 space-time supersymmetric string theories in 4-d are classified by N=2 superconformal field theories with $\tilde{c} = 3$. The study of N=2 superconformal field theories is a very important part of constructing phenomenologically viable string theories. It would be very useful to construct N=2 models which are exactly solvable. Using such models as building blocks we would construct string theories where scattering amplitudes would be calculable.

The study of the unitary representations of the N=2 superconformal algebra showed that there is an analogous structure as in the N=0,1 cases. There is a discreetly infinite set of minimal models, [18,19,26], with,

$$\tilde{c} = 1 - \frac{2}{m} \ , \ m = 2, 3, 4, ... \tag{2.1.8}$$

which contain a finite number of N=2 irreducible representations and which are exactly solvable. For $\tilde{c} \geq 1$ there is a continuum of models which have not been classified yet.

There are also other motivations for studying N=2 superconformal CFTs. They come from condensed matter physics. There are critical 2-d systems that exhibit N=2 superconformal invariance. Such examples will be discussed in chapter 3.

The first step in the study of N=2 theories is the calculation of the Kač determinant. As discussed in the introduction it is very important in the study of questions of unitarity as well as in studying the existence of null states present in some representations. The presence of null states is welcome since they impose extra constraints on the correlation functions. Another important concept is that of a character. It is a generalization of the corresponding concept in finite dimensional algebras and groups. It contains a lot of information about the structure of a representation, in a sense it specifies it unambiguously. There is an extra property of the characters that is crucial both for 2-d critical systems and string theory. This is the fact that characters are closely related to the exact partition function of the system on the torus. This connection will be studied in more detail in the next section.

For an irreducible representation $[h]$ of the conformal algebra the character is defined as,

$$ch_h = Tr_h(z^{L_0}) \tag{2.1.9}$$

where the trace is to be taken over all the states of the irreducible representation $[h]$ and z is a formal variable.

When the representation $[h]$ does not contain any null vectors then the computation of the character is quite easy. Things start to get complicated when there are null vectors. As we already mentioned the Verma module in this case is not irreducible and in order to compute the character one has to subtract the contributions of the extra representations embedded in the original one. Thus there are two stages in the process. The first consists in determining, (using the Kač determinant), the embedding pattern of the representations in the original one. The second consists in using the embedding pattern to subtract their contributions. Both stages will be discussed in more detail in subsequent sections.

There are other issues that need to be examined in N=2 Superconformal models. One has to derive Ward identities and in particular solve the ones that relate to the fact that N=2 supersymmetry is an unbroken symmetry. Also the operator product rules, (fusion rules), in such models have to be worked out.

In this chapter we will discuss the N=2 superconformal field theories with particular emphasis on the minimal ones. In section 2.2 we discuss the relation between characters and partition functions. The concept and consequences of modular invariance will be also touched upon. Section 2.3 is devoted to a study of the unitary irreducible representations of the N=2 superconformal algebras. We will derive the embedding structure of the degenerate irreducible representations and we will derive their characters for any value of the central charge. Section 2.4 deals with a description of (2,0) superspace and its geometry as well as with the group of N=2 global transformations, Osp(2|2). Section 2.5 is devoted to the general description of N=2 CFTs, their primary fields and the structure of their ground states. In section 2.6 we will study Osp(2—2) invariance and the constraints it puts on correlation functions. In section 2.7 we discuss the operator algebra and correlation functions in the NS sector of the unitary minimal N=2 models. Section 2.8 deals with the operator formalism in the Ramond sector. We will point out how we can apply the techniques used in the NS sector to the R sector. Section 2.9 is devoted to the study of the first minimal model with $\tilde{c} = \frac{1}{3}$ using the general techniques we introduced so far. Finally section 2.10 contain conclusions and future prospects on the study of N=2 models.

2.2 Modular Invariance, Characters and Partition Functions on the Torus

Let's consider a 2-d critical model on the torus. The torus can be represented as a parallelogram with sides l, l' and periodic boundary conditions. At the limit $l,\ l' \to \infty$ with $l/l' = \delta$ fixed*, the Hamiltonian operator is,

$$H = \frac{2\pi}{l}(L_0 + \bar{L}_0) \tag{2.2.1}$$

while the momentum operator is,

$$P = \frac{2\pi}{l}(L_0 - \bar{L}_0) \tag{2.2.2}$$

l' will be allowed to take complex values. This will consequently allow the parallelogram representing the torus to be tilted. Then the partition function of the system can be written as,

$$Z(l, l') = e^{-fll' + \frac{\pi c Re\delta}{6}} \sum_n e^{-E_n Re l' - iP_n Im l'} \tag{2.2.3}$$

where the sum is over all the states of the theory. Equation (2.2.3) can be written in a more suggestive form:

$$Z(\delta, \delta^*) = e^{-fA + \frac{\pi c Re\delta}{6}} Tr[z^{L_0} \bar{z}^{\bar{L}_0}] \tag{2.2.4}$$

where $z = e^{2\pi\delta}$, $\bar{z} = e^{-2\pi\delta^*}$ and A is the area of the torus. In a conformally invariant theory the states are assembled in irreducible representations of the conformal group. Then equation (2.2.4) can be written in terms of the characters of the holomorphic (left) and anti-holomorphic (right) representations.

$$Z(\delta, \delta^*) = e^{-fA + \frac{\pi c Re\delta}{6}} \sum_{(h,\bar{h})} N(h, \bar{h}) ch_h(\delta) ch_{\bar{h}}(\delta^*) \tag{2.2.5}$$

where $N(h, \bar{h})$ is the number of times the irreducible representation $(h, \bar{h})$ appears in the theory and ch_h denotes the character of the holomorphic part of the representation $(h, \bar{h})$. Thus knowledge of the characters and the representation content of the theory is enough to determine the partition function.

As already mentioned the characters of the representation can be calculated by purely algebraic means. Thus the only issue to be settled is the representation content. It is here that the concept of modular invariance comes to the rescue, [14], (if we assume that the theory contains a finite number of irreducible representations)†.

⋆ δ is closely related to the modulus of the torus. The exact relation is $\delta = i\tau$.

† For theories invariant under the conformal group only, this implies that the central charge must be $c < 1$. In N=1(2) superconformal field theories $\hat{c}(\tilde{c}) < 1$. In N=3,4 superconformal theories as well as WZW models this is always true. The above are special cases of G/H theories which always contain a finite number of representations of some local algebra.

When we consider a field theory of the torus which we want to be coordinate invariant we first check invariance under infinitesimal coordinate transformations. Sometimes there are coordinate transformations which are not continuously connected to the identity, thus they cannot be built out of infinitesimal ones. Then we have to check that the theory is in fact invariant under such coordinate transformations. The group of coordinate transformations of the torus is known to contain such disconnected components which can be labeled by elements of $PSL(2,Z)$. Such globally non-trivial coordinate transformations, (modular transformations), are generated by two basic transformations, $T:\delta\to\delta+i$ and $S:\delta\to\frac{1}{\delta}$. The partition function of a theory on the torus must be invariant under the modular transformations.

The consequences of invariance under T are easy to determine because the characters are diagonal under its action. It implies that all the states of the theory must have spin $h-\bar{h}$ which is integer. The consequences of invariance under S are more difficult to find. The reason is that the action of S mixes the characters among themselves. One ends up with a linear algebraic system of equations among the numbers $N(h,\bar{h})$. This system has to be supplemented with extra physical requirements. There must be only one unit operator in the theory, $N(0,0)=1$, and $N(h,\bar{h})$ must be non-negative integers‡. In this manner one obtains the representation content of a wide class of theories.

The derivation of the torus partition functions of various CFTs is very useful also for string theory. The partition function, integrated over the modulus of the torus, δ, gives the one-loop contribution to the vacuum energy of string theory.

In the next section we will analyze the structure of the representations of the N=2 superconformal algebras and we will eventually evaluate their characters.

2.3 Character Formulae and the Structure of the Representations of the N=2 Superconformal Algebras

In this section we will consider the unitary degenerate representations of the N=2 superconformal algebra. We will derive their structure and the corresponding characters§.

The N=2 algebra is given by the following (anti-)commutation relations¶

$$[L_m,L_n]=(m-n)L_{m+n}+\frac{\tilde{c}}{4}(m^3-m)\delta_{m+n,0} \tag{2.3.1a}$$

‡ In cases that have been examined so far it seems that the $N(h,\bar{h})$ obtained by solving the system are always integers but there are examples with $N(0,0)\neq 1$ and/or $N(h,\bar{h})$ being negative integers.

§ Character formulae were also derived in [22]. In [23] the characters of the $\tilde{c}<1$ representations were derived.

¶ We have chosen a particular normalization for the central charge of the U(1) sub-algebra. It is worth noting that the most general N=2 superconformal algebra includes, up to the freedom of redefinitions, another free parameter, the U(1) charge of the supercharges. Then the respective commutation relations become: $[J_m,G^i_r]=iq\epsilon^{ij}G^j_{m+r}$ and $[G^i_r,G^j_s]_+=2\delta^{ij}L_{r+s}+\frac{i}{q}\epsilon^{ij}(r-s)J_{r+s}+\tilde{c}(r^2-\frac{1}{4})\delta^{ij}\delta_{r+s,0}$. This new parameter does not change the structure of the irreducible representations. Its only effect is to change the distance between successive relative charge levels.

$$[L_m, G^i_r] = (\frac{m}{2} - r)G^i_{m+r}\ ,\ [L_m, J_n] = -nJ_{m+n} \tag{2.3.1b}$$

$$[J_m, J_n] = \tilde{c}\, m\delta_{m+n,0}\ ,\ [J_m, G^i_r] = i\epsilon^{ij} G^j_{m+r} \tag{2.3.1c}$$

$$[G^i_r, G^j_s]_+ = 2\delta^{ij} L_{r+s} + i\epsilon^{ij}(r-s)J_{r+s} + \tilde{c}(r^2 - \frac{1}{4})\delta^{ij}\delta_{r+s,0} \tag{2.3.1d}$$

The normalization of the conformal anomaly is such that a free N=2 scalar superfield has $\tilde{c} = 1$. It is related to the anomaly of the Virasoro algebra by $\tilde{c} = 3c$.

As already discussed in section 2.1 there are two inequivalent N=2 algebras The twisted one and the untwisted one. There is a continuous family of untwisted N=2 algebras which are related through local U(1) transformations (2.1.17). We will study one of them, the NS algebra. Then through the aforementioned isomorphism we will be able to translate our statements to the general member of the continuous set. Choosing integer moding for L_m, J_n and half-integer for G^i_r we get the NS-type algebra.

We will start our discussion from the NS algebra and focus on the unitary representations with $\tilde{c} < 1$. In [18,19] it was shown that these exist only when :

$$\tilde{c} = 1 - \frac{2}{m}\ ,\ m = 2, 3, 4, ... \tag{2.3.2}$$

and have hwv's with dimension and U(1) charge q given by,

$$h_{j,k}\ =\ \frac{4jk-1}{4m}\ ,\ q = \frac{j-k}{m}\ ,\ j,k \in Z + \frac{1}{2}\ ,\ 0 < j,k,j+k \leq m-1 \tag{2.3.3}$$

Hwv states are labeled by the eigenvalues of the zero modes, L_0 and J_0, which are the dimension h and the U(1) charge q. Then any descendant is labeled by its level (eigenvalue of $L_0 - h$) and its relative charge (eigenvalue of $J_0 - q$).

The Kač determinant at level n and relative charge m is given by [18,19,20,21]

$$det M^{NS}_{n,m}(\tilde{c}, h, q) = \prod_{1 \leq rs \leq 2n}^{s\ even} [f^{NS}_{r,s}]^{P_{NS}(n-rs/2,m)} \times \prod_{k \in Z+\frac{1}{2}} [g^{NS}_k]^{\tilde{P}_{NS}(n-|k|,m-sgn(k);k)} \tag{2.3.4}$$

where :

$$f^{NS}_{r,s} = 2(\tilde{c}-1)h - q^2 - \frac{1}{4}(\tilde{c}-1)^2 + \frac{1}{4}[(\tilde{c}-1)r + s]^2\ ,\ r \in Z^+\ ,\ s \in 2Z^+ \tag{2.3.5a}$$

$$g^{NS}_k = 2h - 2kq + (\tilde{c}-1)(k^2 - \frac{1}{4})\ ,\ k \in Z + \frac{1}{2}, \tag{2.3.5b}$$

while the NS partition functions are defined by,*

$$\sum_{n,m} P_{NS}(n,m) z^n w^m = \prod_{k=1}^{\infty} \frac{(1+z^{k-1/2}w)(1+z^{k-1/2}w^{-1})}{(1-z^k)^2} \tag{2.3.6a}$$

$$\sum_{n,m} \tilde{P}_{NS}(n,m;k) z^n w^m = [1+z^{|k|}w^{sgn(k)}]^{-1} \sum_{n,m} P_{NS}(n,m) z^n w^m \tag{2.3.6b}$$

Equation (2.3.4) implies that whenever there is a vanishing of $f_{r,s}^{NS}$, there exists a unique hwv at level $rs/2$ with the same charge as the initial one, (relative charge zero). When $g_k^{NS} = 0$, there is a hwv at level $|k|$ and relative charge $sgn(k)$.

Consider the representation of dimension $h_{j,k} = (4jk-1)/4m$ and charge $q = (j-k)/m$. We will first search for null hwv's at relative charge zero. $f_{r,s}^{NS}$ vanishes for,

$$r = nm \pm (j+k) \ , \ s = 2n \ \ n = 1,2,.... \tag{2.3.7}$$

Thus there are null vectors at relative charge zero, embedded in the family $(h_{j,k}, q)$ their dimensions being $h_{j,k} + n^2 \pm n(j+k)$. We can show that the above hwv's exhaust all null hwv's at relative charge zero. In fact if we order them in order of increasing dimension,

$$h_{2n-1} = h_{j,k} + n^2 m - n(j+k) \ \ n = 1,2,.... \tag{2.3.8a}$$

$$h_{2n} = h_{j,k} + n^2 m + n(j+k) \ \ n = 0,1,2,.... \tag{2.3.8b}$$

we can show by analyzing the Kač determinant for h_i, that (still at relative charge zero), the families h_j $j > i$ (and only these) are embedded in h_i.

Next we have to look for null vectors of non-zero relative charge. For $h_{j,k}$ g_l^{NS} vanishes for $l = k$ and $l = -j$. This implies the existence of a hwv of dimension $h_{j,k} + k$ and charge $q+1$ as well as a hwv of dimension $h_{j,k} + j$ and charge $q-1$ embedded in $[h_{j,k}]$.

Looking now at the Kač determinant (relative charge zero), of the hwv $h'_1 = h_{j,k} + k$, $q'_1 = q+1$, we can establish that it vanishes for,

$$r = (n+1)m + (j+k) \ , \ s = 2n \ \ n = 1,2,.... \tag{2.3.9a}$$

$$r = nm - (j+k) \ , \ s = 2(n+1) \ \ n = 1,2,.... \tag{2.3.9b}$$

implying the existence of another series of null hwv's with dimensions,

$$h'_{2n-1} = h_{j,k} + n(n+1)m - (n+1)j - nk \ \ n = 1,2,.... \tag{2.3.10a}$$

$$h'_{2n} = h_{j,k} + n(n+1)m + nj + (n+1)k \ \ n = 1,2,.... \tag{2.3.10b}$$

and charge $q+1$.

---

* The derivation of the partition functions can be found in App. 2.B.

This scenario continues so that by using induction we can establish the existence of an embedding pattern shown in fig. 3 . All embedding diagrams are commutative. The maps between sectors of different charge form exact sequences due to the fermionic nature of the generating operators. There is unique hwv at each level and charge since the Kač determinant has a simple zero corresponding to that hwv. The dimensions and charges of the various families depicted on it are,

$$h^l_{2n+l} = h_{j,k} + n(n+l)m + n(j+k) + lk \ , \ l \geq 0 \ , \ n \geq 0 \tag{2.3.11a}$$

$$h^l_{2n+l-1} = h_{j,k} + n(n+l)m - (n+l)(j+k) + lk \ , \ l \geq 0 \ , \ n \geq 1 \tag{2.3.11b}$$

$$h^{-l}_{2n+l} = h_{j,k} + n(n+l)m + n(j+k) + lj \ , \ l \geq 0 \ , \ n \geq 0 \tag{2.3.11c}$$

$$h^{-l}_{2n+l-1} = h_{j,k} + n(n+l)m - (n+l)(j+k) + lj \ , \ l \geq 0 \ , \ n \geq 1 \tag{2.3.11d}$$

$$q^l_n = q + l \ , \ l \in Z \tag{2.3.11e}$$

It is obvious from (2.3.11) that all dimensions in a given charge sector are different so that the corresponding representations are distinct.

We define the character of the irreducible representation generated by the hwv of dimension $h_{j,k} = \frac{4jk-1}{4m}$ and charge $q = \frac{j-k}{m}$ ($m \geq 2$, $0 < j, k, j+k \leq m-1$, $j, k \in Z + \frac{1}{2}$) by :

$$ch(h_{j,k}, \tilde{c}, z, w) \equiv Tr[z^{L_0} w^{J_0}] \tag{2.3.12}$$

The trace over all the descendants of a hwv, (h, q), is given by*

$$\chi(h, q, z, w) = F_{NS}(z, w) z^h w^q \tag{2.3.13}$$

$$F_{NS}(z, w) = \prod_{k=1}^{\infty} \frac{(1 + z^{k-1/2} w)(1 + z^{k-1/2} w^{-1})}{(1 - z^k)^2} \tag{2.3.14}$$

Our task now is to compute the trace by excluding all superconformal families that are embedded in $h_{j,k}$. It is obvious from the embedding pattern pictured in fig. 3 that,

$$[h^0_i] \cap [h^1_i] = [h^0_{i+1}] + [h^1_{i+1}] \ , \ [h^0_i] \cap [h^{-1}_i] = [h^0_{i+1}] + [h^{-1}_{i+1}] \tag{2.3.15a}$$

$$[h^1_i] \cap [h^{-1}_i] = [h^0_{i+1}] \ , \ [h^0_i] \cap [h^1_i] \cap [h^{-1}_i] = [h^0_{i+1}] \tag{2.3.15b}$$

The largest proper submodule of h^0_0 is $[h^0_1] + [h^1_1] + [h^{-1}_1]$. The largest proper submodule of

* See App. 2.B.

$[h_1^0]+[h_1^1]+[h_1^{-1}]$ is given by:

$$[h_1^0]\cap[h_1^1]+[h_1^0]\cap[h_1^{-1}]+[h_1^1]\cap[h_1^{-1}]-2[h_1^0]\cap[h_1^1]\cap[h_1^{-1}]$$

which is equal to $[h_2^0]+[h_2^1]+[h_2^{-1}]$. Inductively, the largest proper submodule of $[h_i^0]+[h_i^1]+[h_i^{-1}]$ is $[h_{i+1}^0]+[h_{i+1}^1]+[h_{i+1}^{-1}]$. Consequently the character for the irreducible representation $[h_0^0]$ is given by:

$$ch[h_0^0]=\chi([h_0^0])+\sum_{i=1}^{\infty}(-1)^i\chi([h_i^0+h_i^1+h_i^{-1}]) \tag{2.3.16}$$

where χ denotes the unrestricted trace defined by (2.3.13).

In order to write down an explicit formula for the character we need also the partition functions for single charged fermions†

Substituting in (2.3.16) we get,

$$ch(h_{j,k},z,w)=F_{NS}(z,w)z^{h_{j,k}}w^q\left[1+f_1(z,w)-f_2(z,w)\right] \tag{2.3.17}$$

$$f_1(z,w)=\sum_{n=1}^{\infty}\left[z^{n^2m+n(j+k)}+\frac{z^{n(n+1)m-(n+1)(j+k)+k}w}{1+z^{nm-j}w}+\frac{z^{n(n+1)m-(n+1)(j+k)+j}w^{-1}}{1+z^{nm-k}w^{-1}}\right]$$

$$f_2(z,w)=\sum_{n=0}^{\infty}\left[z^{(n+1)^2m-(n+1)(j+k)}+\frac{z^{n(n+1)m+n(j+k)+k}w}{1+z^{nm+k}w}+\frac{z^{n(n+1)m+n(j+k)+j}w^{-1}}{1+z^{nm+j}w^{-1}}\right]$$

Let's now consider the rest of the untwisted algebras. The isomorphism described in section 2.1 among the modes of the various algebras is,

$$L_n^{\alpha}=L_n^0-\alpha J_n^0+\frac{\alpha^2}{2}\tilde{c}\delta_{n,0} \tag{2.3.18a}$$

$$J_n^{\alpha}=J_n^0-\alpha\tilde{c}\delta_{n,0} \tag{2.3.18b}$$

$$G_{n+\alpha}^{\alpha}=G_n^0\ ,\ \bar{G}_{n-\alpha}^{\alpha}=\bar{G}_n^0 \tag{2.3.18c}$$

In particular the dimensions and charges of the irreducible representations are related by,

$$h_\alpha=h_0-\alpha q_0+\frac{\alpha^2}{2}\tilde{c}\ ,\ \ q_\alpha=q_0-\alpha\tilde{c} \tag{2.3.19}$$

$\alpha=0$ corresponds to the NS-algebra whereas $\alpha=\pm\frac{1}{2}$ corresponds to the $R^{\pm}$ algebras. Thus

† For a derivation see app. 2.B

the characters are related by,

$$ch_{h_\alpha}(z,w) = z^{\frac{\alpha^2}{2}\tilde{c}} w^{-\alpha\tilde{c}} ch_{h_0}(z, z^{-\alpha}w) \tag{2.3.20}$$

The expression for the characters (2.3.17) can be written in an elegant and compact form using the SU(2) ϑ-functions and the SU(2) string-functions, [24]. Such a form is useful when one desires to study the modular properties of the characters.

The expressions for general α are:

$$ch^{\alpha}_{j,k}(z,w) = \sum_{-N+1\leq\tilde{m}\leq N}^{l-\tilde{m}\ even} c^{l}_{\tilde{m}}(\tau)\vartheta^{\tilde{q},\tilde{m}}_{N,\alpha}(\tau,\theta) \tag{2.3.21}$$

where

$$\vartheta^{\tilde{q},\tilde{m}}_{N,\alpha}(\tau,\theta) = \vartheta_{N(N+2)/2,(\tilde{m}-\tilde{q})(N+2)/2+\tilde{q}-N(\alpha-1/2)}(\tau,\theta) \tag{2.3.22a}$$

$$\vartheta_{a,b}(\tau,\theta) = \sum_{n\in Z+b/2a} e^{2\pi i(\tau n^2-n\theta)} \tag{2.3.22b}$$

$$z = e^{2\pi i\tau}\ ,\ w = e^{2\pi i\theta} \tag{2.3.22c}$$

$$l = j+k-1\ ,\ \tilde{q} = j-k\ ,\ N = m-2 \tag{2.3.22d}$$

and $c^{l}_{m}(\tau)$ are the SU(2) string-functions.

The twisted algebra is defined by imposing anti-periodic boundary conditions on the U(1) current and periodic boundary conditions on $G^1(z)$. For the twisted algebra the zero modes are L_0 and G^1_0. Their eigenvalues characterize hwv's. Each level contains two equal subspaces of fermion number $(-1)^F = \pm 1$. The Kač determinant for the T-algebra is the following, [18,25],

$$det M^{T}_{+,0} = 1\ ,\ det M^{T}_{-,0} = h - \frac{\tilde{c}}{8} \tag{2.3.23a}$$

$$det M^{T}_{\pm,n}(\tilde{c},h) = [h-\frac{\tilde{c}}{8}]^{P_T(n)/2} \prod_{1\leq rs\leq 2n}^{s\ odd} [f^{T}_{r,s}]^{P_T(n-rs/2)} \tag{2.3.23b}$$

$$f^{T}_{r,s} = 2(\tilde{c}-1)(h-\frac{\tilde{c}}{8}) + \frac{1}{4}[(\tilde{c}-1)r+s]^2\ ,\ s=1,3,5,... \tag{2.3.24}$$

$$\sum_n P_T(n) z^n = \prod_{k=1}^{\infty} \frac{(1+z^k)(1+z^{k-1/2})}{(1-z^k)(1-z^{k-1/2})} \equiv \bar{F}_T(z) \qquad (2.3.25)$$

The unitary representations of the T-algebra with $\tilde{c} < 1$ are given by,

$$\tilde{c} = 1 - \frac{2}{m} \ , \ h = \frac{\tilde{c}}{8} + \frac{(m-2r)^2}{16m} \ , \ m = 2,3,\ldots \ , \ r \in Z \ , \ 1 \leq r \leq \frac{m}{2} \qquad (2.3.26)$$

Only even m allows the state $h = \frac{\tilde{c}}{8}$, the presence of which implies that supersymmetry is unbroken.

The vanishing of $f^T_{r,s}$ signals the existence of two hwv's at level $rs/2$ and fermion parity ± 1. At level zero there is only one vanishing whereas for each of the higher levels there are two vanishings corresponding to states of opposite parity. Analyzing the vanishings of $f^T_{r,s}$, we can easily show that the embedding pattern is the one shown in fig.1 with,

$$h_0 = \frac{\tilde{c}}{8} + \frac{(m-2r)^2}{16m} \qquad (2.3.27)$$

$$h_k = \frac{\tilde{c}}{8} + \frac{[(2k-1)m+2r]^2}{16m} \ , \ h'_k = \frac{\tilde{c}}{8} + \frac{[(2k+1)m-2r]^2}{16m} \qquad (2.3.28)$$

The character formula in this case is written down in the same way as in the N=1 case.

$$ch^T_{m,r}(z) = F_T(z) z^{\frac{\tilde{c}}{8}} \left[\sum_{k \in 2Z} (-1)^{k/2} z^{\frac{[(k+1)m-2r]^2}{16m}} \right] \qquad (2.3.29)$$

When $h = \frac{\tilde{c}}{8}$, one of the two states of different chirality is degenerate at the zeroth level and decouples as it can be easily seen from the formula for the Kač determinant. Then supersymmetry is unbroken due to the non-vanishing of the Witten index.

The above complete the derivation of the character formulae for the degenerate representations of the N=2 superconformal algebras with $\tilde{c} < 1$.

A construction of these representations based on the coset space $SU(2) \otimes U(1)/U(1)$ has been given, [26,27], proving their unitarity through an explicit unitary construction of their Hilbert space.

The untwisted algebra contain another class of degenerate representations with $\tilde{c} \geq 1$. We will focus as before on the NS sector*. There we have two distinct sets of degenerate representations.

$\star$ The results then can be extended to the rest by use of the isomorphism (2.1.7)

NS_2 *representations.*(the subscript indicates the dimension of their moduli space). A representation in this class is unitary and degenerate if $g^{NS}_{n_0} = 0$ for some $n_0 \in Z + \frac{1}{2}$, $g^{NS}_{n_0+sgn(n_0)} < 0$ and $f^{NS}_{1,2} \geq 0$. According to (2.3.5b) the first condition implies that,

$$2h = 2n_0 q - (\tilde{c} - 1)(n_0^2 - \frac{1}{4}) \tag{2.3.30}$$

We will suppose for the moment that $n_0 > 0$. Then the second condition implies that,

$$q > (n_0 + \frac{1}{2})(\tilde{c} - 1) \tag{2.3.31}$$

whereas the third condition implies,

$$-\frac{(\tilde{c}+1)}{2} + n_0(\tilde{c} - 1) \leq q \leq \frac{(\tilde{c}+1)}{2} + n_0(\tilde{c} - 1) \tag{2.3.32}$$

Collecting everything together, the three conditions boil down to (2.3.30) and

$$(n_0 + \frac{1}{2})(\tilde{c} - 1) < q \leq (n_0 + \frac{1}{2})(\tilde{c} - 1) + 1 \tag{2.3.33}$$

and it is obvious that both h and q are positive. If $n_0 < 0$ then (2.3.33) is replaced by :

$$(n_0 - \frac{1}{2})(\tilde{c} - 1) - 1 \leq q < (n_0 - \frac{1}{2})(\tilde{c} - 1) \tag{2.3.34}$$

which in particular implies $h > 0$, $q < 0$ in this case. In the following we will discuss the $n_0 > 0$ case and we will point out in the end the appropriate changes for $n_0 < 0$.

As it turns out to be, the embedding structure of these representations depends crucially on the values of $\tilde{c}$ and q, (constrained already by (2.3.33)). We have to distinguish the following cases:

(A) . $\tilde{c} > 1$, $\tilde{c}$ *irrational*. We will analyze first the interior of the interval (2.3.33).

(i) The $U(1)$ charge q has the form, $q = \frac{1}{2}n(\tilde{c} - 1) - m$, $n \in Z$, $m \in Z_0^+$ with n constrained from (2.3.33) :

$$2n_0 + 1 + \frac{2m}{\tilde{c} - 1} < n \leq 2n_0 1 + \frac{2(m+1)}{\tilde{c} - 1} \tag{2.3.35}$$

Then it is easy to show that the embedding pattern is the one shown in fig. 4 with,

$$h_k = h_0 + kn_0 \ , \ h'_{m+k} = h_0 + k(n - n_0) \ , \ q_k = q'_k = q + k \tag{2.3.36}$$

It is obvious that in a given charge sector the various dimensions are distinct and thus the corresponding representations different. Also the maps fron one charged sector to another generate exact sequences due to the fermionic nature of the operators generating the relevant hwv's. Another remark is in order here concerning the embedding diagrams: embedding maps that are factorizable have been omitted from the figures. For example in fig. 4 the family h_m contains also a degenerate vector generating h'_{m+1}. Thus the embedding map f : $h_m \to h'_{m+1}$ is the composition of the maps $g_1 : h_m \to h_{m+1}$ and $g_2 : h_{m+1} \to h'_{m+1}$, that is $f(x) = g_2(g_1(x))$. Similar remarks are true for the rest of the embedding diagrams.

The trace over all the descendants of the primary state $|h, q>$ is given*

$$Tr[z^{L_0} w^{J_0}] = F_{NS}(z, w) z^h w^q \tag{2.3.37a}$$

whereas the trace, for example, over all the descendants of the family (h_1, q_1) is given by,

$$Tr_{h_1}[z^{L0} w^{J_0}] = \frac{F_{NS}(z, w)}{1 + z^{n_0} w} \tag{2.3.38b}$$

To compute the character in this case we have to subtract the contribution from the family (h_1, q_1) so that,

$$ch(h, q, z, w) = F_{NS}(z, w) z^h w^q \left[1 - \frac{z^{n_0} w}{1 + z^{n_0} w} \right] = F_{NS}(z, w) \frac{z^h w^q}{1 + z^{n_0} w} \tag{2.3.39}$$

(ii) q has any other allowable value except the ones mentioned in (i). In this case the embedding pattern is shown in fig.2. The relevant dimensions are,

$$h_k = h + kn_0 \ , \ q_k = q + k \tag{2.3.40}$$

so that the character is given again by (2.3.39).

Let's now consider the representation which lies on the vanishing surface $f_{1,2}^{NS} = 0$, whose charge is given by $q = (\tilde{c} - 1)(n_0 + \frac{1}{2}) + 1$. In this case there is also a null hwv at relative charge zero embedded in the initial representation at the first level. The relevant diagram is given in fig. 5. The corresponding dimensions are,

$$h_k = h + kn_0 \ , \ h'_k = h + k(n_0 + 1) + 1 \ , \ q_k = q'_k = q + k \tag{2.3.41}$$

To evaluate the character in this case we subtract first the family h_1 so that we factor out everything else except the irreducible family h'_0. This is given by subtracting h'_1 off h'_0.

⋆ See App. 2.B.

Consequently,

$$ch(z,w) = \chi([h_0] - [h_1] - [h_0'] + [h_1']) = F_{NS}(z,w)\frac{z^h w^q (1-z)}{(1+z^{n_0}w)(1+z^{n_0+1}w)} \qquad (2.3.42)$$

(B) $\tilde{c} > 1$, $\tilde{c}$ *rational.* Then there is a unique way to write $\tilde{c}$ as,

$$\tilde{c} = 1 + \frac{2r_2}{r_1}\ ,\ r_1, r_2 \in Z\ , r_1 \geq 1,\ r_2 \geq 1 \qquad (2.3.43)$$

and with r_2 being the least positive integer such that (2.3.43) is true. For $r_2 = 1$ this corresponds to the special class of representations found in [1], which are identified by triple intersections of vanishing surfaces.

We will focus first on representations which are contained in the interior of the interval (2.3.33).

(i) If $q = \frac{1}{2}n(\tilde{c}-1) - m$, $n \in Z$, $m \in Z_0^+$ with the integer n constrained by (2.3.35), then there are three possible embedding patterns corresponding to the following situations.

(ia) $r_2 > 1$. The corresponding diagram in this case is displayed in fig. 6. The pattern repeats itself with "period" r_2, and the relevant dimensions are,

$$h_k = h + kn_0\ ,\ h'_{m+k} = h + k(n-n_0)\ ,\ q_k = q'_k = q + keqno(2.3.44a)$$

$$h''_{k+m+r_2} = h + (r_2 - k)n_0 + k(n + r_1)\ ,\ q''_k = q + k\ ,\ k \leq r_2 \qquad (2.3.44b)$$

$$h'''_{k+m+r_2} = h + r_2(n-n_0) + k(n_0 + r_1)\ ,\ q'''_k = q + k\ ,\ k \leq r_2 (2.3.44c)$$

At each relative charge level all the dimensions are different and correspond to different hwv's.

(ib) $r_2 = 1$, $n \neq 2n_0 + r_1$. Then the diagram of fig. 6 simplifies to the one shown in fig. 7. The dimensions and charges are given by,

$$h^{2l-1}_{m+k} = h + (k-l+1)[n + (l-1)r_1] + (m+2l-k-2)n_0\ ,\ 1 \leq l \leq [k+\frac{1}{2}] \qquad (2.3.45a)$$

$$h^{2l}_{m+k} = h + l[n + (k-l)r_1] + (m+k-2l)n_0\ ,\ 1 \leq l \leq [\frac{k}{2}] \qquad (2.3.45b)$$

$$h^0_k = h + kn_0\ ,\ k \geq 0\ ,\ q^l_k = q + k \qquad (2.3.45c)$$

(ic) $r_2 = 1$, $n = 2n_0 + r_1$. In this case the diagram on fig. 7 collapses even further to the diagram shown in fig. 8, the relevant dimensions being,

$$h^l_{m+k} = h + l(k - l + 1)n + [2(l - k)(l - 1) + m - k]n_0 \, , \, q^l_k = q + k \tag{2.3.46}$$

(*ii*) The charge q is not of the form (i). Then the embedding diagram is very simple and it is shown in fig. 2.

In all the cases discussed above the character can be computed by subtracting the contribution of the first embedded family. Consequently the character is given by (2.3.39).

Let's now consider the representation that lies on the $f^{NS}_{1,2} = 0$ surface with $q = (n_0 + \frac{1}{2})(\tilde{c} - 1) + 1$.

(*a*) For $r_1 > 1$, $r_2 > 1$ the embedding pattern is shown in fig. 9, the relevant dimensions being,

$$h_k = h + kn_0 \, , \, h'_k = h + kn_0 + k + 1 \, , \, q_k = q'_k = q + k \tag{2.3.47a}$$

$$h''_{k+r_2} = h + (r_2 + k)n_0 + (k+1)r_1 + r_2 \, , \, h'''_{k+r_2} = h + (r_2 + k)n_0 + (k+1)(r_1 + 1) \tag{2.3.47b}$$

(*b*) $r_2 = 1$, $r_1 > 1$. The corresponding diagram is shown in figure 10 with the following dimensions and charges,

$$h^{2l-1}_k = h + kn_0 + (k - l - 2)[(l - 1)r_1 + 1] \, , \, k \geq 0 \, , \, l \geq 1 \tag{2.3.48a}$$

$$h^{2l}_k = h + kn_0 + l[(k - l + 1)r_1 + 1] \, , \, k \geq 0 \, , \, l \geq 0 \tag{2.3.48b}$$

$$q^l_k = q + k \tag{2.3.48c}$$

(*c*) $r_1 = 1$, $r_2 > 1$. In this case the embedding diagram becomes the one shown in fig. 11 where the periodicity of the pattern is again set by r_2. The corresponding dimensions are,

$$h_k = h + kn_0 \, , \, h'_k = h + kn_0 + k + 1 \, , \, q_k = q'_k = q + k \tag{2.3.49}$$

(*d*) $r_1 = r_2 = 1$, $\tilde{c} = 3$. Then the previous diagram collapses to the one shown in fig. 12,

$$h^k_l = h + ln_0 + k(l - k + 2) \, , \, q^k_l = q + k \, , \, k \geq 2l - 2 \tag{2.3.50}$$

In all of the above cases the character can be computed in the same way as in the respective case where $\tilde{c}$ was irrational. Consequently the character is given by (2.3.42).

The only case left to consider for the NS_2 representations is $\tilde{c}=1$ which is not included in (B).

(C) $\tilde{c}=1$.

(i) $0<q<1$. In this case the embedding diagram becomes fairly simple and it is shown in fig. 2,

$$h_k=(q+k)n_0\ ,\ q_k=q+k \tag{2.3.51}$$

and the character is given by (2.3.39).

(ii) $q=1$. The Kač determinant simplifies enormously, its factors becoming,

$$f^{NS}_{r,s}=-q^2+\frac{s^2}{4}\ ,\ g^{NS}_k=h-qk \tag{2.3.52}$$

This gives rise to the pattern pictured in fig. 13 with

$$h_{k,l}=k[n_0+l-1]\ q_{k,l}=k\ ,\ k,l\geq 1 \tag{2.3.53}$$

The character is given again by (2.3.42).

We will now focus on the degenerate representations of NS_3. They are characterized by the following conditions,

$$\tilde{c}\geq 1\ ,\ g^{NS}_n\geq 0\ \forall n\in Z+\frac{1}{2} \tag{2.3.54}$$

For a fixed $\tilde{c}$ this is a convex region in the (h,q) plane bounded by pieces of the $g^{NS}_n=0$ lines. The degenerate representations lie on the boundary of the region above and can be labeled by n_0 such that $g^{NS}_{n_0}=0$ and their charge. This implies that their dimensions and charges are given by,

$$(\tilde{c}-1)(n_0-\frac{1}{2})<q\leq(\tilde{c}-1)(n_0+\frac{1}{2}) \tag{2.3.55a}$$

$$h=n_0q-\frac{(\tilde{c}-1)}{2}(n_0^2-\frac{1}{4}) \tag{2.3.55b}$$

We will focus again on $n_0>0$.

(A') $\tilde{c}>1$ *rational.*

(i) $q=(n_0+\frac{1}{2})(\tilde{c}-1)$. In this case the embedding diagram is shown in fig. 15 with,

$$h_k=h+kn_0\ ,\ h'_k=h+k(n_0+1)\ ,\ q_k=q'_k=q+k \tag{2.3.56}$$

For the other allowed values of q we have to distinguish the following two cases

(ii) $q = \frac{n}{2}(\tilde{c}-1) - m$ with $n \in Z$, $m \in Z_0^+$. The embedding diagram in this case is shown in fig.14 with,

$$h_k = h + kn_0 \ , \ h'_{m+k+r_2} = h + (r_2+k)n_0 + kr_1 \ , \ q_k = q'_k = q + k \qquad (2.3.57)$$

(iii) q has any other allowed valued except the ones mentioned in (i), (ii). Then the embedding structure is the one shown in fig. 2.

(B') $\tilde{c} > 1$ *irrational.*

(i') $q = (n_0 + \frac{1}{2})(\tilde{c}-1)$. Then the embedding diagram is the one shown in fig. 16 with,

$$h_k = h + kn_0 \ , \ h'_k = h + k(n_0+1) \ , \ q_k = q'_k = q + k \qquad (2.3.58)$$

(ii') For all the other allowed values of q the embedding pattern is the one of fig. 2.

The above exhaust all possible degenerate representations belonging to NS_3. In the $\tilde{c} = 1$ case the only degenerate representation is given by the unit operator. From the structure of the representations of NS_3 we can conclude that their characters are given by (2.3.39).

Thus we can distinguish representations for $\tilde{c} \geq 1$ in those that have only degeneracies related to g_n with their corresponding characters given by (2.3.39) and in those that have additional degeneracies related to $f_{1,2}$ whose characters are given by (2.3.42).

The same results apply in the case $n_0 < 0$ with the following substitutions in the relevant formulae : $n_0 \to |n_0|$, $w \to w^{-1}$, $w^q \to w^q$.

The null hwv's which correspond to the representations studied above degenerate at relative charge ± 1 do not generate full Verma modules. There exist lowering operators which annihilate them.*

The above complete the derivation of the characters for all the unitary degenerate representations of the N=2 algebras. The characters of the non-degenerate representations are given in Appendix 2.B.

The special values of $\tilde{c}$ mentioned in [18], namely $\tilde{c} = 1 + \frac{2}{n}$, $n = 1, 2, 3, \ldots$ also contain the interesting case of $\tilde{c} = 3(2)$ arising in the string theory compactification on a compact six(four) dimensional Ricci flat manifold. In particular the (anti-)holomorphic ϵ-tensor realizes the representations of the NS_2 algebra, (since it is a space-time boson), with $q = \pm\tilde{c}$ and $h = \frac{\tilde{c}}{2}$ corresponding to our notation to $r_1 = 1(2)$, $r_2 = 1$, $n_0 = \pm 1/2$, $n = \pm 3(\pm 4)$, $m = 0$. The embedding structure of their Verma module is depicted in fig. 12. The covariantly constant spinors on the internal manifold correspond to degenerate representations of the $R_2^{\pm}$, (space-time fermions), which are degenerate at level $n_0 = 0$ with $h = \frac{\tilde{c}}{8}$ and $q = sgn(0)\frac{\tilde{c}+1}{2}$ (lying on the intersection of $g_0^R = 0$ and $f_{1,2}^R = 0$). These representations are important in the construction of the four generators of the four-dimensional N=1 supersymmetry. The dimensions and charges

⋆ For explicit examples see App. 2.A.

of these operators should not be renormalized even non-perturbatively since the spectrum for this class of representations is discrete. Their partition functions can be read-off immediately from (2.3.42), and they provide the means to study questions of modular invariance in the corresponding σ-model.

2.4 N=2 Supersymmetry and the Analytic Geometry of (2,0) Superspace

In this section we are going to discuss the local geometry of (2,0) superspace†.

N=2 supersymmetry is a natural extension of N=1 supersymmetry. In this case we have two different supersymmetry generators (supercharges), as well as an O(2) (or U(1)) current which manifests the symmetry of the theory under an O(2) rotation of the two supersymmetries. The natural space to define the fields of the theory is N=2 superspace, (or more precisely (2,0) superspace). In a theory with (super)conformal invariance the left and right sectors of the theory completely decouple, so that the structure of the theory is that of a tensor product of the left and right sectors. From now on we will restrict ourselves to the left sector only, keeping in mind the previous remarks.

(2,0) superspace includes, apart from the complex analytic coordinate z, two other fermionic coordinates, θ and $\bar{\theta}$ corresponding to the two supersymmetries.

$$\theta^2 = \bar{\theta}^2 = \{\theta, \bar{\theta}\} = 0. \tag{2.4.1}$$

A point in superspace will be denoted by $\mathbf{z} \equiv (z, \theta, \bar{\theta})$.

A superfield is an analytic function in $\mathbf{z}$ defined through its power series expansion in the fermionic coordinates:

$$\Phi(\mathbf{z}) \equiv \phi(z) + \theta\bar{\psi}(z) + \bar{\theta}\psi(z) + \theta\bar{\theta}g(z). \tag{2.4.2}$$

The two supersymmetry transformations can be written as:

$$(z, \theta, \bar{\theta}) \to (z - \epsilon\bar{\theta}, \theta + \epsilon, \bar{\theta}) \tag{2.4.3a}$$

$$(z, \theta, \bar{\theta}) \to (z - \bar{\epsilon}\theta, \theta, \bar{\theta} + \bar{\epsilon}) \tag{2.4.3b}$$

where $\epsilon, \bar{\epsilon}$ are anticommuting variables which are the parameter of the transformation. Under

† Global issues have been discussed in [30].

the two supersymmetry transformations, (2.4.3a,b), a superfield transforms as:

$$\Phi(z,\theta,\bar\theta) \rightarrow \Phi(z-\epsilon\bar\theta, \theta+\epsilon, \bar\theta) = \phi(z-\epsilon\bar\theta) + (\theta+\epsilon)\bar\psi(z-\epsilon\bar\theta) + \bar\theta\psi(z-\epsilon\bar\theta)$$

$$+(\theta+\epsilon)\bar\theta g(z-\epsilon\bar\theta) = \phi(z) + \epsilon\bar\psi(z) + \bar\theta[\epsilon\partial_z\phi(z) - \epsilon g(z) + \psi(z)]$$

$$+\theta\bar\psi(z) + \theta\bar\theta[g(z) + \epsilon\partial_z\bar\psi(z)] \tag{2.4.4a}$$

$$\Phi(z,\theta,\bar\theta) \rightarrow \Phi(z-\bar\epsilon\theta, \theta, \bar\theta+\epsilon) = \phi(z-\bar\epsilon\theta) + \theta\bar\psi(z-\bar\epsilon\theta) + (\bar\theta+\bar\epsilon)\psi(z-\bar\epsilon\theta)$$

$$+\theta(\bar\theta+\bar\epsilon)g(z-\bar\epsilon\theta) = \phi(z) + \bar\epsilon\psi(z) + \theta[\bar\psi(z) + \bar\epsilon\partial_z\phi(z) + \bar\epsilon g(z)]$$

$$+\bar\theta\psi(z) + \theta\bar\theta[g(z) - \bar\epsilon\partial_z\psi(z)] \tag{2.4.4b}$$

which implies the following transformation laws for the component fields:

$$\begin{array}{ll} \delta_\epsilon\phi(z) = \epsilon\bar\psi(z) & \bar\delta_{\bar\epsilon}\phi(z) = \bar\epsilon\psi(z) \\ \delta_\epsilon\psi(z) = \epsilon[\partial_z\phi(z) - g(z)] & \bar\delta_{\bar\epsilon}\psi(z) = 0 \\ \delta_\epsilon\bar\psi(z) = 0 & \bar\delta_{\bar\epsilon}\bar\psi(z) = \bar\epsilon[\partial_z\phi(z) + g(z)] \\ \delta_\epsilon g(z) = \epsilon\partial_z\bar\psi(z) & \bar\delta_{\bar\epsilon}g(z) = -\bar\epsilon\partial_z\psi(z) \end{array} \tag{2.4.5}$$

It is easy to verify the global supersymmetry algebra:

$$[\delta_\epsilon, \delta_{\bar\epsilon}] = 2\epsilon\bar\epsilon\,\frac{\partial}{\partial z}, \qquad [\delta_\epsilon, \delta_\epsilon] = [\delta_{\bar\epsilon}, \delta_{\bar\epsilon}] = 0. \tag{2.4.6}$$

The covariant derivatives in superspace are defined by:

$$D \equiv \frac{\partial}{\partial\theta} + \bar\theta\frac{\partial}{\partial z} \quad , \quad \bar D \equiv \frac{\partial}{\partial\bar\theta} + \theta\frac{\partial}{\partial z} \tag{2.4.7a}$$

$$D^2 = \bar D^2 = 0 \quad , \quad \{D, \bar D\} = 2\frac{\partial}{\partial z}. \tag{2.4.7b}$$

We introduce here the notion of a chiral N=2 superfield, as a superfield satisfying one of the following conditions:

$$D\Phi(\mathbf{z}) = 0 \Longrightarrow \Phi(\mathbf{z}) = \phi(z) + 2\bar{\theta}\psi(z) - \theta\bar{\theta}\partial_z\phi(z) \tag{2.4.8a}$$

$$\bar{D}\bar{\Phi}(\mathbf{z}) = 0 \Longrightarrow \Phi(\mathbf{z}) = \bar{\phi}(z) + 2\theta\bar{\psi}(z) + \theta\bar{\theta}\partial_z\bar{\phi}(z) \tag{2.4.8b}$$

The Grassman integration is defined through the usual standard rules:

$$\int d\theta d\bar{\theta} = \int d\theta d\bar{\theta}\theta = \int d\theta d\bar{\theta}\bar{\theta} = 0 \quad , \quad \int d\theta d\bar{\theta}\,\bar{\theta}\theta = 1. \tag{2.4.9}$$

If we call the generators of the two supersymmetries $G_{-1/2}, \bar{G}_{-1/2}$ then eq. (2.4.6) is translated into:

$$\{G_{-1/2}, G_{-1/2}\} = \{\bar{G}_{-1/2}, \bar{G}_{-1/2}\} = 0, \qquad \{G_{-1/2}, \bar{G}_{-1/2}\} = 2L_{-1} \tag{2.4.10}$$

L_{-1} being the usual translation operator on the complex plane. The full superconformal symmetry is generated by the usual Virasoro generators L_n, the supersymmetry generators,

$$G_r \equiv \frac{2}{n+1}[L_n, G_{-1/2}]\,,\ \bar{G}_r \equiv \frac{2}{n+1}[L_n, \bar{G}_{-1/2}]\,,\ r = n - 1/2 \tag{2.4.11}$$

and the U(1) current generators, J_n, which implement the U(1) symmetry, under which the two supercurrents are in complex conjugate representations. The full N=2 superconformal algebra then takes the form:

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{\tilde{c}}{4}(m^3 - m)\delta_{m+n,0} \\
&[L_m, G_r] = (\frac{m}{2} - r)G_{m+r} \quad , \quad [L_m, \bar{G}_r] = (\frac{m}{2} - r)\bar{G}_{m+r} \\
&[J_m, J_n] = \tilde{c}m\,\delta_{m+n,0} \quad , \quad [J_m, G_r] = G_{m+r}, [J_m, \bar{G}_r] = \bar{G}_{m+r} \\
&\{G_r, G_s\} = \{\bar{G}_r, \bar{G}_s\} = 0 \quad , \quad [L_m, J_n] = -nJ_{m+n} \\
&\{G_r, \bar{G}_s\} = 2L_{r+s} + (r-s)J_{r+s} + \tilde{c}(r^2 - \frac{1}{4})\delta_{r+s,0}
\end{aligned} \tag{2.4.12}$$

It is the generating algebra of N=2 superanalytic transformations in N=2 superspace. We should, at this point, define what we mean by an extended superanalytic transformation. The most general coordinate transformation in N = 2 superspace has the form*

$$\begin{aligned}
z' &= f_0(z) + \theta f_1(z) + \bar{\theta}\bar{f}_1(z) + \theta\bar{\theta}f_2(z) \\
\theta' &= g_0(z) + \theta g_1(z) + \bar{\theta}\bar{g}_1(z) + \theta\bar{\theta}g_2(z) \\
\bar{\theta}' &= h_0(z) + \theta h_1(z) + \bar{\theta}\bar{h}_1(z) + \theta\bar{\theta}h_2(z)
\end{aligned} \tag{2.4.13}$$

A natural definition for an extended superanalytic transformation is one under which the covariant derivatives transform homogeneously. Under (2.4.13) the covariant derivatives transform

* $f_0, f_2, g_1, \bar{g}_1, h_1, \bar{h}_1$ are commuting functions, whereas $f_1, \bar{f}_1, g_0, g_2, h_0, h_2$ are anticommuting ones.

as:

$$D = (D\theta')D' + (D\bar{\theta}')\frac{\partial}{\partial\bar{\theta}'} + [Dz' - (D\theta')\bar{\theta}']\frac{\partial}{\partial z'} \tag{2.4.14a}$$

$$\bar{D} = (\bar{D}\bar{\theta}')\bar{D}' + (\bar{D}\theta')\frac{\partial}{\partial\theta'} + [\bar{D}z' - (\bar{D}\bar{\theta}')\theta']\frac{\partial}{\partial z'}. \tag{2.4.14b}$$

Consequently the conditions for (2.4.13) to be a superanalytic transformation are:†

$$\bar{D}\theta' = D\bar{\theta}' = Dz' - (D\theta')\bar{\theta}' = \bar{D}z' - (\bar{D}\bar{\theta}')\theta' = 0. \tag{2.4.15}$$

Solving (2.4.15) we arrive at the most general form of an extended superanalytic transformation:

$$\begin{aligned} z' &= f_0(z) + \theta g_1(z)h_0(z) + \bar{\theta}\bar{h}_1(z)g_0(z) + \theta\bar{\theta}[g_0(z)h_0(z)]' \\ \theta' &= g_0(z) + \theta g_1(z) + \theta\bar{\theta}g_0'(z) \\ \bar{\theta}' &= h_0(z) + \bar{\theta}h_1(z) - \theta\bar{\theta}h_0'(z) \end{aligned} \tag{2.4.16}$$

along with the supplementary condition:

$$f_0'(z) = g_0'(z)h_0(z) - g_0(z)h_0'(z) + g_1(z)\bar{h}_1(z) \tag{2.4.17}$$

where in (2.4.17) and in the left-hand side of (2.4.16) a prime means differentiation with respect to z.

In particular the global supersymmetry transformations are special cases of (2.4.16) with $f_0(z) = z, g_0(z) = \epsilon, h_0(z) = 0, g_1(z) = \bar{h}_1(z) = 1$ and $f_0(z) = z, g_0(z) = 0, h_0(z) = \bar{\epsilon}, g_1(z) = \bar{h}_1(z) = 1$ respectively.

We define the two abelian N=2 superdifferentials through their transformation properties under analytic superconformal transformations:

$$d\mathbf{z}' \equiv (D\theta')d\mathbf{z}\ ,\ \tilde{\mathbf{z}}' \equiv (\tilde{D}\tilde{\theta}')d\tilde{\mathbf{z}} \tag{2.4.18}$$

The superconformal tensor fields are defined by the condition that

$$\Phi(\mathbf{z})(d\mathbf{z})^{\Delta+Q/2}(d\tilde{\mathbf{z}})^{\Delta-Q/2}$$

is an N=2 superconformal invariant quantity, where Δ, Q are the dimensions and charge of the lowest component field. They are the primary superfields generating the highest weight irreducible representations of the N=2 superconformal algebra. Globally defined tensor superfields must have dimensions and charges which are integers or half integers. They can be constructed as composite operations from locally defined fields.

† In fact, even if we demand that D transforms in general as $D = (D\theta')D + (D\bar{\theta}')\bar{D}'$ we end up at (2.4.15). There is a dual requirement, $D = (D\bar{\theta}')\bar{D}'$ which gives conditions conjugate to (2.4.15)

We can also extend the Cauchy integral formulas in superspace. If we define the invariant "distances," $\mathbf{z}_{ij} \equiv z_i - z_j - \theta_i\bar{\theta}_j - \bar{\theta}_i\theta_j, \theta_{ij} \equiv \theta_i - \theta_j, \bar{\theta}_{ij} \equiv \bar{\theta}_i - \bar{\theta}_j$, and the "volume" element $d\mathbf{z} = dz d\theta d\bar{\theta}$, then, [28],

$$
\begin{aligned}
&\frac{1}{2\pi i}\oint_C d\mathbf{z}_1\, \Phi\,(\mathbf{z}_1)\bar{\theta}_{12}\theta_{12} = 0 \\
&\frac{1}{2\pi i}\oint_C d\mathbf{z}_1\, \Phi\,(\mathbf{z}_1)\frac{\bar{\theta}_{12}\theta_{12}}{\mathbf{z}_{12}^m} = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial z_2^{m-1}}\Phi\,(\mathbf{z}_2) \\
&\frac{1}{2\pi i}\oint_C d\mathbf{z}_1\, \Phi\,(\mathbf{z}_1)\frac{\bar{\theta}_{12}}{\mathbf{z}_{12}^m} = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial z_2^{m-1}}D\Phi(\mathbf{z}_2) \qquad . \\
&\frac{1}{2\pi i}\oint_C d\mathbf{z}_1\, \Phi\,(\mathbf{z}_1)\frac{\theta_{12}}{\mathbf{z}_{12}^m} = -\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial z_2^{m-1}}\bar{D}\,\Phi\,(\mathbf{z}_2) \\
&\frac{1}{2\pi i}\oint_C d\mathbf{z}_1\, \Phi\,(\mathbf{z}_1)\frac{1}{\mathbf{z}_{12}^m} = \frac{1}{2}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial z_2^{m-1}}[D,\bar{D}]\,\Phi(\mathbf{z}_2)
\end{aligned}
\tag{2.4.19}
$$

The prescription to evaluate the integrals above is the following: First, do the Grassman integrations using eq. (2.4.9) and then perform the complex integrations in the usual way. The contour C is winding around the point z_2.

The N=2 superanalytic transformations are generated by the stress-energy superfield, which in component form can be written as:

$$
\mathbf{J}(\mathbf{z}) \equiv J(z) + i\theta\bar{G}(z) + i\bar{\theta}G(z) + 2\theta\bar{\theta}T(z). \tag{2.4.20}
$$

The Fourier modes of the generators are defined in the usual way:

$$
\begin{aligned}
J(z) &\equiv \sum_{n\in\mathbf{Z}}\frac{J_n}{z^{n+1}} \qquad , \qquad & T(z) &\equiv \sum_{n\in\mathbf{Z}}\frac{L_n}{z^{n+2}} \\
G(z) &\equiv \sum_{n\in\mathbf{Z}}\frac{G_{n-1/2}}{z^{n+1}} & \bar{G}(z) &\equiv \sum_{n\in\mathbf{Z}}\frac{\bar{G}_{n-1/2}}{z^{n+1}}
\end{aligned}
\tag{2.4.21}
$$

These generators are represented in the space of superfield functions in the following way:

$$
\begin{aligned}
L_n &\equiv -z^{n+1}\frac{\partial}{\partial z} - \frac{n+1}{2}z^n[\theta\frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial\bar{\theta}}] \\
J_n &\equiv z^n[\bar{\theta}\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial\theta}] \\
G_{n-1/2} &\equiv z^n[\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial z}] + nz^{n-1}\theta\bar{\theta}\frac{\partial}{\partial\theta} \\
\bar{G}_{n-1/2} &\equiv z^n[\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial z}] - nz^{n-1}\theta\bar{\theta}\frac{\partial}{\partial\bar{\theta}}
\end{aligned}
\tag{2.4.22}
$$

It is straightforward to check that the generators in equations (2.4.22) satisfy the N=2 superconformal loop algebra, (as in (2.4.12) with $\tilde{c}=0$), which is the algebra of N=2 superconformal transformations over S^1. The explicit representation (2.4.22) will be useful later on in this paper, to analyze the correlation functions of N=2 superconformal invariant theories.

The stress-energy tensor has an operator product expansion with itself:

$$\mathbf{J}(\mathbf{z}_1)\mathbf{J}(\mathbf{z}_2) = \frac{\theta_{12}}{\mathbf{z}_{12}} D\mathbf{J}(\mathbf{z}_2) - \frac{\bar{\theta}_{12}}{\mathbf{z}_{12}} \bar{D}\mathbf{J}(\mathbf{z}_2) + 2\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}^2}\mathbf{J}(\mathbf{z}_2)$$

$$+2\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\mathbf{J}'(\mathbf{z}_2) + \frac{\tilde{c}}{\mathbf{z}_{12}^2} \tag{2.4.23}$$

where the anomaly $\tilde{c}$ is normalized, so that a free scalar N=2 superfields has $\tilde{c}=1$. Eq. (2.4.23) corresponds to a change of the stress-energy tensor under a superconformal transformation

$$\delta_v\mathbf{J}(\mathbf{z}) = [\partial_z v]\mathbf{J}(\mathbf{z}) + v\partial_z\mathbf{J}(\mathbf{z}) + \frac{1}{2}[\bar{D}v]D\mathbf{J}(\mathbf{z}) + \frac{1}{2}[Dv]\bar{D}\mathbf{J}(\mathbf{z})$$

$$+\frac{\tilde{c}}{4}\partial_z[\bar{D}, D]v \tag{2.4.24}$$

v being an infinitesimal N=2 superfield.

The change in the stress-energy tensor under a finite superconformal transformation is given by:

$$\mathbf{J}(\mathbf{z}) = \mathbf{J}'(\mathbf{z}')[D\theta'][\bar{D}\bar{\theta}'] + \frac{\tilde{c}}{2}S(\mathbf{z},\mathbf{z}') \tag{2.4.25}$$

where the N=2 super-Schwarzian derivative is defined through:

$$S(\mathbf{z},\mathbf{z}') \equiv \frac{\partial\bar{D}\bar{\theta}'}{\bar{D}\bar{\theta}'} - \frac{\partial D\theta'}{D\theta'} - 2\frac{\partial\bar{\theta}'\,\partial\theta'}{(\bar{D}\bar{\theta}')\,(D\theta')} \tag{2.4.26}$$

It satisfies the following composition law:

$$S(\mathbf{z}_1,\mathbf{z}_3) = S(\mathbf{z}_1,\mathbf{z}_2) + (D\theta_2)(\bar{D}\bar{\theta}_2)S(\mathbf{z}_2,\mathbf{z}_3) \tag{2.4.27}$$

On the sphere for a vector field to be globally defined, it must have a vanishing "anomaly," that is under an infinitesimal transformation generated by it, the anomalous part in (2.4.24) must vanish, that is, $\partial_z[\bar{D}, D]v = 0$, which gives an eight-parameter family of globally defined vector fields on the sphere:

$$v(\mathbf{z}) = v_{-1} + v_0 z + v_1 z^2 + \theta[u_{-1/2} + u_{1/2}z] + \bar{\theta}[\bar{u}_{-1/2} + \bar{u}_{1/2}z] + q_0\theta\bar{\theta}. \qquad (2.4.28)$$

These vector fields generate the global N=2 superconformal algebra, $osp(2|2)$. (In fact they generate half of $osp(2|2)$, its holomorphic part.) The global N=2 superconformal algebra is the maximal, finite dimensional, subalgebra of the N=2 superconformal algebra. It contains the generators of the ordinary projective transformations, L_1, L_0, L_{-1}, the supercharges $G_{\pm 1/2}, \bar{G}_{\pm 1/2}$ and the zero mode of the U(1) current. It is easy to check using (2.4.12) that this set of generators closes into itself, and it contains as a subalgebra, the N=1 superconformal algebra, $osp(2|1)$. Since the Schwarzian derivative transforms as in (2.4.27), the fact that it vanishes for infinitesimal global N=2 transformations continues to be true for finite transformations belonging to the identity component of the group.

The $OSP(2|2)$ group transformations can be found either by exponentiating the generators of the algebra given in (2.5.22) or using the general form of superanalytic transformations (2.4.16), and some analyticity arguments [29]. Another way is to solve the equation $S(\mathbf{z}, \mathbf{z}') = 0$. There are three parameters associated with the subgroup $SL(2, C)$, four supersymmetry parameters, (Grassman), $\epsilon_1, \epsilon_2, \bar{\epsilon}_1, \bar{\epsilon}_2$ and a parameter q associated with the zero mode of the U(1) current. The group transformations are:

$$z' = \frac{az+b}{cz+d} + e^{q}\theta\frac{(1-\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\bar{\epsilon}_1 z + \bar{\epsilon}_2(1+\frac{1}{2}\epsilon_2\bar{\epsilon}_1)}{(cz+d)^2} + e^{-q}\bar{\theta}\frac{(1+\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\epsilon_1 z + (1-\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\epsilon_2}{(cz+d)^2}$$

$$+\frac{[2d\epsilon_1\bar{\epsilon}_1 - 2c(\bar{\epsilon}_1\epsilon_2 + \bar{\epsilon}_2\epsilon_1)]z + d(\bar{\epsilon}_1\epsilon_2 + \bar{\epsilon}_2\epsilon_1) - 2c\bar{\epsilon}_2\epsilon_2}{(cz+d)^3} \qquad (2.4.29a)$$

$$\theta' = \frac{\epsilon_1 z + \epsilon_2}{cz+d} + e^{q}\theta\frac{1+\frac{1}{2}(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2) + \frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2}{(cz+d)} + \theta\bar{\theta}\frac{\epsilon_1 d - \epsilon_2 c}{(cz+d)^2} \qquad (2.4.29b)$$

$$\bar{\theta}' = \frac{\bar{\epsilon}_1 z + \bar{\epsilon}}{cz+d} + e^{-q}\bar{\theta}\frac{1+\frac{1}{2}(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2) + \frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2}{(cz+d)} + \theta\bar{\theta}\frac{\bar{\epsilon}_2 c - \bar{\epsilon}_1 d}{(cz+d)^2} \qquad (2.4.29c)$$

The N=2 superconformal vector field generates the group of N=2 super-diffeomorphisms on the circle, $\hat{\hat{D}}iff(S^1)$. The Schwarzian derivative is the globally invariant generator of the second cohomology group of $\hat{\hat{D}}iff(S^1)$. It generates a non-trivial transformation on the stress-energy tensor viewed as a connection on moduli space.

As can be seen from (2.4.12), the subalgebra does not have an anomaly even if $\tilde{c} \neq 0$. This is of crucial importance in a superconformal theory as we will see later. It implies that all correlation functions are invariant under $OSP(2|2)$ constraining in such a way their form. Along with some supplementary constraints on the correlation functions, present when the theory has degenerate representations, it helps to determine the correlation functions completely, rendering the theory exactly solvable.

2.5 The Ground States and Primary Fields in N=2 Superconformal Field Theories

An N=2 superconformal field theory is a field theory invariant under the N=2 superanalytic transformations described in the previous section, which form the N=2 superconformal group. The infinitesimal transformations are generated by an infinitesimal local superfield $v(\mathbf{z})$:

$$v(\mathbf{z}) \equiv v_0(z) + \theta\, v_1(z) + \bar{\theta}\bar{v}_1(z) + \theta\bar{\theta}\, v_2(z) \tag{2.5.1}$$

$$\begin{aligned} z' &= z + v(\mathbf{z}) + \frac{1}{2}[(\bar{D}v)\theta + (Dv)\bar{\theta}] \\ \theta' &= \theta + \frac{1}{2}\bar{D}v \quad , \quad \bar{\theta}' = \bar{\theta} + \frac{1}{2}Dv \end{aligned} \tag{2.5.2}$$

The function $v_1, \bar{v}_1$ are Grassman functions anticommuting among themselves and with $\theta, \bar{\theta}$, whereas v_0, v_2 are usual meromorphic functions. The superconformal transformations are generated by the super-stress-energy tensor, see (2.4.20). Using the Cauchy formulas of the previous section we can write the change of a local superfield under a superconformal transformation as:

$$\delta_v \Phi(\mathbf{z}) = -\frac{1}{4\pi i}\oint_{C_z} d\mathbf{z}' v(\mathbf{z}')\mathbf{J}(\mathbf{z}')\Phi(\mathbf{z}) \tag{2.5.3}$$

where the contour C_z surrounds the point z in the complex plane.

The variation (2.5.3) is determined by the singularities of the OPE, of the stress-energy tensor with the superfield. In particular a superfield function transforms under an infinitesimal transformation as:

$$\delta_v \Phi = v\partial\Phi + \frac{1}{2}Dv\bar{D}\Phi + \frac{1}{2}\bar{D}vD\Phi. \tag{2.5.4}$$

It is usually convenient to use radial quantizations going, (through a superanalytic transformation), from the cylinder to the plane, $(lnz, z^{-1/2}\theta, z^{-1/2}\bar{\theta}) \longleftrightarrow (\tau + i\sigma, \theta, \bar{\theta})$.

The fermionic fields on the cylinder can have two possible boundary conditions*, periodic or antiperiodic. On the plane, this is translated to $G, \bar{G}(ze^{2\pi i}) = \pm G, \bar{G}(z)$, the corresponding subspaces of the full Hilbert spaces being the NS and R sectors. In the NS sector $G(ze^{2\pi i}) = G(z)$ whereas in the Ramond sector, $G(ze^{2\pi i}) = -G(z)$, that is the fermionic fields are double valued on the plane.

$\star$ We will postpone for the moment the discussion of more general boundary conditions.

The operator product expansion for the stress-energy tensor was given in (2.4.23). The terms that appear in (2.4.23) are the most general terms that are allowed in a Euclidean N=2 supersymmetric quantum field theory, satisfying the standard constructive field theory axioms. The proof of [10] can be extended easily in our case, to guarantee (2.4.23) provided the theory has scale invariance and global N=2 supersymmetry. Using the mode expansions (2.4.21) we can derive (2.4.12) from (2.4.23). The stress-energy tensor must be a Hermitian operator, implying some hermiticity conditions among its components:

$$L_n^\dagger = L_{-n},\ J_n^\dagger = J_{-n},\ G_r^\dagger = \bar{G}_{-r}\ \bar{G}_r^\dagger = G_{-r}. \tag{2.6.5}$$

We define the in-vacuum $|0\rangle$ of the theory at time $\tau = -\infty, (z = 0)$, to be $OSP(2|2)$ invariant. This means that it is annihilated by $L_n, n \geq -1$, $J_n, n \geq 0$, $G_r, \bar{G}_r, r \geq -1/2$, ($NS$ sector) or $G_n, \bar{G}_n, n \geq 0$ in the R sector. In the same way the out-vacuum is defined at $z \to \infty$. The vacuum state belongs to the NS sector and it is the ground state of the theory. The unitary irreducible representations of the N=2 superconformal algebra are generated from highest weight vectors, (hwv), by the action of the lowering operators of the algebra , $L_n, J_n, G_r, \bar{G}_r, n, r < 0$.

In the NS sector the hwv's are generated by the action of primary conformal superfields on the vacuum state. Their defining relations are their transformation properties under superconformal transformations encoded in their OPE with the stress-energy tensor:

$$\begin{aligned}\mathbf{J}(\mathbf{z}_1)\Phi(\mathbf{z}_2) = {} & 2\Delta\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}^2}\Phi(\mathbf{z}_2) + 2\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\Phi'(\mathbf{z}_2) + \frac{\theta_{12}}{\mathbf{z}_{12}}D\Phi(\mathbf{z}_2) \\ & - \frac{\bar{\theta}_{12}}{\mathbf{z}_{12}}\bar{D}\Phi(\mathbf{z}_2) + \frac{Q\Phi(\mathbf{z}_2)}{\mathbf{z}_{12}}\end{aligned} \tag{2.5.6}$$

Using (2.5.3) and (2.5.6) we can derive the transformation law for a primary superfield operator:

$$\begin{aligned}\delta_v\Phi(\mathbf{z}) = {} & \Delta(\partial_z v)\Phi(\mathbf{z}) + v\partial_z\Phi(\mathbf{z}) + \frac{1}{2}[\bar{D}v]D\Phi(\mathbf{z}) + \frac{1}{2}[Dv]\bar{D}\Phi(\mathbf{z}) \\ & - \frac{Q}{4}\{[D,\bar{D}]v\}\Phi(\mathbf{z})\end{aligned} \tag{2.5.7}$$

Under a finite transformation $\Phi(\mathbf{z})$ transforms as:

$$\Phi(\mathbf{z}) = \Phi(\mathbf{z}')[D\theta']^{\Delta+\frac{Q}{2}}[\bar{D}\bar{\theta}']^{\Delta-\frac{Q}{2}} \tag{2.5.8}$$

where (Δ, Q) are its dimension and $U(1)$ charge. The hwv in the NS sector are characterized by their eigenvalues under the zero modes of the algebra:

$$L_0|\Phi\rangle = \Delta|\Phi\rangle \quad , \quad J_0|\Phi\rangle = Q|\Phi\rangle \tag{2.5.9}$$

Being hw states they must be annihilated by the raising operators of the algebra:

$$L_n|\Phi\rangle = J_n|\Phi\rangle = G_n|\Phi\rangle = \bar{G}_n|\Phi\rangle, n > 0. \tag{2.5.10}$$

The OPE (2.5.6) can be written also as commutation relations which will be useful later on:

$$\begin{aligned}
[L_n, \Phi(\mathbf{z})] &= z^{n+1}\frac{\partial}{\partial z}\Phi(\mathbf{z}) + (n+1)z^n[\Delta + \frac{1}{2}[\theta\frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial\bar{\theta}}]]\Phi(\mathbf{z}) \\
&+ \frac{Q}{2}n(n+1)z^{n-1}\theta\bar{\theta}\Phi(\mathbf{z}) \\
[J_n, \Phi(\mathbf{z})] &= z^n[Q + \bar{\theta}\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial\theta}]\Phi(\mathbf{z}) + 2n\Delta z^{n-1}\theta\bar{\theta}\Phi(\mathbf{z}) \\
[G_r, \Phi(\mathbf{z})] &= z^{r+\frac{1}{2}}[\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial z}]\Phi(\mathbf{z}) - (r+\frac{1}{2})z^{r-\frac{1}{2}}[(2\Delta + Q)\theta + \theta\bar{\theta}\frac{\partial}{\partial\theta}]\Phi(\mathbf{z}) \\
[\bar{G}_r, \Phi(\mathbf{z})] &= z^{r+\frac{1}{2}}[\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial z}]\Phi(\mathbf{z}) - (r+\frac{1}{2})z^{r-\frac{1}{2}}[(2\Delta - Q)\bar{\theta} - \theta\bar{\theta}\frac{\partial}{\partial\bar{\theta}}]\Phi(\mathbf{z})
\end{aligned} \tag{2.5.11}$$

In the R-sector the zero modes are L_0, J_0, and $\bar{G}_0, G_0$, their eigenvalues characterizing hwv's. There are two kinds of hwv's, $|\Delta, Q \mp 1/2\rangle_\pm$, [18],

$$\begin{aligned}
L_0|\Delta, Q \mp \frac{1}{2}\rangle_\pm &= \Delta|\Delta, Q \mp \frac{1}{2}\rangle_\pm \\
J_0|\Delta, Q \mp \frac{1}{2}\rangle_\pm &= (Q \mp \frac{1}{2})|\Delta, Q \mp \frac{1}{2}\rangle_\pm
\end{aligned} \tag{2.5.12}$$

which satisfy an additional hwv condition with respect to the supercharges:

$$G_0|\Delta, Q + \frac{1}{2}\rangle_- = 0 \quad , \quad \bar{G}_0|\Delta, Q - \frac{1}{2}\rangle_+ = 0. \tag{2.5.13}$$

Consequently there are two kinds of representations, $R^\pm$. The two representations are isomorphic under charge conjugations ($G_n \leftrightarrow \bar{G}_n, J_n \to -J_n$).

From now on we will restrict to one of them, say R^+, our statements being valid for R^- as well.

In the R-sector the ground state is not unique. There are two ground states degenerate in energy, (i.e., having the same dimension). $|\Theta^+\rangle$ and $G_0|\Theta^+\rangle \equiv |\Theta^-\rangle$. They are generated from the vacuum $|0\rangle$, (which belongs to the NS sector), by primary fields $\Theta^\pm(\mathbf{z})$, much like the spin fields of the N=1 superconformal theories. The spin fields have double-valued OPE with the stress-energy tensor, for example:

$$G(z)\Theta^\pm(\omega) = \frac{1}{2}\alpha_\pm\frac{\Theta^\mp(\omega)}{(z-\omega)^{3/2}} \tag{2.5.14}$$

where $\alpha_+ = 1, \quad \alpha_- = \Delta - \tilde{c}/8$. This happens in order for the spin field to be able to change the boundary conditions of the fermionic parts of the superfields. We can view the spin fields as opening and closing cuts on the cylinder. The states in the R-sector are generated by ordinary conformal superfields acting on the Ramond ground states. The generators of global N=2 supersymmetry transformations in the R-sector are $G_0, \bar{G}_0$.

Unbroken N=2 supersymmetry is implied by the existence of a ground state which is annihilated by the global N=2 supersymmetry generators. The state $|\Theta^+\rangle$ is annihilated by $\bar{G}_0$ due to (2.5.13). Applying $\{G_0, \bar{G}_0\}$ to it we obtain:

$$\{G_0, \bar{G}_0\}|\Theta^+\rangle = \bar{G}_0 G_0|\Theta^+\rangle = (2L_0 - \tilde{c}/4)|\Theta^+\rangle = (2\Delta - \tilde{c}/4)|\Theta^+\rangle. \tag{2.5.15}$$

Consequently, in order for G_0 to annihilate $|\Theta^+\rangle$, its dimension must be $\Delta_+ = \tilde{c}/8$. The operator $\{G_0, \bar{G}_0\}$ is a hermitian positive operator, thus any dimension in the R-sector has to be $\geq \tilde{c}/8$. This is the reason that the vacuum $|0\rangle$, the lowest energy state must belong to the NS-sector. In the same way $\bar{G}_0|\Theta^-\rangle = 0$, implies $\Delta_- = \tilde{c}/8$. Therefore, the existence of a state in the R-sector with $\Delta = \tilde{c}/8$ implies unbroken N=2 supersymmetry on the cylinder. On the other hand if such a state does not exist in the theory the one supersymmetry out of the two is broken.

So far we have been discussing the two sectors of the N=2 superconformal theory that parallel the situation in ordinary N=1 superconformal theories. In the N=2 case though, unlike the N=1, there is another sector present in general due to the fact that N=2 superfields contain two fermionic components, so there is also the possibility of choosing periodic boundary conditions for one of them, and antiperiodic for the other one. This can be seen easier if we write the algebra (2.4.13) in an O(2) basis:

$$G_r^1 \equiv \frac{G_r + \bar{G}_r}{\sqrt{2}} \quad , \quad G_r^2 = \frac{G_r - \bar{G}_r}{i\sqrt{2}}. \tag{2.5.16}$$

In this basis the algebra (2.4.12) becomes:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\tilde{c}}{4}(m^3 - m)\delta_{m+n,0} \\
[L_m, G_r^i] &= (\frac{m}{2} - r)G_{m+r}^i, \quad [L_m, J_n] = -nJ_{m+n} \\
[J_m, J_n] &= \tilde{c}m\delta_{m+n,0} \quad [J_m, G_r^i] = i\epsilon^{ij}G_{m+r}^j \\
\{G_r^i, G_s^j\} &= 2\delta^{ij}L_{r+s} + i\epsilon^{ij}(r-s)J_{r+s} + \tilde{c}(r^2 - \frac{1}{4})\delta^{ij}\delta_{r+s,0}
\end{aligned} \quad . \tag{2.5.17}$$

The twisted (T) N=2 algebra is defined by choosing integer modes for G_m^1, L_m and half integer modes for G_r^2, J_r choices, compatible with the commutation relations (2.5.17). In the O(2) basis the stress-energy tensor becomes:

$$\mathbf{J}(\mathbf{z}) \equiv J(z) + \epsilon^{ij}\theta^i G^j(z) + \epsilon^{ij}\theta^i\theta^j T(z) \tag{2.5.18}$$

where θ^i is an O(2) doublet of Grassmann coordinates. A twisted superfield:

$$\Phi(\mathbf{z}) \equiv \phi(z) + \epsilon^{ij}\theta^i\psi^j(z)(z) + \frac{1}{2}\epsilon^{ij}\theta^i\theta^j g(z) \tag{2.5.19}$$

has antiperiodic boundary conditions for $\phi(z)$ and $\psi^2(z)$ and periodic boundary conditions for

$g(z)$ and $\psi^1(z)$, on the cylinder, that is ϕ and ψ^1 are $\mathbf{Z}_2$ twisted. Again here, G_0^1 is a hermitian operator. Its square, acting on a primary state must give positive eigenvalues, which implies that all the dimensions in the T-sector satisfy: $\Delta \geq \tilde{c}/8$. In particular it implies that if there is a state with $\Delta = \tilde{c}/8$ this is then the ground state, and it is doubly degenerate since this state $|H^+\rangle$ and $|H^-\rangle \equiv G_0^1|H^+\rangle$, have the same energies. One of the two supersymmetries, namely the one generated by G_0^1 is then unbroken, since G_0^1 annihilates the ground states:

$$\begin{aligned} (G_0^1)^2|H^+\rangle &= \frac{1}{2}\{G_0^1, G_0^1\}|H^+\rangle = (L_0 - \tilde{c}/8)|H^+\rangle = 0 \\ G_0^1|H^-\rangle &= (G_0^1)^2|H^+\rangle = 0 \end{aligned} \tag{2.5.20}$$

The global supersymmetry generated by $G^2_{-1/2}$ is broken since $G^2_{-1/2}$ fails to annihilate the ground states. This is obvious since in order for $G^2_{1/2}$ to annihilate a primary state, its dimension has to be zero, and as we argued above, states with zero dimension do not exist in the T-sector. Thus in the T-sector we have at most a remnant N=1 supersymmetry. The ground states are generated from the NS vacuum by the "twist" fields $H^{\pm}(\mathbf{z})$, the presence of which induces cuts on the complex plane such that $\phi(z)$ and $\psi^1(z)$ are double valued around the point where the twist field lies. In the T-sector there is a parity operator, $(-1)^F$, which commutes with L_m, J_m and anticommutes with G^i_m. In particular:

$$(-1)^F|H^+\rangle = |H^+\rangle\,,\ (-1)^F|H^-\rangle = -|H^-\rangle, \tag{2.5.21}$$

In the R-sector the two-spin fields are non-local with respect to each other. Their operator product expansion contains square root singularities in the complex plane which induce non-locality when we project to Euclidean space. The same is true in the T-sector. In order to obtain a local theory we must suitably project out one fermion parity, the same way as in the N=1 case.

2.6 Global OSP(2|2) Invariance

As it was mentioned earlier in this work, the invariance of the vacuum under the global $N=2$ superconformal group, $OSP(2|2)$, turns out to be very useful towards the evaluation of the correlation functions. From now on we restrict ourselves to the NS sector. Similar techniques though apply to the $R^{\pm}$ and T sectors although the analysis is somewhat more complicated.

Using the commutations relations (2.5.11), derived in the previous section, we can write the Ward identities for global superconformal invariance. Their derivation is obvious. For example L_{-1} annihilates the in-vacuum. But we can move it to the left using (2.6.11), so we end up with a differential equation for the correlation function. Thus the n-point function:

$$F_n(\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n) \equiv \langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(\mathbf{z}_3)\cdots\Phi_n(\mathbf{z}_n)|0\rangle \tag{2.7.1}$$

satisfies the following Ward identities:

$$L_{-1} : \left[\sum_{i=1}^{n} \frac{\partial}{\partial z_i}\right] F_n = 0 \tag{2.6.2a}$$

$$L_0 : \sum_{i=1}^{n} \left\{ z_i \frac{\partial}{\partial z_i} + \Delta_i + \frac{1}{2}[\theta_i \frac{\partial}{\partial \theta_i} + \bar{\theta}_i \frac{\partial}{\partial \bar{\theta}_i}] \right\} F_n = 0 \tag{2.6.2b}$$

$$L_1 : \sum_{i=1}^{n} \left\{ z_i^2 \frac{\partial}{\partial z_i} + 2z_i \left\{ \Delta_i + z_i(\theta_i \frac{\partial}{\partial \theta_i} + \bar{\theta}_i \frac{\partial}{\partial \bar{\theta}_i}) \right\} + Q_i \theta_i \bar{\theta}_i \right\} F_n = 0 \tag{2.6.1c}$$

$$J_0 : \sum_{i=1}^{n} \left(Q_i + \bar{\theta}_i \frac{\partial}{\partial \bar{\theta}_i} - \theta_i \frac{\partial}{\partial \theta_i} \right) F_n = 0 \tag{2.6.2d}$$

$$G_{-1/2}, \bar{G}_{-1/2} : \sum_{i=1}^{n} \left(\frac{\partial}{\partial \bar{\theta}_i} - \theta_i \frac{\partial}{\partial z_i} \right) F_n = \sum_{i=1}^{n} \left(\frac{\partial}{\partial \theta_i} - \bar{\theta}_i \frac{\partial}{\partial z_i} \right) F_n = 0 \tag{2.6.2e}$$

$$G_{1/2} : \sum_{i=1}^{n} \left[z_i [\frac{\partial}{\partial \bar{\theta}_i} - \theta_i \frac{\partial}{\partial z_i}] - (2\Delta_i + Q_i)\theta_i - \theta_i \bar{\theta}_i \frac{\partial}{\partial \theta_i} \right] F_n = 0 \tag{2.6.2f}$$

$$\bar{G}_{1/2} : \sum_{i=1}^{n} \left[z_i [\frac{\partial}{\partial \theta_i} - \bar{\theta}_i \frac{\partial}{\partial z_i}] - (2\Delta_i - Q_i)\bar{\theta}_i + \theta_i \bar{\theta}_i \frac{\partial}{\partial \theta_i} \right] F_n = 0 \tag{2.6.2g}$$

where Δ_i, Q_i are dimensions and charges of the various fields appearing in the correlation function (2.6.1).

A superfield operator in terms of components has the form:

$$\Phi(\mathbf{z}) \equiv \phi(z) + \theta\psi(z) + \bar{\theta}\psi(z) + \theta\bar{\theta}g(z). \tag{2.6.3}$$

The two-point function is completely fixed by the Ward identities, up to an irrelevant normalization constant.

$$\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)|0\rangle = \mathbf{z}_{12}^{-(\Delta_1+\Delta_2)} \exp\{Q_2 \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\} \delta_{Q_1+Q_2,0}\delta_{\Delta_1,\Delta_2} \tag{2.6.4}$$

It is a function of the supersymmetry invariant distances in super space, $\mathbf{z}_{12} = z_1 - z_2 - \theta_1\bar{\theta}_2 - \bar{\theta}_1\theta_2$, $\theta_{12} = \theta_1 - \theta_2$, $\bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2$. The three-point function depends on nine independent variables $(z_i, \theta_i, \bar{\theta}_i)$. Since $OSP(2|2)$ has eight generators we can fix at most eight of them, so there must be a unique combination invariant under $OSP(2|2)$. This is a commuting combination which turns out to be nilpotent:

$$\hat{R} = \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} - \frac{\theta_{13}\bar{\theta}_{13}}{\mathbf{z}_{13}} + \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}}, \hat{R}^2 = 0 \tag{2.6.5}$$

So, for any particular solution of the Ward identities, we can obtain the general solution by multiplying it with $(1 + \alpha\hat{R})$, α being an arbitrary commuting constant. Solving the Ward identities for the three-point function we obtain:

$$\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(\mathbf{z}_3)|0\rangle = C\left[\prod_{i<j}^{3} \mathbf{z}_{ij}^{-\Delta_{ij}}\right] \exp\left[\sum_{i<j}^{3} A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{\mathbf{z}_{ij}}\right] \delta_{Q_1+Q_2+Q_3,0} \tag{2.6.6}$$

where the constants

$$A_{ij} = -A_{ji}, \qquad \sum_{\substack{j=1\\ j\neq i}}^{3} A_{ij} = -Q_i \tag{2.6.7}$$

It is easily seen from (2.6.7), that the equations defining the constants A_{ij}, are not fixing all of them because of the change neutrality condition, for the correlation function. In particular, if A_{ij} is some solution of (2.6.7) then $A_{12}+\alpha, A_{31}+\alpha, A_{23}+\alpha$, is also a solution. Of course this is expected. It corresponds to multiplying the three-point function by the $OSP(2|2)$ invariant, $(1+\alpha\hat{R})$. For the three-point function to be non-zero, the OPE of the operators Φ_1, Φ_2 must contain the family Φ_3. Then the normalization constant C of the three-point function is the Glebsch-Gordan coefficient for the decomposition $[\Phi_1] \otimes [\Phi_2] \rightarrow [\Phi_3]$. In the $N = 2$ case, like the $N = 1$, there is another operator product coefficient to be determined, namely one of the A_{ij}, due to the existence of the $OSP(2|2)$ invariant $\hat{R}$.

In general $OSP(2|2)$ invariance constraints the n-point function to have the form:

$$\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2), \ldots \Phi_n(\mathbf{z}_n)|0\rangle \sim \prod_{i<j}^{n}[\mathbf{z}_{ij}^{-\Delta_{ij}}] \exp[\sum_{i<j}^{n} A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{\mathbf{z}_{ij}}]$$

$$\times F_n[x_1, x_2, \ldots, x_{3n-8}] \times \delta_{\sum_{i=1}^{n} Q_i, 0} \tag{2.6.8}$$

$$A_{ij} = -A_{ji}, \qquad \Delta_{ij} = \Delta_{ji}, \qquad \sum_{\substack{j=1\\ i\neq j}}^{n} A_{ij} = -Q_i, \qquad \sum_{\substack{i=1\\ j\neq i}}^{n} \Delta_{ij} = 2\Delta_i \tag{2.6.9}$$

where $x_i, i = 1, 2 \ldots, 3n-8$ are the combinations of the coordinates, with dimension zero, invariant under $OSP(2|2)$. They are functions of the invariant distances, $\mathbf{z}_{ij}, \theta_{ij}, \bar{\theta}_{ij}$. All the non-trivial information about the theory is encoded in the functions F_n. In most cases they are determined by the specific details of the theory. In certain cases though, that will be discussed in the next section, they can be evaluated, just by knowing the representation content of the theory.

2.7 Operator Algebra and Correlation Functions in N=2 Unitary Minimal Superconformal Models (NS Sector)

From now on we will focus on unitary minimal N=2 superconformal models. As mentioned in section 2.3 these exist for the following values of the central charge,

$$\tilde{c} = 1 - \frac{2}{m}\ ,\ m = 2, 3, ... \tag{2.7.1}$$

They contain degenerate representations only and we will show that they are exactly solvable.

The strategy is the following. Consider a *hw* unitary irreducible representation of the $N = 2$ superconformal algebra. It is generated by a *hwv*, $|\Delta, Q\rangle$, the primary state, satisfying the usual *hwv* conditions. The full representation is obtained from $|\Delta, Q\rangle$ by applying the lowering operations of the algebra. In some special situations it may turn out that one of the secondary states satisfies the *hwv* conditions. This means that the representation generated by $|\Delta, Q\rangle$ is not irreducible, but there is another representation, (the one generated by the secondary vector), embedded in it. The secondary *hwv*, $|\chi\rangle$, has the interesting property, that it is null, (i.e. $\langle\chi|\chi\rangle = 0$), and orthogonal to any other state in the Hilber space. We may thus consistently set $|\chi\rangle$ to be equal to zero, a condition that decouples all its family from the correlation functions of the theory. In fact this condition will generate constraints on the correlation functions, of the primary state $|\Delta, Q\rangle$. To see how such constraints arise we have to remember that $|\chi\rangle$ is given by some operator $\hat{O}$, constructed out of the lowering operators of the algebra, acting on $|\Delta, Q\rangle$, thus:

$$0 \equiv \langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\cdots\Phi_{n-1}(\mathbf{z}_{n-1})|\chi\rangle = \langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\ldots\Phi_{n-1}(\mathbf{z}_{n-1})\hat{O}|\Delta, Q\rangle \tag{2.7.2}$$

Moving the operator $\hat{O}$ to the left using the commutation relations (2.6.11) we end up with a super-differential equation for the correlation function. Solving these equations we can determine all the correlation functions that the degenerate family is participating in.

A necessary and sufficient condition for the existence of such systems is the closure of the operator algebra of a set of unitary degenerate representations. In fact we will show that the operator algebra of the unitary degenerate representations of the $N = 2$ superconformal algebra, with $\tilde{c} < 1$, does close. We will derive also the “fusion” rules for the operator algebra.

Consider the OPE of two primary operators:

$$\Phi_1(\mathbf{z})\Phi_2(0) = \sum_i \Phi_i(0)\mathbf{z}^{\Delta_1+\Delta_2-\Delta_i} \tag{2.7.3}$$

where the notation in the right hand side of (2.7.3) is symbolic, meaning the product can be written as a sum of primary operators and/or their descendants, and the $(z, \theta, \bar{\theta})$ dependence can be easily substituted back. What we want to know is which irreducible representation can appear in the operator product of two given representations. There is a simple criterion for representations which are not allowed, and this is the vanishing of the appropriate 3-point function.

The strategy is to use the superdifferential equations stemming from the degeneracy of the representations to derive selection rules for the operator product algebra. Let's consider a concrete example. Take a representation which has a null vector at the first level. Such a representation is for example one with $\Delta = \frac{m-2}{2m}, Q = -\frac{m-2}{m}$, when $\tilde{c} = 1 - \frac{2}{m}, m = 2, 3 \ldots$. The null vector at level one is given by:

$$|\chi_1^0\rangle = [(Q-1)L_{-1} - (2\Delta+1)J_{-1} + G_{-1/2}\bar{G}_{-1/2}]|\Delta, Q\rangle \tag{2.7.4}$$

It is easy to verify, using the commutation relations (2.4.12), that $|\chi_1^0\rangle$ satisfies all the *hwv* conditions. Consider now the n-point function where this state is participating in. We've mentioned already that such a correlation function is identically zero.

$$0 \equiv \langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\ldots\Phi_n(\mathbf{z}_n)|\chi_1^0\rangle = \langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\ldots\hat{O}\Phi(0)|0\rangle \tag{2.7.5}$$

Commuting $\hat{O}$ through to the left we arrive at the following superdifferential equation

$$\left\{(1-Q)\sum_{n=1}^{n}\frac{\partial}{\partial z_i} + (2\Delta+1)\sum_{i=1}^{n}\left[\frac{Q_i}{z_i} + \frac{1}{z_i}(\bar{\theta}_i\frac{\partial}{\partial\theta_i} - \theta_i\frac{\partial}{\partial\bar{\theta}_i}) - \frac{2\Delta_i}{z_i^2}\theta_i\bar{\theta}_i\right]\right.$$

$$\left.+\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial z_i}\right]\left[\frac{\partial}{\partial\theta_j} - \bar{\theta}_i\frac{\partial}{\partial z_i}\right]\right\}\langle 0|\Phi(\mathbf{z}_1)\cdots\Phi_n(\mathbf{z}_n)\Phi(0)|0\rangle = 0 \tag{2.7.6}$$

We will specialize (2.7.6) to the 3-point function $< 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(\mathbf{z}_3)|0 >$ where Φ_3 is the degenerate operator mentioned above.

Doing a translation and two global supersymmetry transformations (we have the freedom to do that, thanks to the $OSP(2|2)$ invariance of the correlation function), we can write the three-point function in the form $< 0|\Phi_1(\tilde{\mathbf{z}}_1)\Phi_2(\tilde{\mathbf{z}}_2)\Phi_3(0)|0 >$, where:

$$\tilde{\mathbf{z}}_1 \equiv (z_1 - z_3 - \theta_1\bar{\theta}_3 - \bar{\theta}_1\theta_3, \theta_1 - \theta_3, \bar{\theta}_1 - \bar{\theta}_3)$$

$$\tilde{\mathbf{z}}_2 \equiv (z_2 - z_3 - \theta_2\bar{\theta}_3 - \bar{\theta}_2\theta_3, \theta_2 - \theta_3, \bar{\theta}_2 - \bar{\theta}_3) \tag{2.7.7}$$

Using the form of the three-point function found earlier, in (2.6.6) we arrive at the following set of conditions for the dimension Δ_{ij} and the constants A_{ij}:

$$\Delta_{13}(Q_3 - 1) + Q_1(2\Delta_3 + 1) + A_{13} + \Delta_{13} = 0$$

$$(\Delta_{12} - A_{12})(A_{13} + \Delta_{13}) = 0, \qquad (\Delta_{12} + A_{12})(A_{13} + \Delta_{13}) = 0$$

$$(Q_3 - 1)A_{13} - 2\Delta_1(2\Delta_3 + 1) + (\Delta_{13} - A_{13} + 1)(\Delta_{13} + A_{13}) = 0$$

$$(2\Delta_3 + 1)(\Delta_{12} - A_{12}) + (\Delta_{13} - A_{13})(\Delta_{23} + A_{23}) = 0$$

$$(2\Delta_3 + 1)(A_{12} + \Delta_{12}) + (\Delta_{23} - A_{23})(\Delta_{13} + A_{13}) = 0 \tag{2.7.8}$$

The state mentioned above happens to be also degenerate at level 1/2 and relative charge -1, the null vector being

$$|\chi^-_{1/2}\rangle = \bar{G}_{-1/2}|\Delta, Q\rangle \tag{2.7.9}$$

In the same way we derive another equation:

$$\sum_{i=1}^{n}\left[\frac{\partial}{\partial\theta_i} - \bar{\theta}_i\frac{\partial}{\partial\theta_i}\right] < 0|\Phi_1(\mathbf{z}_1)\ldots\Phi_n(\mathbf{z}_n)\Phi(0)|0 >= 0 \tag{2.7.10}$$

which for the three-point function in particular implies

$$A_{13} = -\Delta_{13}, A_{23} = -\Delta_{23} \tag{2.7.11}$$

Solving (2.7.8) and (2.7.11) we obtain

$$2\Delta_1 = Q_1, 2\Delta_2 = Q_2, \Delta_1 = \Delta_3 - \Delta_2 \tag{2.7.12}$$

Consequently in the operator product of Φ_2, with $2\Delta_2 = Q_2$, and Φ_3, only fields with $2\Delta_1 = Q_1$, and $\Delta_1 = \Delta_3 - \Delta_2$ can appear.

The superdifferential equations for the three-point functions are solved in Appendix 2.D. Here we will present the "fusion" rules for the NS sector of degenerate theories with $\tilde{c} < 1$.

As was mentioned in section 2.3, the unitary irreducible representations in the NS sector with $\tilde{c} < 1$, exist when $\tilde{c} = 1 - \frac{2}{m}, m\epsilon\mathbf{Z}^+ - \{1\}$ and their dimensions and changes are given by (2.3.3). It was shown that for the family (j, k), there are three independent null hwvs embedded in it, one at relative change zero and level $m - (j + k)$, another at relative change 1 and level k and another one at relative change -1 and level j. Consequently the correlation functions of (j, k) satisfy three superdifferential equations of orders $j, k, m - (j + k)$, simultaneously. The existence of three null vectors in the N=2 case is qualitatively different from the N=0,1 cases.

The "fusion" rules coming from the consideration of the two charged null vectors at levels j_1, k_1 of the family (j_1, k_1) are the following:

$$(j_1, k_1) \otimes (j_2, k_2) = \sum_{n=\frac{1}{2}-k_2}^{j_2-\frac{1}{2}} (j_1 + n, n - j_2 + k_1 + k_2) \ , \ j_1 + k_1 \geq j_2 + k_2 \qquad (2.7.13a)$$

$$(j_1, k_1) \otimes (j_2, k_2) = \sum_{n=\frac{1}{2}-k_1}^{j_1-\frac{1}{2}} (j_2 + n, n - j_1 + k_1 + k_2) \ , \ j_1 + k_1 \leq j_2 + k_2 \qquad (2.7.13b)$$

The strategy to derive the fusion rules in their general form (2.7.13), is parallel to the one used in the N=0,1 cases. The representations $(\frac{3}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{3}{2})$ are the shifting up and down operators and the following relations are proven using the results of Appendix 2.D,

$$(\frac{3}{2}, \frac{1}{2}) \otimes (j, k) = (j, k - 1) \oplus (j + 1, k) \qquad (2.7.14a)$$

$$(\frac{1}{2}, \frac{3}{2}) \otimes (j, k) = (j - 1, k) \oplus (j, k + 1) \qquad (2.7.14b)$$

Then (2.7.13) is proven by induction using (2.7.14) and the commutativity and associativity of the operator algebra.

As was mentioned above, the family (j, k) is also degenerate at relative charge zero and level $m - (j + k)$. The extra conditions from this new null hwv have the effect of truncating the sums in (2.7.13) into the "unitary bounds", $0 < j, k, j + k \leq m - 1$, where $(j_1, k_1) \otimes (j_2, k_2) \sim (j, k)$. This truncation phenomenon is known already to happen in the analogous minimal theories of the $N = 0, 1$ algebras. Thus it is consistent to built $N = 2$ unitary minimal systems, with $\tilde{c} < 1$, where there is a finite number of representations, all degenerate, and all the correlation functions calculable.

We present the two explicit examples of the operator algebra of the first two non-trivial theories with $\tilde{c} = 1/3, (m = 3), \tilde{c} = 1/2, (m = 2)$. In the $\tilde{c} = 1/3$ theory the operator algebra is the following:

$$(\frac{1}{6}, \pm\frac{1}{3}) \otimes (0,0) \sim (0,0), \qquad (\frac{1}{6}, \pm\frac{1}{3}) \otimes (\frac{1}{6}, \mp\frac{1}{3}) \sim (0,0). \tag{2.7.15}$$

This system, is somewhat special and it will be analyzed in more detail in the next section.

The fusion rules of the $\tilde{c} = 1/2$ system are:

$$\begin{array}{ll} (\frac{1}{8}, \pm\frac{1}{4}) \otimes (\frac{1}{8}, \mp\frac{1}{4}) \sim (0,0) & , (\frac{1}{8}, \pm\frac{1}{4}) \otimes (\frac{1}{4}, \mp\frac{1}{2}) \sim (\frac{1}{8}, \mp\frac{1}{4}) \\ (\frac{1}{8}, \pm\frac{1}{4}) \otimes (\frac{1}{2}, 0) \sim (\frac{1}{8}, \pm\frac{1}{4}) & , (\frac{1}{4}, \pm\frac{1}{2}) \otimes (\frac{1}{8}, \mp\frac{1}{4}) \sim (\frac{1}{8}, \mp\frac{1}{4}) \\ (\frac{1}{4}, \pm\frac{1}{2}) \otimes (\frac{1}{2}, 0) \sim (\frac{1}{4}, \pm\frac{1}{2}) & , (\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) \sim (\frac{1}{2}, 0) \oplus (0,0) \end{array} \tag{2.7.16}$$

We should remind the reader that the "fusion" rules we have derived, give the maximum possible set of operators that can appear in an operator product expansion. To determine exactly which of them contribute and to evaluate their Glebsch-Gordon coefficients one has to evaluate the 4-point function. This is what we will do for the $\tilde{c} = 1/3$ system in the next section.

2.8 The Operator Formalism in The Ramond Sector

In section 2.5 we gave a brief description of the Ramond sector and its ground states. We will continue this discussion and develop in a parallel way the structure that we outlined in sections 2.6, 2.7 for the NS sector⋆.

The ground states that preserves N=2 supersymmetry has $\Delta = \frac{\tilde{c}}{8}$. The rest of the primary states are generated from the ground state by the action of NS superfield operators. Since primary operators are also labeled by their charge there is a non-trivial question to answer: What is the charge of the ground state? To find a plausible answer to that we will employ the isomorphism between the NS and the R sector. We will focus for concreteness on the R^+ sector. It is natural to consider as the Ramond ground state the image of the vacuum state (in the NS sector) under the isomorphism (2.1.7). From (2.3.19) it can be seen that it has dimension $\Delta = \frac{\tilde{c}}{8}$ and charge $Q = -\frac{\tilde{c}}{2}$. The "out" ground state $|R_-\rangle$ then must have charge $Q = \frac{\tilde{c}}{8}$. The states $|R_+\rangle$ and $|R_-\rangle$ being hwvs of the R^+ algebra must satisfy among others the following hwv conditions,

$$\bar{G}_0|R_+\rangle = \bar{G}_0|R_-\rangle = 0 \tag{2.8.1}$$

The representations corresponding to $|R_+\rangle$ and $|R_-\rangle$ are degenerate for any value of $\tilde{c}$. By looking at the Kač determinant in the R sector we can easily verify that $|R_+\rangle$ is degenerate at

⋆ The isomorphism (2.1.7) cannot provide complete information about correlation functions in the R sector.

level zero, relative charge one as well as level one and relative charge -1. On the other hand $|R_-\rangle$ is degenerate at level zero and relative charge $1^\dagger$. The vanishing conditions for the null vectors mentioned above are,

$$G_0|R_+\rangle = \bar{G}_{-1}|R_+\rangle = G_0|R_-\rangle = 0 \tag{2.8.2}$$

We define the correlation functions in the R sector as ,

$$F_n(\mathbf{z}_1, \mathbf{z}_2, ...\mathbf{z}_n) \equiv \frac{\langle R_-|\Phi_1(\mathbf{z}_1)....\Phi_n(\mathbf{z}_n)|R_+ >}{< R_-|R_+\rangle} \tag{2.8.3}$$

where $\Phi_i(\mathbf{z}_i)$ is a NS superfield operator. Then the correlation function (2.8.3) satisfy Ward identities due to (2.8.1,2) which parallel the global N=2 Ward identities in the NS sector:

$$\sum_{i=1}^{n}\left[Q_i + \bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial\theta_i}\right] F_n(\mathbf{z}_1, \cdots, \mathbf{z}_n) = 0 \tag{2.8.4a}$$

$$\sum_{i=1}^{n}\left[z_i\frac{\partial}{\partial z_i} + \Delta_i + \frac{1}{2}\left[\theta_i\frac{\partial}{\partial\theta_i} + \bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\right]\right] F_n(\mathbf{z}_1, \cdots, \mathbf{z}_n) = 0 \tag{2.8.4b}$$

$$\sum_{i=1}^{n}\left[\sqrt{z_i}\left[\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial z_i}\right] - \frac{1}{2\sqrt{z_i}}\left[(2\Delta_i + Q_i)\theta_i + \theta_i\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\right]\right] F_n(\mathbf{z}_1, \cdots, \mathbf{z}_n) = 0 \tag{2.8.4c}$$

$$\sum_{i=1}^{n}\left[\sqrt{z_i}\left[\frac{\partial}{\partial\theta_i} - \bar{\theta}_i\frac{\partial}{\partial z_i}\right] - \frac{1}{2\sqrt{z_i}}\left[(2\Delta_i - Q_i)\bar{\theta}_i - \theta_i\bar{\theta}_i\frac{\partial}{\partial\theta_i}\right]\right] F_n(\mathbf{z}_1, \cdots, \mathbf{z}_n) = 0 \tag{2.8.4d}$$

$$\sum_{i=1}^{n}\left[\frac{1}{\sqrt{z_i}}\left[\frac{\partial}{\partial\theta_i} - \bar{\theta}_i\frac{\partial}{\partial z_i}\right] + \frac{1}{2z_i\sqrt{z_i}}\left[(2\Delta_i - Q_i)\bar{\theta}_i - \theta_i\bar{\theta}_i\frac{\partial}{\partial\theta_i}\right]\right] F_n(\mathbf{z}_1, \cdots, \mathbf{z}_n) = 0 \tag{2.8.4e}$$

where Δ_i, Q_i are the (NS) dimension and charge of Φ_i. Equations (2.8.4) can be used to constraint the form of the Ramond correlation functions. We will solve as an example the constraints (2.8.4) on the 2-point function, $F_2(\mathbf{z}_1, \mathbf{z}_2)$.

† At special values of $\tilde{c}$ there are additional degeneracies

Equation (2.8.4a) implies that $Q_1 + Q_2 = 0$ and

$$F_2(\mathbf{z}_1, \mathbf{z}_2) = f_0(z_1, z_2) + \theta_1\bar{\theta}_1 f_1(z_1, z_2) + \theta_2\bar{\theta}_2 f_2(z_1, z_2) +$$

$$+\theta_1\bar{\theta}_2 f_3(z_1, z_2) + \bar{\theta}_1\theta_2 f_4(z_1, z_2) + \theta_1\bar{\theta}_1\theta_2\bar{\theta}_2 g(z_1, z_2) \tag{2.8.5}$$

Define the variables $u = \sqrt{z_1/z_2}$, $v = \sqrt{z_1 z_2}$ in order to split the dimensional dependence. Equation (2.8.4b) implies,

$$f_0(u, v) = \frac{f_0(u)}{v^{2\Delta}} \ , \ g(u, v) = \frac{g(u)}{v^{2\Delta+2}} \tag{2.8.6a}$$

$$f_i(u, v) = \frac{f_i(u)}{v^{2\Delta+1}} \ , \ i = 1, 2, 3, 4 \tag{2.8.6b}$$

The rest of the equations are solved by,

$$f_0(u) = \frac{u^{2\Delta - Q_1}}{(u^2-1)^{2\Delta}} \ , \ f_1(u) = \frac{\Delta - (\Delta + Q_1)u^2}{u(u^2-1)} f_0(u) \tag{2.8.7a}$$

$$f_2(u) = -\frac{\Delta u^2 + Q_1 - \Delta}{u^2 - 1} u f_0(u) \tag{2.8.7b}$$

$$f_3(u) = 2\frac{2\Delta u^2 - 2\Delta + Q_1}{u^2 - 1} f_0(u) \tag{2.8.7c}$$

$$f_4(u) = 2\frac{\Delta - Q_1}{u^2 - 1} f_0(u) \tag{2.8.7d}$$

$$g(u) = 2\Delta\left[\frac{(2\Delta+1)u^4}{(u^2-1)^2} - \frac{(2\Delta - Q_1 + 1)u^2}{u^2 - 1} + \frac{\Delta - Q_1}{2}\right] f_0(u) \tag{2.8.7e}$$

The 2-point function is asymmetric due to the asymmetry in the charge assignments of the "in" and "out" states. The 2-point function with $|R_+\rangle \leftrightarrow |R_-\rangle$ is given by (2.8.7) with the following substitutions:$f_0 \to f_0$, $g \to g$, $f_1 \to -f_1$, $f_2 \to -f_2$, $f_3 \leftrightarrow f_4$ and $Q_1 \to -Q_1$. In a similar way higher correlation functions can be constrained by (2.8.4).

Let's now discuss the fusion rules in the R sector. It is important to note that the isomorphism (2.1.7) preserves the structure of the Kač determinant (the relations (2.3.19) have to be taken into account). Thus it preserves the form of the fusion rules derived in the NS sector. Consider the set of hwvs of the R^+ algebra, $|\Delta, Q - \frac{1}{2}\rangle$ with dimensions and charges given by,

$$\Delta = \frac{\tilde{c}}{8} + \frac{jk}{2}(1-\tilde{c}) \ , \ Q = \frac{j-k}{2}(1-\tilde{c}) \ , \ j,k \in Z \qquad (2.8.8)$$

Using (2.3.19) we can establish the correspondence,

$$NS \ni (j,k) \leftrightarrow (j+\frac{1}{2}, k-\frac{1}{2}) \in R^+ \ , \ j,k \in Z^+ + \frac{1}{2} \qquad (2.8.9)$$

which along with (2.7.13) implies the following fusion rules in the R^+ sector,

$$(j_1,k_1) \otimes (j_2,k_2) = \sum_{n=-k_2}^{j_2-1} (j_1+n, n+k_1+k_2-j_2+1) \ , \ j_1+k_1 \geq j_2+k_2 \qquad (2.8.10a)$$

$$(j_1,k_1) \otimes (j_2,k_2) = \sum_{n=-k_1}^{j_1-1} (j_2+n, n+k_1+k_2-j_1+1) \ , \ j_1+k_1 \leq j_2+k_2 \qquad (2.8.10b)$$

where j_1, j_2, k_1, k_2 are integers.

The preceding results in the Ramond sector are verified explicitly in App. 2.E.

2.9 The $\tilde{c} = \frac{1}{3}, N = 2$ Superconformal Theory

This theory has the simplest operator content compared to the other unitary minimal $N = 2$ theories. It is also the only member of the $N = 2$ discrete series which has the same central element with a member of the $N = 1$ discrete series. The model is also interesting since it describes a point* in the A-T model phase diagram, [6,33]. The operator content of the theory as well as it decomposition into N=1 representations is discussed in Appendix 2.C.

The general discussion of the previous section can be specialized in this situation. The model contains the unit (superfield) operator and a conjugate pair of primary operators, representing the $\Delta = \frac{1}{6}, Q = \pm\frac{1}{3}$ states of the model. We will denote by $\Phi_\pm$ and Φ_0 the corresponding superfield operators. The two point function is:

$$\langle 0|\Phi_+(\mathbf{z}_1)\Phi_-(\mathbf{z}_2)|0\rangle = \mathbf{z}_{12}^{-1/3} \exp\left\{-\frac{1}{3}\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\right\} \qquad (2.9.1)$$

where we suppressed the antianalytic part and we've chosen a particular convenient normalization for the two-point function. The only three-point function which is non-zero is,

$\star$ In fact, as explained in the next chapter, it describes three points in the phase diagram of the A-T model.

$\langle 0|\Phi_0(\mathbf{z}_1)\Phi_+(\mathbf{z}_2)\Phi_-(\mathbf{z}_3))|0\rangle$. It is fixed up to a normalization constant by the $OSP(2|2)$ invariance and the extra differential equations that it is satisfying due to the fact that it contains degenerate fields.

$$\langle 0|\Phi_0(\mathbf{z}_1)\Phi_+(\mathbf{z}_2)\Phi_-(\mathbf{z}_3)|0\rangle = C\mathbf{z}_{23}^{-1/3}\exp\left\{-\frac{1}{3}\frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}}\right\} \tag{2.9.2}$$

It implies the following operator product expansions for the component fields

$$\Phi_\pm(\mathbf{z}) \equiv \phi_\pm(z) + \theta\bar{\psi}_\pm(z) + \bar{\theta}\psi_\pm(z) + \theta\bar{\theta}g_\pm(z) \tag{2.9.3a}$$

$$\phi_+\phi_- \sim J, \quad \phi_+g_- \sim -\frac{1}{3}J, \quad \phi_-g_+ \sim -\frac{1}{3}J \tag{2.9.3b}$$

$$\bar{\psi}_+\psi_- \sim \frac{1}{3}J, \quad \psi_+\bar{\psi}_- \sim \frac{1}{3}J, \quad g_+g_- \sim \frac{4}{9}J(2.9.3c)$$

which are determined up to an overall normalization constant. The first non-trivial correlation function is the 4-point function. Its evaluation enables us to fix the Glebsch-Gordon coefficient in the OPE, in (2.9.3).

There are two ways to evaluate the 4-point function. One is to solve the superdifferential equations that it satisfies due to degeneracy of the operators contained in it. The other is to use the Feigin-Fuks construction. The only non-trivial 4-point function is $\langle 0|\Phi_-(\mathbf{z}_1)\Phi_+(\mathbf{z}_2)\Phi_-(\mathbf{z}_3)\Phi_+(\mathbf{z}_4)|0\rangle$. The operator $\Phi_+(\mathbf{z})$ is degenerate at level 1, relative change zero, at level 1/2, relative charge one and at level $\frac{3}{2}$, relative change -1. The relevant superdifferential equation for the 4-point function $F_4(\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2, \tilde{\mathbf{z}}_3)$ are:

$$\left[\sum_{i=1}^{3} G^i_{1/2}\right] F_4(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = 0 \tag{2.9.4a}$$

$$\sum_{i=1}^{3}\left[L^i_1 - J^i_1\right] F_4(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = 0 \tag{2.9.4b}$$

$$\left[\sum_{i=1}^{3}\bar{G}^i_{3/2} - \sum_{i,j=1}^{3}(J^i_1 + L^i_1)\bar{G}^j_{1/2}\right] F_4(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = 0 \tag{2.9.4c}$$

where the relevant differential operators are defined in (2.D.4) in Appendix 2.D, and we have simplified (2.9.4b) using (2.9.4a). The variables $\mathbf{z}_i$ are the shifted variables we mentioned in section 2.7.

Global $N = 2$ superconformal invariance constrains the four-point function to be of the form:

$$F_4(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4) = C(\mathbf{z}_{12}\mathbf{z}_{34})^{-1/3} \exp\left[\frac{1}{3}\left(\frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}} - \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}}\right)\right] \times$$

$$\times G_4(x_1, x_2, x_3, x_4) \tag{2.9.5}$$

where $x_1, i = 1, 2, 3, 4$ are the four independent combinations of the coordinates invariant under the OSP(2|2) group. The obvious (dependent) invariants are:

$$x_1 = \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}} - \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}}$$

$$x_2 = \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} + \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}} - \frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}}$$

$$x_3 = \frac{\theta_{13}\bar{\theta}_{13}}{\mathbf{z}_{13}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}} - \frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}}$$

$$y_1 = \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} + \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}} - \frac{\theta_{13}\bar{\theta}_{13}}{\mathbf{z}_{13}}$$

$$y_2 = \frac{\mathbf{z}_{14}\mathbf{z}_{23}}{\mathbf{z}_{12}\mathbf{z}_{34}}, \quad y_3 = \frac{\mathbf{z}_{13}\mathbf{z}_{24}}{\mathbf{z}_{12}\mathbf{z}_{34}} \tag{2.9.6}$$

Since $y_1 = x_1 + x_2 - x_3, y_1$ can be deleted. We have also the additional relations:

$$x_1^2 = x_2^2 = x_3^2 = y_1^2 = 0, \quad x_1x_2 = (x_1 + x_3)x_2, \quad x_1x_2x_3 = 0 \tag{2.9.7a}$$

$$(y_2 - y_3 + 1)^2 = 2y_2x_1x_2, \quad x_2x_3 = y_2x_1x_3, \quad x_1x_2 = y_3x_1x_3 \tag{2.9.7b}$$

The relations above imply that in fact x_1, x_2, x_3 and $x_4 \equiv y_2$ are independent invariants. Solving equations (2.9.4) we arrive at a four-point function of the form:

$$G_4(x_1, x_2, x_3, x_4) = C\left(\frac{x_4 + 1}{x_4}\right)^{1/3} \exp\left[\frac{1}{3(x_4 + 1)}(y - x_1 + x_4x_2)\right] \tag{2.9.8}$$

where $y \equiv y_2 - y_3 + 1$.

The four-point function, (2.9.8), is powerlike, something to be expected since the primary fields of the $\tilde{c} = 1/3$ theory can be constructed as vertex operators of a single $c = 1$ scalar field (see Appendix 2.E). We have performed the same calculations using the vertex operator method, [20,25,31]. We find the same result as in (2.9.8). It is difficult though in this method to obtain the result as a super meromorphic function in $N = 2$ superspace.

By factorizing over two-point functions we can find that $C = 1$. This implies that the OPE coefficient in (2.9.3) is in fact unity.

The full construction of the four-point function, including its anti-holomorphic part does not involve any subtleties related to monodromy invariance, (locality in the Euclidean domain). We simply have to multiply the holomorphic and antiholomorphic pieces which have the same form. Knowledge of the four-point function (2.9.8) is enough to determine any $n-$point function using the OPE coefficient for the degenerate operators.

2.10 Conclusions and Prospects

In this chapter we delved into a detailed analysis of N=2 superconformal field theories. We described the structure of the representations of the N=2 superconformal algebras and we calculated their characters. We also discussed Ward identities, and in the case of minimal models we derived their fusion rules as well as some correlation functions.

There is construction of the N=2 minimal models that gives a lot of information by relating them to critical SU(2) WZ-models, [26]. The N=2 minimal models can be constructed as G/H CFTs where $G = SU(2) \otimes U(1)$ and $H = U(1)$, a linear combination of the initial $U(1)$ and the Cartan generator of $SU(2)$. Thus these theories can be constructed out of a free boson and SU(2) parafermions. In this way their correlation functions can be calculated using the known correlation functions of the parafermionic theory.*

N=1 space-time supersymmetric string theories in 4(6) dimensions have been constructed where the CFT describing the internal degrees of freedom is a tensor product of N=2 minimal models with the right value of the central charge, ($\tilde{c} = 3(2)$), [32]. It is argued that such models describe exactly string propagation on a subclass of Calabi-Yau manifolds. One can then use marginal perturbations in these models to obtain the solutions corresponding to (hopefully) all Calabi-Yau manifolds.

This shows one of the main advantages of CFT. The handling of non-linear σ-models on Calabi-Yau manifolds, (in particular their exact solution at their critical points), is a hopelessly difficult task using the methods of conventional quantum field theory.

Of course the effort in this respect has to be concentrated in classifying all N=2 superconformal models. This will provide with all possible classical solutions to string theory having N=1 supersymmetry.

$\star$ This is true for the untwisted sector. For the twisted sector one needs the correlation functions of the C-disorder fields which are presently unknown.

APPENDIX 2.A

Examples of Null States in N=2 Superconformal Algebras

In this appendix we give explicit examples of null hwv's of the N=2 algebras which , we think, are helpful to visualize several properties that we stated in the main body of the paper. Their explicit form is also very useful in deriving superdifferential equations for the correlation functions of the degenerate primary fields. We remind the reader that a null hwv is a secondary state, $|\chi\rangle$, in a Verma module which has also the properties of a hwv, namely,

$$L_n|\chi\rangle = J_n|\chi\rangle = G_r|\chi\rangle = \bar{G}_r|\chi\rangle = 0 \ , \ n, r > 0 \tag{2.A.1}$$

It is easy to deduce that such states have zero norm and the Verma module they generate is orthogonal to all other states contained in the initial Verma module. So they can consistently set to zero and this condition implies superdifferential equations for correlation functions of the initial hwv with other operators. These equations provide us with the means to solve the theory exactly. Such a theory must contain only degenerate representations.

(i) *NS algebra*, relative charge zero. An example of a null vector belonging to the superconformal family generated by $|h, q\rangle$ at the first level and relative charge zero is given by:

$$|\chi\rangle = [(q-1)L_{-1} - (2h+1)J_{-1} + G_{-1/2}\bar{G}_{-1/2}]|h, q\rangle \tag{2.A.2}$$

when $2h(\tilde{c}-1) = q^2 - \tilde{c}$. The only non-trivial hwv condition that one has to check is the action of L_{-1}, J_{-1}, $G_{-1/2}$, $\bar{G}_{-1/2}$. The others are trivially satisfied.

NS algebra relative charge ± 1.

Let's first consider a state which is degenerate at the $n_0 = 1/2$ level. Then, $g^{NS}_{1/2} = 2h - q$ so that a state with $h = q/2$ is an example of a primary state that generates such a representation. The null state in this representation is given by,

$$|\chi^+_{1/2}\rangle = G_{1/2}|h, q\rangle \tag{2.A.3}$$

which is obviously annihilated by any of L_n, J_n, G_n, $\bar{G}_n$ for $n \geq 1$. The only non-trivial condition is $\bar{G}_{1/2}|\chi^+_{1/2}\rangle = (2h-q)|\chi^+_{1/2}\rangle = 0$ due to the previously mentioned relation between his dimension and charge. It is obvious that this null vector does not generate a full Verma module since it is annihilated by $G_{-1/2}$. For $n_0 = -1/2$ the corresponding null state is $|\chi^-_{1/2}\rangle = \bar{G}_{-1/2}|h, q\rangle$. At higher levels the degenerate states involve also generators of the Virasoro or the

U(1) algebra. For example at $n_0 = \pm 3/2$ the corresponding states are,

$$|\chi^+_{3/2}\rangle = [(h - \frac{q}{2} + 1)G_{-3/2} + G_{-1/2}(J_{-1} - L_{-1})]|h, q\rangle \tag{2.A.4a}$$

$$|\chi^-_{3/2}\rangle = [(h + \frac{q}{2} + 1)\bar{G}_{-3/2} - \bar{G}_{-1/2}(J_{-1} + L_{-1})]|h, q\rangle \tag{2.A.4b}$$

Again these null hwv's do not generate full Verma modules. There exist lowering operators that annihilate them.

$$[(h - \frac{q}{2} + 1)G_{-3/2} + (J_{-1} - L_{-1})G_{-1/2}]|\chi^+_{3/2}\rangle = 0 \tag{2.A.5a}$$

$$[(h + \frac{q}{2} + 1)\bar{G}_{-3/2} - (J_{-1} + L_{-1})\bar{G}_{-1/2}]|\chi^-_{3/2}\rangle = 0 \tag{2.A.5b}$$

Finally at level 5/2 and relative charge one, when $2h - 5q + 6(\tilde{c} - 1) = 0$, the null hwv is,

$$|\chi^+_{5/2}\rangle = [(2h - q + 4)(q + 3 - 2\tilde{c})G_{-5/2} + (2h - q + 4)G_{-3/2}\hat{\Lambda}_{-1} + G_{-1/2}\hat{\Lambda}_{-2}]|h, q\rangle$$

$$\hat{\Lambda}_{-1} = (2J_{-1} - L_{-1}) \tag{2.A.6}$$

$$\hat{\Lambda}_{-2} = [(q + 3 - 2\tilde{c})(3J_{-2} - 2L_{-2}) - 4J_{-2} + 2(L_{-1})^2 + 4(J_{-1})^2 - 6J_{-1}L_{-1} + G_{-3/2}\bar{G}_{-1/2}]$$

(ii) $R^\pm$ *algebra*, null states with the same charge as the initial hwv.

An example of a null hwv of the representation of the $R^\pm$ algebra generated by $|h, q \pm 1/2\rangle_\pm$ at the first level is given by :

$$|\chi_+\rangle = [(q+1)(2h - \frac{\tilde{c}}{4})L_{-1} - (2h + \frac{3}{4})(2h - \frac{\tilde{c}}{4})J_{-1} - (2h - \frac{q}{2} + \frac{1}{4})\bar{G}_{-1}G_0]|h, q - 1/2\rangle_+ \tag{2.A.7a}$$

$$|\chi_-\rangle = [(q-1)(2h - \frac{\tilde{c}}{4})L_{-1} - (2h + \frac{3}{4})(2h - \frac{\tilde{c}}{4})J_{-1} + (2h + \frac{q}{2} + \frac{1}{4})G_{-1}\bar{G}_0]|h, q + 1/2\rangle_- \tag{2.A.7b}$$

satisfying all the hwv conditions provided $h = \frac{\tilde{c}}{8} + \frac{q^2 - (\tilde{c}+1)^2/4}{2(\tilde{c}-1)}$.

$R^\pm$ *algebra*, null states having charges differing by ± 1 from the initial charge.

In the R^+ algebra the null state at $n_0 = 0$ and relative charge +1/2 is,

$$|\chi_0^+\rangle = G_0|h, q - 1/2\rangle_+ \tag{2.A.8}$$

which is annihilated by G_0 provided $h = \frac{\tilde{c}}{8}$. At level one and relative charge +1/2 and -3/2, $(n_0 = \pm 1)$, the null states are :

$$|\chi_1^+\rangle = [(2h + 2 - \frac{\tilde{c}}{4})G_{-1} + G_0(J_{-1} - 2L_{-1})]|h, q - 1/2\rangle_+ \tag{2.A.9a}$$

$$|\chi_1^-\rangle = \bar{G}_{-1}|h, q - 1/2\rangle_+ \tag{2.A.9b}$$

The state $|\chi_1^+\rangle$ is annihilated by the operator $(2h + 2 - \frac{\tilde{c}}{4})G_{-1} + (J_{-1} - 2L_{-1})G_0$, whereas $|\chi_1^-\rangle$ is annihilated by $\bar{G}_{-1}$. At level two and relative charge +1/2, $(n_0 = 2)$, the null state is,

$$|\chi_2^+\rangle = [2(q - \tilde{c} + 2)(2q - 3\tilde{c} + 5)G_{-2} + 2(q - \tilde{c} + 2)G_{-1}\bar{\Lambda}_{-1} + G_0\bar{\Lambda}_{-2}]|h, q - 1/2\rangle_+$$

$$\bar{\Lambda}_{-1} = (3J_{-1} - 2L_{-1}) \tag{2.A.10}$$

$$\bar{\Lambda}_{-2} = [(2q - 3\tilde{c} + 5)(J_{-2} - L_{-2}) - 3J_{-2} + 2(L_{-1})^2 + \frac{3}{2}(J_{-1})^2 - 4J_{-1}L_{-1} + G_{-1}\bar{G}_{-1}]$$

At $n_0 = -2$ the null hwv of relative charge -3/2 is,

$$|\chi_2^-\rangle = [(2q + 3\tilde{c} - 5)\bar{G}_{-2} + \bar{G}_{-1}(2L_{-1} + 3J_{-1})]|h, q - 1/2\rangle_+ \tag{2.A.11}$$

The corresponding null state of the R^- algebra at level zero is,

$$|\chi_0^-\rangle = \bar{G}_0|h, q + 1/2\rangle_- \tag{2.A.12}$$

annihilated by $\bar{G}_0$, whereas at level one, $(n_0 = \pm 1)$, they are,

$$|\chi_1^+\rangle = [(2h + 2 - \frac{\tilde{c}}{4})\bar{G}_{-1} - \bar{G}_0(2L_{-1} + J_{-1})]|h, q + 1/2\rangle_- \tag{2.A.13a}$$

$$|\chi_1^-\rangle = G_{-1}|h, q + 1/2\rangle_- \tag{2.A.13b}$$

annihilated by $[(2h + 2 - \frac{\tilde{c}}{4})\bar{G}_{-1} - (2L_{-1} + J_{-1})\bar{G}_0]$ and G_{-1} respectively.

(*iii*) *T algebra*. When $h=\frac{\tilde{c}}{8}$, one of the two states of opposite parity is degenerate at level zero and decouples from the spectrum. The explicit form of the null hwv is,

$$|\chi_0^-\rangle = G_0^1|h\rangle \tag{2.A.14}$$

which has negative parity.(We define the parity or fermion number operator, $(-1)^F$, so that it commutes with L_{-n}, J_{-n} and anticommutes with G_{-n}^1, G_{-n}^2. It is obvious that it counts the number of fermionic operators modulo two.) The existence of the state with $h=\frac{\tilde{c}}{8}$ implies the non-vanishing of the Witten index and thus that supersymmetry is unbroken on the cylinder.

At level 1/2 there are two null hwv's of opposite parity when $h\tilde{c}=h-\frac{\tilde{c}}{8}$,

$$|\chi_{1/2}^-\rangle = [2iJ_{-1/2}G_0^1 + \tilde{c}G_{-1/2}^2]|h\rangle \tag{2.A.15a}$$

$$|\chi_{1/2}^+\rangle = [2ihJ_{-1/2} + G_{-1/2}^2G_0^1]|h\rangle \tag{2.A.15b}$$

At level one there are again two null hwv's provided $2h=-\frac{3\tilde{c}^2-3\tilde{c}+1}{4(\tilde{c}-1)}$,

$$|\chi_1^+\rangle = [(2\tilde{c}-1)(2(\tilde{c}-1)L_{-1}+(J_{-1/2})^2)+(\tilde{c}-1)(8iJ_{-1/2}G_{-1/2}^2G_0^1-4\tilde{c}G_{-1}^1G_0^1)]|h\rangle \tag{2.A.16a}$$

$$|\chi_1^-\rangle = [4(\tilde{c}-1)L_{-1}G_0^1 - 2i(2\tilde{c}-1)J_{-1/2}G_{-1/2}^2 + 2(J_{-1/2})^2G_0^1 + \tilde{c}(2\tilde{c}-1)G_{-1}^1]|h\rangle \tag{2.A.16b}$$

The examples presented above are also very important in the derivation of the super-differential equations satisfied by the correlation functions of the corresponding degenerate hwv's.

APPENDIX 2.B

Derivation of the Partition Functions for the N=2 Superconformal Algebras

In this appendix we will evaluate the partition functions for the N=2 superconformal algebras.

For the NS and $R^{\pm}$ algebras the partition functions are defined as:

$$F(z,w) = z^{-h} w^{-q} Tr[z^{L_0} w^{J_0}] \tag{2.B.1}$$

whereas for the T-algebra :

$$F(z) = z^{-h} Tr[z^{L_0}] \tag{2.B.2}$$

where the trace is taken over all the secondary states of a non-degenerate representation of dimension h and charge q.

(i) NS $algebra$. A basis of states is given by,

$$|(n),(m),(k),(r)\rangle \;=\; L(n)J(m)G(k)\bar{G}(r)|h,q\rangle \tag{2.B.3}$$

where the respective operators are defined as,

$$L(n) \equiv (L_{-1})^{n_1}(L_{-2})^{n_2}\ldots \quad n_i \in N_0 \tag{2.B.4a}$$

$$J(m) \equiv (J_{-1})^{m_1}(J_{-2})^{m_2}\ldots \quad m_i \in N_0 \tag{2.B.4b}$$

$$G(k) \equiv (G_{-1/2})^{k_1}(G_{-3/2})^{k_2}\ldots \quad k_i \in (0,1) \tag{2.B.4c}$$

$$\bar{G}(r) \equiv (\bar{G}_{-1/2})^{r_1}(\bar{G}_{-3/2})^{r_2}\ldots \quad r_i \in (0,1) \tag{2.B.4d}$$

$$G_r \equiv \frac{1}{\sqrt{2}}(G_r^1 + iG_r^2)\ ,\ \bar{G}_r \equiv \frac{1}{\sqrt{2}}(G_r^1 - iG_r^2)$$

Any other permutation in (2.B.4) can be expressed, using the commutation relations of the algebra, as a linear combination of the above. The range of the exponents in (2.B.4c,d) is such because the squares of G_r and $\bar{G}_r$ are zero due to the anti-commutation relations.

The next step is to evaluate the expectation value,

$$F[(n),(m),(k),(r)) \equiv \langle (n),(m),(k),(r)|z^{L_0}w^{J_0}|(n),(m),(k),(r)\rangle \tag{2.B.5}$$

where the basis states are assumed to be normalized. J_0 commutes with L_{-n} , J_{-n} for every $n \in Z$ and

$$[J_0, G_{-r}] = G_{-r} \ , \ [J_0, \bar{G}_{-r}] = -\bar{G}_{-r}$$

To evaluate the commutators of w^{J_0} with the supercharge operators we have to consider:

$$f(\delta) \equiv e^{\delta J_0}(G_{-r})^k \, e^{-\delta J_0}$$

$$\frac{df}{d\delta} = r f(\delta) \tag{2.B.6}$$

Solving the differential equation and setting $w = e^{\delta}$, we obtain:

$$w^{J_0}(G_{-r})^k = (G_{-r})^k w^{J_0+k} \ , \ k \in (0,1) \tag{2.B.7a}$$

$$w^{J_0}(\bar{G}_{-r})^k = (\bar{G}_{-r})^k w^{J_0-k} \ , \ k \in (0,1) \tag{2.B.7b}$$

The same procedure for the z^{L_0} factor gives

$$z^{L_0}(L_{-n})^k = (L_{-n})^k z^{L_0+nk} \ , \ z^{L_0}(J_{-n})^k = (J_{-n})^k z^{L_0+nk} \tag{2.B.8a}$$

$$z^{L_0}(G_{-n})^k = (G_{-n})^k z^{L_0+nk} \ , \ z^{L_0}(\bar{G}_{-n})^k = (\bar{G}_{-n})^k z^{L_0+nk} \tag{2.B.8b}$$

Taking into account all the above we obtain :

$$F[(n),(m),(k),(r)] = z^h w^q \left[z^{\sum_{j=1}^{\infty}(jn_j+jm_j)}(z^{\frac{1}{2}}w)^{k_1}(z^{\frac{3}{2}}w)^{k_2}...(\frac{z^{\frac{1}{2}}}{w})^{r_1}(\frac{z^{\frac{3}{2}}}{w})^{r_2}... \right] \tag{2.B.9}$$

It remains to sum over all the permissible sets of integers $(n),(m),(k),(r)$.

$$\sum_{(n_i)} z^{\sum_{j=1}^{\infty} jn_j} = \sum (n_i) \prod_{j=1}^{\infty} z^{jn_j} = \prod_{j=1}^{\infty} \sum_{(n_i)} z^{jn_j} = \prod_{j=1}^{\infty} \frac{1}{(1-z^j)} \tag{2.B.10a}$$

$$\sum_{k_i=0,1} (z^{\frac{2i-1}{2}} w)^{k_i} = (1 + z^{\frac{2i-1}{2}} w) \qquad (2.B.10b)$$

so that finally,

$$F_{NS}(z,w) = \prod_{n=1}^{\infty} \frac{(1+z^{n-1/2}w)(1+z^{n-1/2}w^{-1})}{(1-z^n)^2} \qquad (2.B.11)$$

For the R^+ algebra the modding of the supercharges is integral. The derivation goes along the same lines with the following minor modifications. There is the additional contribution of G_0, ($\bar{G}_0$ annihilates the primary state $|h, q-1/2\rangle_+$), which amounts to a factor $(1+w)$,there is another factor of $w^{-1/2}$ coming from the incomplete cancellation of $w^{q-1/2}$ and since we have integer modding, $n-1/2$ in (2.B.11) is replaced by n. Consequently the partition function for the R^+ algebra is,

$$F_R(z,w) = (w^{1/2} + w^{-1/2}) \prod_{n=1}^{\infty} \frac{(1+z^n w)(1+z^n w^{-1})}{(1-z^n)^2} \qquad (2.B.12)$$

In the R^- algebra we have to replace G_0 with $\bar{G}_0$ and $q-1/2$ with $q+1/2$. The partition function is identical to (2.B.12).

We have also to discuss the partition functions of single charged fermions. Some particular examples in this case are the incomplete Verma modules generated by the null vectors of the degenerate representations of the NS and $R^{\pm}$ algebras with $\tilde{c} \geq 1$. To motivate the discussion, let's look at the simplest example of such a module generated by the null hwv at level 1/2, ($n_0 = 1/2$), of the NS algebra, given explicitly by (2.A.3). This state, as it was mentioned before is annihilated by $G_{-1/2}$. So, in our previous computation of the partition functions, basis states with a $G_{-1/2}$ operator in them do not contribute. This in turn means that a factor $(1+z^{1/2}w)$ is absent from the corresponding partition function. The first non-trivial example comes at level 3/2, ($n_0 = 3/2$), the null hwv given explicitly by (2.A.4a). Instead of choosing the $G_{-3/2}$, $G_{-1/2}J_{-1}$, $G_{-1/2}L_{-1}$ as basis operators, we can choose the annihilating operator, $(2h-q/2+1)G_{-3/2}+(J_{-1}-L_{-1})G_{-1/2}$, giving a zero contribution, and the remaining $G_{-1/2}J_{-1}$, $G_{-1/2}L_{-1}$. Thus, effectively, the contribution of $G_{-3/2}$ is absent, causing a loss of a factor $(1+z^{3/2}w)$ from the corresponding partition function. For the null hwv at $n_0 = -3/2$, given by (2.A.4b), following the previous argument, the contribution of $\bar{G}_{-3/2}$ is again effectively missing, and consequently a factor $(1+z^{3/2}w^{-1})$ is absent from the partition function.

Now the general situation is evident. For a null hwv at some level $|n_0|$, (n_0 being an integer or half-integer, corresponding to $R^{\pm}$ or NS respectively), the partition function lacks the contribution of G_{-n_0}, $sgn(n_0) > 0$ or $\bar{G}_{-n_0}$, $sgn(n_0) < 0$. Thus the partition function is given by :

$$\tilde{F}_X(z,w;n_0) = [1 + z^{|n_0|} w^{sgn(n_0)}]^{-1} \bar{F}_X(z,w) \qquad (2.B.13)$$

where X stands for either R or NS.

In the T-algebra the situation is now clear. There is no w^{J_0} factor . The contribution from the Virasoro and U(1) operators is $\prod_{n=1}^{\infty}(1-z^n)^{-1}(1-z^{n-1/2})^{-1}$ (the U(1) generators have half-integer modding). The contribution from the G^1_{-r} operators, (integer modding), is $\prod_{n=1}^{\infty}(1+z^n)$, whereas for the G^2_{-r} operators, (half-integer modding), it is $\prod_{n=1}^{\infty}(1+z^{n-1/2})$. Collecting everything :

$$F_T(z,w)=\prod_{n=1}^{\infty}\frac{(1+z^n)(1+z^{n-1/2})}{(1-z^n)(1-z^{n-1/2})} \qquad (2.B.14)$$

This concludes the derivation of the partition functions of the N=2 superconformal algebras.

APPENDIX 2.C

Proof of the Equivalence between the $\tilde{c} = \frac{1}{3}$ N=2 Model and the $\hat{c} = \frac{2}{3}$ N=1 Model

In this section we will show that the first member of the discreet series of N=2 superconformal models coincides with the second member of the corresponding N=1 discreet series.

The $\tilde{c} = 1/3$ theory constitutes a subsector of the $\hat{c} = 2/3$ N=1 superconformal theory. It is the only member of the $\tilde{c} < 1$ N=2 series which has the same anomaly with a member of the $\hat{c} < 1$ N=1 series. For example the N=2 unit operator, $(0)_2$, decomposes into the unit operator of the N=1 theory, $(0)_1$, (containing the unit operator and one of the N=2 supercharges), and a dimension-one operator, $(1)_1$, (containing the U(1) current of dimension one and the second N=2 supercharge). The representation of the NS sector with $h = \frac{1}{6}$, $q = \pm\frac{1}{3}$ decomposes into $(\frac{1}{6})_1$ of the N=1 NS sector. The operator $(\frac{3}{8})_2$ belonging to the Ramond sector, decomposes as $(\frac{3}{8})_2 \rightarrow (\frac{3}{8})_1$ whereas the two $(\frac{1}{24})_2$ representations of the $R^{\pm}$ sector decompose as $(\frac{1}{24})_2 \rightarrow (\frac{1}{24})_1$ in the R sector of the N=1 theory. Finally in the twisted sector of the $\tilde{c} = 1/3$, N=2 system the representation of dimension $h = \frac{1}{16}$ decomposes into $(\frac{1}{16})_1$ in the NS sector of the N=1 system. These decompositions can be easily justified by checking the validity of the equalities between the appropriate characters:

$$ch_1^{NS}(h=0,z) + ch_1^{NS}(h=1,z) = ch_2^{NS}(h=0,q=0,z,w=1) \tag{2.C.1}$$

$$ch_1^{NS}(h=\frac{1}{6},z) = ch_2^{NS}(h=\frac{1}{6},q=\pm\frac{1}{3},z,w=1) \tag{2.C.2}$$

$$ch_1^{R}(h=\frac{3}{8},z) = ch_2^{R}(h=\frac{3}{8},q=0,z,w=1) \tag{2.C.3}$$

$$ch_1^{R}(h=\frac{1}{24},z) = ch_2^{R}(h=\frac{1}{24},q=\pm\frac{1}{3},z,w=1) = ch_2^{R}(h=\frac{1}{24},q=\pm\frac{2}{3},z,w=1) \tag{2.C.4}$$

$$ch_1^{NS}(h=\frac{1}{16},z) = ch_1^{R}(h=\frac{1}{16}) + ch_1^{R}(h=\frac{9}{16}) = ch_2^{T}(h=\frac{1}{16},z) \tag{2.C.5}$$

APPENDIX 2.D

Solution of the Degeneracy Equations Up to Level $\frac{5}{2}$

In this appendix we solve the first few superdifferential equations for the three-point function and derive the conditions leading to the "fusion" rules discussed in section 2.7.

For the representation (Δ_3, Q_3), degenerate at level 1/2 and relative change 1 the null *hwv* is:

$$|\chi^{+}_{1/2}\rangle = G_{-1/2}|\Delta_3, Q_3\rangle \tag{2.D.1}$$

It implies the following equation for the three-point function

$$\sum_{i=1}^{2}\Big[\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial z_i}\Big]\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(0)|0\rangle = 0 \tag{2.D.2}$$

Substituting the general form of the three-point function in (2.D.2) we obtain:

$$A_{13} = \Delta_{13}, \quad A_{23} = \Delta_{23} \tag{2.D.3}$$

Before we continue, it is convenient to introduce some notations concerning the superdifferential operations we use. We define:

$$\hat{L}^i_n = z_i^{1-n}\frac{\partial}{\partial z_i} - (n-1)z_i^{-n}\Big[\Delta_i + \frac{1}{2}(\theta_i\frac{\partial}{\partial\theta_i} + \bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\Big] + \frac{Q_i}{2}n(n-1)z_i^{-n-1}\theta_i\bar{\theta}_i$$

$$\hat{J}^i_n = z_i^{-n}\Big[Q_i + \bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial\theta_i} - 2nz_i^{-1}\Delta_i\theta_i\bar{\theta}_i\Big] \tag{2.D.4}$$

$$\hat{G}^i_r = z_i^{\frac{1}{2}-r}\Big[\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial z_i}\Big] + (r-1/2)(2\Delta_i + q_i)z_i^{-r-1/2}\theta_i + (r-1/2)z_i^{-r-1/2}\theta_i\bar{\theta}_i\frac{\partial}{\partial\theta_i}$$

$$\hat{\bar{G}}^i_r = z_i^{\frac{1}{2}-r}\Big[\frac{\partial}{\partial\theta_i} - \bar{\theta}_i\frac{\partial}{\partial z_i}\Big] + (r-1/2)(2\Delta_i - Q_i)z_i^{-r-1/2}\bar{\theta}_i + (r-1/2)z_i^{-r-1/2}\theta_i\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}$$

The conditions coming from the null *hwv* at level one and relative change zero have been discussed in the main body of the paper. At level 3/2 and relative change 1 the null *hwv* is

$$|\chi^{+}_{3/2}\rangle = \Big[(\Delta_3 - \frac{Q_3}{2})G_{-3/2} + (J_{-1} - L_{-1})G_{-1/2}\Big]|\Delta_3, Q_3\rangle \qquad (2.D.5)$$

implying the following equations for the three point function

$$\Big[\Big(\Delta_3 - \frac{Q_3}{2}\Big)\sum_{i=1}^{2}\hat{G}^i_{3/2} + \sum_{i=1}^{2}\sum_{j=1}^{2}(\hat{J}^i_1 - \hat{L}^i_1)\hat{G}^j_{1/2}\Big]\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(0)|0\rangle = 0 \qquad (2.D.6)$$

which give after substituting the three-point function in:

$$(2\Delta_3 - Q_3)(\Delta_{12} - A_{12}) + 2(Q_2 + \Delta_{23})(\Delta_{13} - A_{13}) = 0 \qquad (2.D.7)$$

Finally at level 5/2 and relative change 1 the differential equation is:

$$\Big[(2\Delta_3 - Q_3 + 4)(2\Delta_3 - 2Q_3 + 3)\sum_{i=1}^{2}\hat{G}^i_{5/2} - 3(2\Delta_3 - Q_3 + 4)\sum_{i=1}^{2}\sum_{j=1}^{2}\hat{G}^i_{3/2}(2\hat{J}^j_1 - 2\hat{L}^j_1)$$

$$+\hat{\Lambda}\Big]\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(0)|0\rangle = 0 \qquad (2.D.8)$$

where

$$\hat{\Lambda} \equiv \sum_{i=1}^{2}\hat{G}^i_{1/2}\sum_{j=1}^{2}\Big[(2\Delta_3 - 2Q_3 + 3)(2\hat{L}^j_2 - 3\hat{J}^j_2) + 12\hat{J}^j_2 + \sum_{k=1}^{2}\Big\{6\hat{L}^j_1\hat{L}^k_1$$

$$+12\hat{J}^j_1\hat{J}^k_1 - 18\hat{J}^j_1\hat{L}^k_1 + 3G^j_{3/2}\bar{G}^k_{1/2}\Big\}\Big] \qquad (2.D.8')$$

implying the following set of conditions

$$\begin{aligned}(\Delta_{13} - A_{13})[&(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) - 3(\Delta_{13} + 1)(2\Delta_3 - Q_3 - 2)\\ &- 6(Q_1 - 1)(2\Delta_3 - Q_3) - 3(Q_1 - 1)(2\Delta_3 - 2Q_3 - 1)\\ &- (2\Delta_3 - 2Q_3 + 3)(2\Delta_{13} + 2\Delta_1 + 3) + 6(\Delta_{13} + 1)(\Delta_{13} + 2)\\ &+ 18(Q_1 - 1)(\Delta_{13} + 1) + 12(Q_1 - 1)^2] + (2\Delta_1 + Q_1)\\ &\times [2(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) - 3\Delta_{13}(2\Delta_3 - Q_3 - 2) - 6(Q_1 - 1)\\ &\times (2\Delta_3 - Q_3) - 3(\Delta_{13} + A_{13})] = 0\end{aligned} \qquad (2.D.9a)$$

$$
\begin{aligned}
&(\Delta_{12} - A_{12})[-(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) + 3\Delta_{13}(2\Delta_3 - Q_3 - 2) \\
&\quad + 6(Q_1 - 1)(2\Delta_3 - Q_3) + 3(\Delta_{13} + A_{13})] + (\Delta_{13} - A_{13}) \\
&\quad \times [-3(2\Delta_3 - Q_3 - 2)\Delta_{23} - 6Q_2(2\Delta_3 - Q_3) + 2\Delta_{12}(2\Delta_3 - 2Q_3 + 3) \\
&\quad - 3(\Delta_{23} + A_{23}) + 12\Delta_{23}(\Delta_{13} + 1) + 18Q_2(\Delta_{13} + 1) + 18(Q_1 - 1) \\
&\quad \times \Delta_{23} + 24(Q_1 - 1)Q_2] + (2\Delta_1 + Q_1)[-3\Delta_{23}(2\Delta_3 - Q_3 - 2) \\
&\quad - 6Q_2(2\Delta_3 - Q_3) - 3(\Delta_{23} + A_{23})] = 0
\end{aligned}
\tag{2.A.9b}
$$

$$
\begin{aligned}
&(\Delta_{12} - A_{12})[-(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) + 3\Delta_{23}(2\Delta_3 - Q_3 - 2) + 6Q_2 \\
&\quad \times (2\Delta_3 - Q_3) + 3(\Delta_{23} + A_{23})] + (\Delta_{13} - A_{13})[-3Q_2 \\
&\quad (2\Delta_3 - 2Q_3 - 1) - 2(2\Delta_3 - 2Q_3 + 3)(\Delta_2 + \Delta_{23}) + 6\Delta_{23}(\Delta_{23} + 1) \\
&\quad + 18Q_2\Delta_{23} + 12Q_2^2 + 3(A_{23} + \Delta_{23})] = 0
\end{aligned}
\tag{2.A.9c}
$$

$$
\begin{aligned}
&(\Delta_{12} + A_{12})[-(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) + 3\Delta_{13}(2\Delta_3 - Q_3 - 2) \\
&\quad + 6Q_1(2\Delta_3 - Q_3) + 3(\Delta_{13} + A_{13})] + (\Delta_{23} - A_{23})[-3Q_1 \\
&\quad (2\Delta_3 - 2Q_3 - 1) - 2(2\Delta_3 - 2Q_3 + 3) \times (\Delta_1 + \Delta_{13}) \\
&\quad + 6\Delta_{13}(\Delta_{13} + 1) + 18Q_1\Delta_{13} + 12Q_1^2 + 3(\Delta_{13} + A_{13})] = 0
\end{aligned}
\tag{2.D.9d}
$$

$$
\begin{aligned}
&(\Delta_{23} - A_{23})[(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) - 3(2\Delta_3 - Q_3 - 2)(\Delta_{23} + 1) \\
&\quad - 6(Q_2 - 1)(2\Delta_3 Q_3) - 3(Q_2 - 1)(2\Delta_3 - 2Q_3 - 1)(2\Delta_3 - 2Q_3 + 3) \\
&\quad (2\Delta_{23} + 2\Delta_2 + 3) + 6(\Delta_{23} + 1)(\Delta_{23} + 2) + 18(Q_2 - 1)(\Delta_{23} + 1) \\
&\quad + 12(Q_2 - 1)^2] + (2\Delta_2 + Q_2)[2(2\Delta_3 - Q_3)(2\Delta_3 - 2\Delta_3 - 3) \\
&\quad - 3\Delta_{23}(2\Delta_3 - Q_3 - 2) - 6(Q_2 - 1)(2\Delta_3 - Q_3) \\
&\quad - 3(\Delta_{23} + A_{23})] = 0
\end{aligned}
\tag{2.D.9e}
$$

$$
\begin{aligned}
&(\Delta_{12} + A_{12})[-(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 + 3) + 3\Delta_{23}(2\Delta_3 - Q_3 - 2) + 6(Q_2 - 1) \\
&\quad (2\Delta_1 - Q_3) + 3(\Delta_{23} + A_{23})] + (2\Delta_2 + Q_2)[-3\Delta_{13}(2\Delta_3 - Q_3 - 2) \\
&\quad - 6Q_1(2\Delta_3 - Q_3) - 3(\Delta_{13} + A_{13})] + (\Delta_{23} - A_{23}) \\
&\quad [-3\Delta_{13}(2\Delta_3 - Q_3 - 2) - 6Q_1(2\Delta_3 - Q_3) + 2\Delta_{12}(2\Delta_3 - 2Q_3 + 3) \\
&\quad + 12\Delta_{13}(\Delta_{23} + 1) + 18Q_1(\Delta_{23} + 1) + 18\Delta_{13}(Q_2 - 1) \\
&\quad + 24Q_1(Q_2 - 1) - 3(\Delta_{13} + A_{13})] = 0
\end{aligned}
\tag{2.D.9f}
$$

The null hwv at level $n_0 \epsilon \mathbf{Z}^+ + \frac{1}{2}$ and relative change -1 are obtained from those with relative change 1 by making the following substitutions: $J_n \to -J_n, G_r \leftrightarrow \bar{G}_r$ and $Q \to -Q$. Consequently the conditions derived from the three-point function are those of relative change 1 with the additional substitution $Q_i \to -Q_i, A_{ij} \to -A_{ij}$.

APPENDIX 2.E

The Bosonic Construction of the $\tilde{c} = \frac{1}{3}$ N=2 Superconformal Model

In this appendix we construct the components of the primary superfields of the $\tilde{c} = 1/3$ $N = 2$ superconformal system (NS, R sector) using a single $c = 1$ scalar field. We use these operators to give an alternative calculation of the four-point function (2.9.8) which was computed in the main body of this paper. We will also illustrate some results that were derived in section 2.8 concerning the R sector.

We consider a scalar field $\phi(z)$ with a two-point function given by:

$$\left\langle 0 \,|\, \phi(z)\phi(w) \,|\, 0 \right\rangle = -ln(z-w) \qquad (2.E.1)$$

We define the standard energy momentum tensor $T(z) = -\frac{1}{2} : \partial_z\phi\partial_z\phi :$ satisfying:

$$T(z)T(w) = \frac{\frac{1}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + ... \qquad (2.E.2)$$

A vertex operator defined by $V_a(z) \equiv : e^{ia\phi(z)} :$ has dimension $\Delta_a = \frac{a^2}{2}$:

$$T(z)V_a(w) = \frac{a^2}{2}\frac{V_a(w)}{(z-w)^2} + \frac{\partial_w V_a(w)}{z-w} + ... \qquad (2.E.3)$$

In this system the $N = 2$ superconformal algebra is realized by $T(z)$ and, [33],

$$J(z) \equiv \frac{i}{\sqrt{3}}\partial_z\phi(z), \quad G(z) \equiv \sqrt{2/3} : e^{i\sqrt{3}\phi(z)} :, \quad \bar{G}(z) = \sqrt{2/3} : e^{-i\sqrt{3}\phi(z)} : \qquad (2.E.4)$$

We can evaluate operator product expansions of vertex operators using the familiar formula:

$$V_a(z)V_b(w) = (z-w)^{ab} : e^{ia\phi(z)+ib\phi(w)} : \qquad (2.E.5)$$

by expanding the second exponential around $z = w$ and keeping the singular terms. Since:

$$J(z)V_a(w) = \frac{a}{\sqrt{3}}\frac{V_a(w)}{z-w}$$

we can easily establish that $T(z)$, $G(z)$, $\bar{G}(z)$ and $J(z)$ satisfy the standard $N = 2$ superconformal algebra (2.5.12) with $\tilde{c} = 1/3$.

Candidates for the lowest components of the primary superfields $\phi_{\pm}(z)$ with dimension 1/6 and charge $\pm 1/3$ are the vertex operators:

$$\phi_+(z) \equiv : e^{\frac{i}{\sqrt{3}}\phi(z)} :, \quad \phi_-(z) =: e^{-\frac{i}{\sqrt{3}}\phi(z)} : \tag{2.E.7}$$

which by (2.E.3) and (2.E.6) have the correct dimension and U(1) charge. We have now to find the superpartners of $\phi_{\pm}$. Using the relations (2.6.6) in component form we have that:

$$G(z)\phi_{\pm}(w) = \frac{\psi_{\pm}(w)}{z-w} + ... \tag{2.E.8a}$$

$$\bar{G}(z)\phi_{\pm}(w) = \frac{\bar{\psi}_{\pm}(w)}{z-w} + ... \tag{2.E.8b}$$

Applying (2.E.8a,b) to (2.E.7) we find

$$\psi_+(z) = 0, \quad \bar{\psi}_+(z) = \sqrt{2/3} : e^{-\frac{2i}{\sqrt{3}}\phi(z)} :$$

$$\psi_-(z) = \sqrt{2/3} : e^{\frac{2i}{\sqrt{3}}\phi(z)} : \quad \bar{\psi}_-(z) = 0 \tag{2.E.9}$$

Using then:

$$G(z)\psi_{\pm}(w) = 0, \quad \bar{G}(z)\bar{\psi}_{\pm}(w) = 0$$

$$G(z)\bar{\psi}_{\pm}(w) = (2\Delta + Q)\frac{\phi_{\pm}(w)}{(z-w)^2} + \frac{\partial_w\phi_{\pm}(w)}{z-w} + \frac{g_{\pm}(w)}{z-w}$$

$$\bar{G}(z)\psi_{\pm}(w) = (2\Delta - Q)\frac{\phi_{\pm}(w)}{(z-w)^2} + \frac{\partial_w\phi_{\pm}(w)}{z-w} - \frac{g_{\pm}(w)}{z-w} \tag{2.E.10}$$

we find that they are satisfied if: $g_+(z) = \partial_z\phi_+(z)$ and $g_-(z) = -\partial_z\phi_-(z)$.

The fact that one of the fermionic components is zero and the fourth component is a descendant of the first component explains the group theoretic result*, that the family $(\Delta = \frac{1}{6}, Q = \pm\frac{1}{3})$ decomposes to the $N = 1$ family with $\Delta = \frac{1}{6}$ and half the apparent degrees of freedom.

This means, using our definition (2.4.8a,b) that $\phi_{\pm}$ are chiral primary operators of opposite chirality. In fact, looking at (2.6.11) we can establish that any primary superfield, degenerate at $n_0 = \pm 1/2$, is chiral in the sense of (2.4.8a,b) and thus contains half the apparent degrees of freedom.

$\star$ See Appendic 2.C.

Computing correlation functions of Φ_+ and Φ_- is now trivial. Using:

$$\left\langle 0 \,|\, V_{a_1}(z_1)V_{a_2}(z_2)\ldots V_{a_n}(z_n) \,|\, 0 \right\rangle = \prod_{i<j}^{n}(z_{ij})^{a_i a_j}\delta_{a_1+\cdots+a_n,0} \qquad (2.E.11)$$

We can evaluate the different components of (2.9.8). Such a correlation is non-zero only if $\sum_i a_i = 0$, otherwise IR divergences force it to vanish. Such a calculation has been performed for the four-point function and as expected it agrees with the result (2.9.8).

Let's also illustrate the situation in the R^+ sector of the model. We have two operators of dimension $\frac{1}{24}$ and charge $\pm\frac{1}{6}$ and two operators of dimension $\frac{3}{8}$ and charge $\pm\frac{1}{2}$. the ground states can be represented by the $\Delta = \frac{1}{24}$ vertex operators:

$$R_-(z) =: e^{(i/2\sqrt{3})\phi(z)} : \, , \;\; R_+(z) =: e^{-(i/2\sqrt{3})\phi(z)} : \qquad (2.E.12)$$

The operators of dimension $\frac{3}{8}$ are represented by $: e^{\pm(i\sqrt{3}/2)\phi(z)} :$. It is easy to see that it is generated from the Ramond vacuum by the action of the $\Delta = \frac{1}{6}$ operators of the NS sector due to the following OPE:

$$: e^{(i/\sqrt{3})\phi(z)} :: e^{(i/2\sqrt{3})\phi(w)} := (z-w)^{1/6}[: e^{(i\sqrt{3}/2)\phi(w)} : +O(z-w)] \qquad (2.E.13a)$$

$$: e^{-(i/\sqrt{3})\phi(z)} :: e^{-(i/2\sqrt{3})\phi(w)} := (z-w)^{1/6}[: e^{-(i\sqrt{3}/2)\phi(w)} : +O(z-w)] \qquad (2.E.13b)$$

The 2-point function,

$$F_2(\mathbf{z}_1, \mathbf{z}_2) \equiv \frac{< R_-(\infty)|\Phi^+_{1/6}(\mathbf{z}_1)\Phi^-_{1/6}(\mathbf{z}_2)R_+(0) >}{< R_-(\infty)R_+(0) >} \qquad (2.E.14)$$

can be computed the same way and agrees with the result (2.8.7) obtained through purely group theoretic means.

CHAPTER 3

Some Applications of CFT to 2-d Critical Statistical Models

3.1 Introduction

Conformal Field Theory is a very promising approach in order to obtain an exact solution to known 2-d critical statistical models, or to find wider classes of models that have not been known before.

There are a lot of CFT models that describe known universality classes of critical behavior in 2-d. The unitary models with $c < 1$ are known to describe the critical behavior of an infinite series of models introduced by Andrews, Baxter and Forrester, [34]. The first model in the series describes the universality class of the Ising model, $(c = \frac{1}{2})$. The second model, $(c = \frac{7}{10})$, describes the tri-critical Ising model (which has also N=1 superconformal invariance), [35]. The third member, $(c = \frac{4}{5})$, describes two different statistical models. The solution of the modular invariance constraints shows that there are two consistent subsets of operators for $c = \frac{4}{5}$. One of them describes the three-state, (Z_3), Potts model, [36], whereas the other one describes the "generic" tetra-critical model, (in field theory language this corresponds to the critical point of a ϕ^8 scalar field theory). Finally the $c = \frac{6}{7}$ model describes the tri-critical Potts model, [37]. There are also other such examples. All the critical lines of the Askin-Teller, (A-T), model have been described using conformal field theory. On the $c = 1$ line there are points that correspond to the first minimal N=2 superconformal model $(\tilde{c} = \frac{1}{3})$, [6,38], two decoupled Ising models, a Z_4 parafermionic model, [39] and an SU(2) WZW model with central charge $k = 1$, [7,40]. The critical behavior of isotropic spin-s anti-ferromagnetic chains is described by the SU(2) WZW model with central charge $k = 2s$, [40].

New integrable models and universality classes have been found due to CFT. A whole new class of $SU(N) \otimes SU(N)/SU(N)$ RSOS models has a critical behavior described by the corresponding G/H CFT, [41].

3.2 The CFT of a Free Scalar Field

In this section we will examine in some detail the CFT of a free scalar field, which will prove useful in the next sections. The theory is defined on the complex plane, (Riemann sphere) and the target manifold will taken to be a circle of radius R, (in order to deal with a discreet spectrum*. In complex coordinates the free action is,

$$S = \frac{1}{2\pi} \int d^2 z \partial_z \phi \partial_{\bar{z}} \phi \tag{3.2.1}$$

The classical equation of motion is the 2-d wave equation,

$$\partial_z \partial_{\bar{z}} \phi(z, \bar{z}) = 0 \tag{3.2.2}$$

which is solved by $\phi(z, \bar{z}) = f_1(z) + f_2(\bar{z})$ with f_1, f_2 arbitrary functions. The 2-point function is,

$$\langle \phi(z, \bar{z}) \phi(w, \bar{w}) \rangle = -log|z - w|^2 = -log(z - w) - log(\bar{z} - \bar{w}) \tag{3.2.3}$$

From now on we will talk separately about holomorphic and anti-holomorphic parts of correlation functions. We will put them together at the end. The holomorphic stress-energy tensor is given by,

$$T(z) = -\frac{1}{2} : \partial_z \phi \partial_z \phi : \tag{3.2.4}$$

The theory has a central charge $c = 1$ as can be seen from the OPE,

$$T(z)T(w) = \frac{1}{2} \frac{1}{(z - w)^4} + 2 \frac{T(z)}{(z - w)^2} + \frac{\partial_w T(w)}{(z - w)} + \cdots \tag{3.2.5}$$

The (anti-) holomorphic part of ϕ has the following Fourier expansion,

$$\phi(z) = \frac{1}{2} \phi_0 + plog(z) + \sum_{n \neq 0} \frac{a_n}{n} z^n \tag{3.2.6a}$$

$$\bar{\phi}(\bar{z}) = \frac{1}{2} \phi_0 + \bar{p} log(\bar{z}) + \sum_{n \neq 0} \frac{\bar{a}_n}{n} \bar{z}^n \tag{3.2.6b}$$

where p and $\bar{p}$ talke the following values,

$$(p, \bar{p}) = (\frac{n}{R} + \frac{1}{2} mR, \frac{n}{R} - \frac{1}{2} mR) \ , \ m, n \in Z \tag{3.2.7}$$

The quantized value for the winding number, $(p - \bar{p})$, is due to the fact that since ϕ is defined modulo $2\pi R$ the boundary condition is $\phi(\sigma + 2\pi, \tau) = \phi(\sigma, \tau) + 2\pi mR$. The quantized value for the momentum, $(\frac{1}{2}(p + \bar{p}))$, is due to requiring the single valuedness of $e^{iq\phi}$

⋆ This model will be referred to as the torus model since its target space is a one-dimensional torus.

As it can be seen from (3.2.3) the field ϕ is not a good conformal field, (its correlation functions grow with distance). Among operators that can be constructed out of polynomials in ϕ and derivatives the only primary conformal field is $\partial_z\phi$. This operator is a current with $(\Delta,\bar{\Delta}) = (1,0)$. It generates a $U(1)_L$ symmetry. There is also a right current, $\partial_{\bar{z}}\phi$ which generates $U(1)_R$. There are two additional Z_2 symmetries,

$$(\phi,\bar{\phi}) \rightarrow (-\phi,-\bar{\phi}) \ , \ (\phi,\bar{\phi}) \rightarrow (\bar{\phi},\phi) \tag{3.2.8}$$

Thus the total symmetry of the theory is $O(2)_L \otimes O(2)_R$. There are other primary fields in the theory apart from the currents. They are exponentials of ϕ. They are the so called vertex operators, $V_p(z) =: e^{ip\phi(z)} :$. They are primary operators of dimension $\Delta = \frac{p^2}{2}$ since they satisfy,

$$T(z)V_p(w) = \frac{p^2}{2}\frac{V_p(w)}{(z-w)^2} + \frac{\partial_w V_p(w)}{(z-w)} + \cdots \tag{3.2.9}$$

The $U(1)_L$ current $J(z) = i\partial_z\phi$ generates a $U(1)$ Kač-Moody algebra,

$$J(z)J(w) = \frac{1}{(z-w)^2} \tag{3.2.10}$$

The vertex operator V_p is a primary field of this $U(1)$ algebra with $U(1)$ charge p,

$$J(z)V_p(w) = p\frac{V_p(w)}{(z-w)} \tag{3.2.11}$$

Operator products of vertex operators are obtained using the well-known formula,

$$V_p(z)V_q(w) = (z-w)^{pq} : e^{ip\phi(z)+iq\phi(w)} : \tag{3.2.12}$$

and by expanding the exponential in the right hand side in powers of $z-w$. Using (3.2.12) correlation functions of vertex operators are easily obtained,

$$\langle V_{p_1}(z_1)V_{p_2}(z_2)...V_{p_N}(z_N)\rangle = \prod_{i<j}(z_i-z_j)^{p_ip_j}\delta_{p_1+..+p_N,0} \tag{3.2.13}$$

Charge conservation forces correlation functions with non-zero $U(1)$ charge to vanish.

We can construct vertex operators with well defined momentum and winding number,

$$V^+_{m,n} = \sqrt{2} : cos[p\phi(z) + \bar{p}\phi(\bar{z})] : \tag{3.2.14a}$$

$$V^-_{m,n} = \sqrt{2} : sin[p\phi(z) + \bar{p}\phi(\bar{z})] : \tag{3.2.14b}$$

where $p,\bar{p}$ are given in (3.2.7). The operators $V_{m,0}$ and $V_{0,m}$ are known as electric and magnetic operators.

The scalar field theories above for different compactification radius R are related. There is an exactly marginal operator that drives a particular theory to different values of R. Let's digress a little and discuss marginal perturbations in critical models.

Consider a set of operators M_i which are marginal, in other words primary fields of dimension (1,1). We need marginality so that criticality and the value of the central charge are preserved under marginal perturbations. A perturbation generated by such operators can be written as an extra term in the action,

$$\delta S = \sum_i \frac{\delta g_i}{2\pi} \int d^2 z M_i(z, \bar{z}) \tag{3.2.15}$$

Correlation functions of operators X are modified as follows,

$$\frac{\delta}{\delta g_i} \langle X \rangle = \frac{1}{2\pi} \int d^2 z \langle M_i(z, \bar{z}) X \rangle \tag{3.2.16}$$

Dimensions and operator product coefficients change in general under such perturbations. Using the standard formulas,

$$\langle \phi_i(z, \bar{z}) \phi_j(w, \bar{w}) \rangle = \delta_{i,j} (z - w)^{-2\Delta_i} (\bar{z} - \bar{w})^{-2\bar{\Delta}_i} \tag{3.2.17}$$

$$\phi_i(z, \bar{z}) \phi_j(w, \bar{w}) \sim \sum_k C_{ijk} (z - w)^{\Delta_k - \Delta_i - \Delta_j} (\bar{z} - \bar{w})^{\bar{\Delta}_k - \bar{\Delta}_i - \bar{\Delta}_j} \phi_k(w, \bar{w}) \tag{3.2.18}$$

along with (3.2.16) we find,

$$\frac{\delta}{\delta g_i} \langle \phi_j(z, \bar{z}) \phi_j(w, \bar{w}) \rangle = C_{ijj} (z - w)^{-2\Delta_j} (\bar{z} - \bar{w})^{-2\bar{\Delta}_j} log|z - w|^2 \tag{3.2.19}$$

By looking back to (3.2.17) we find that,

$$\delta \Delta_j = \delta \bar{\Delta}_j = - \sum_i C_{ijj} \delta g_i \tag{3.2.20}$$

For criticality to be preserved the dimension of the marginal operator must not change under the perturbation. If there is only one marginal perturbation then from (3.2.20) we obtain $C_{iii} = 0$. If there are more than one marginal perturbations then $C_{iij} = 0$ where the indices refer to marginal operators.

Thus the neighborhood of a CFT can be parametrized by the couplings g_i of the marginal operators. It has the structure of a manifold. Sometimes it happens that at certain points there is a larger number of marginal operators. Such points are called multi-critical. At such points there are extra directions that the theory can be deformed. There is usually an enhanced symmetry at such points that relates some of the marginal perturbations. Then the manifold develops an orbifold-type singularity at the multi-critical point.

Let's now return to the scalar theory. For any R there is always a (1,1) operator in the theory, $\partial_z\phi\partial_{\bar{z}}\phi$. It is truly marginal in the sense discussed above. The perturbation it induces changes the action by,

$$\delta S = \frac{\delta g}{2\pi}\int d^2z\partial_z\phi\partial_{\bar{z}}\phi \tag{3.2.21}$$

which amounds to a change in the compactification radius by $\delta R^2 = \delta g R^2$. This is true due to (3.2.20) which in this case reads,

$$\frac{\delta}{\delta g}\Delta_{m,n} = -(\frac{m^2}{R^2} - \frac{n^2R^2}{4}) = \frac{1}{2}R\frac{\delta}{\delta R}\Delta_{m,n} \tag{3.2.22}$$

Another interesting concept present in the theory is duality. This is the statement that the theory at radius R is equivalent to the theory at radius $\frac{2}{R}$. This can be easily seen from the partition function,

$$Z(R) = Tr\left[q^{L_0-\frac{1}{24}}\bar{q}^{\bar{L}_0-\frac{1}{24}}\right] = |\eta(q)|^{-2}\sum_{p,\bar{p}} q^{\frac{p^2}{2}}\bar{q}^{\frac{\bar{p}^2}{2}} \tag{3.2.23}$$

where $\eta(q)$ is the Dedekind η-function, [42]. It is easy to see that $Z(R) = Z(2/R)$. Under a duality transformation,

$$V_{m,n} \leftrightarrow V_{n,m} \ , \ \ \partial_z\phi\partial_{\bar{z}}\phi \leftrightarrow -\partial_z\phi\partial_{\bar{z}}\phi \tag{3.2.24}$$

Correlations functions and operator products are invariant under duality.

Let's now look for multicritical points. In order to have extra marginal operators R, (or by duality $\frac{2}{R}$), must be a multiple of $\sqrt{2}$. But in order for the marginal operators to remain marginal under the perturbation we end up with only two possible candidates, the self-dual point $R = \sqrt{2}$, and the point $R = \frac{1}{\sqrt{2}}$. They will be analyzed in the next sections.

3.3 Local SU(2) Invariance in the Scalar Theory

At $R = \sqrt{2}$ the model has an enhanced local symmetry. This point is the fixed point of the duality transformation. There are extra operators of dimension (1,0) that appear in the spectrum, as it can be seen from (3.2.7,9). They generate an $SU(2)_L \otimes SU(2)_R$ Kač-Moody algebra with a central charge $k = 1$. The left currents are,

$$J^1 =: cos[\sqrt{2}\phi] : \ , \ J^2 =: sin[\sqrt{2}\phi] : \ , \ J^3 = \frac{i}{\sqrt{2}}\partial_z\phi \tag{3.3.1}$$

They generate the following algebra,

$$J^a(z)J^b(w) = i\epsilon^{abc}\frac{J^c(w)}{(z-w)} + \frac{1}{2}\frac{\delta^{ab}}{(z-w)^2} + \cdots \tag{3.3.2}$$

The stress-energy tensor is of the Sugawara form,

$$T(z) = \frac{1}{3} : J^a(z) J^a(z) : \tag{3.3.3}$$

The infinite set of primary operators of the conformal algebra in this case is contained in just two irreducible representations of $SU(2)_L \otimes SU(2)_R$. The currents and the stress-energy tensor belong to the family of the unit operator. There is only one nontrivial representation with spin $\frac{1}{2}$. It is generated by the following $SU(2)_L \otimes SU(2)_R$ multiplet,

$$\phi^{\pm} =: e^{\frac{1}{\sqrt{2}}(\pm i\phi \pm i\bar{\phi})} : \tag{3.3.4}$$

of dimension $(\frac{1}{4}, \frac{1}{4})$. The partition function (3.2.23) can be written in terms of the $SU(2)$ characters,

$$Z(\sqrt{2}) = |\eta(q)|^{-2} \sum_{(m,n)\in Z^2} q^{\frac{(m+n)^2}{4}} \bar{q}^{\frac{(m-n)^2}{4}} = \chi_0(\tau)\overline{\chi_0(\tau)} + \chi_{\frac{1}{2}}(\tau)\overline{\chi_{\frac{1}{2}}(\tau)} \tag{3.3.5}$$

where $q = e^{2\pi i\tau}$ and*

$$\chi_0(\tau) = \frac{\vartheta_3(0|2\tau)}{\eta(\tau)} \; , \; \chi_{\frac{1}{2}}(\tau) = \frac{\vartheta_2(0|2\tau)}{\eta(\tau)} \tag{3.3.6}$$

There are nine marginal operators at this point. They can be constructed out of the $SU(2)$ currents as $J^a \bar{J}^b$. But due to SU(2) invariance they are equivalent among themselves and in particular to $J^3 \bar{J}^3 = \partial_z\phi\partial_{\bar{z}}\phi$. Thus there is still one marginal operator that changes the value of the radius. At this point we also have electric-magnetic duality since $\partial_z\phi\partial_{\bar{z}}\phi$ and $-\partial_z\phi\partial_{\bar{z}}\phi$ are related by an $SU(2)$ transformation.

3.4 The One Dimensional Orbifold

As it was mentioned in section 3.2 the torus model has a Z_2 symmetry: $(\phi, \bar{\phi}) \to (-\phi, -\bar{\phi})$. This is also a symmetry of the target space, S^1. One could mode-out by this symmetry, that is consider configurations with opposite values for ϕ as equivalent. This, when applied to the target manifold itself gives rise to a singular manifold, the Z_2 orbifold, [43]. The singularities appear at the fixed points of the symmetry transformation, $\phi = 0, \pi R$. Now apart from the

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⋆ Details about ϑ-functions can be found in [42].

states where the field ϕ is periodic we will have also states which are periodic up tp a symmetry transformation,

$$\phi(\sigma + 2\pi, \tau) = -\phi(\sigma, \tau) + 2\pi m R \ , \ m \in Z \tag{3.4.1}$$

The mode expansion in the twisted sector is,

$$\phi(z, \bar{z}) = \phi_0 + \sum_n \left[\frac{a_{n+1/2}}{n+\frac{1}{2}} z^{n+1/2} + \frac{\bar{a}_{n+1/2}}{n+\frac{1}{2}} \bar{z}^{n+1/2} \right] \ , \ n \in Z \tag{3.4.2}$$

where ϕ_0 can take only the values 0 or πR. Thus ϕ is double valued around the origin, $z = 0$, of the z-plane. In analogy with the free fermion in the Ramond sector we introduce fields, (the twist fields), in the presence of which the field ϕ is double valued around the origin. There are two such twist fields, H^0 and H^1 corresponding to the two different values for the zero mode of ϕ, ($\phi_0 = 0, \pi R$). A twist field at zero and another ar infinity create a branch cut on the plane around which the boson is double valued.

The interpretation above can be made more specific in terms of correlation functions as follows,

$$\langle \prod_i O_i(z_i) \rangle_t \equiv \frac{\langle H^{0,1}(\infty) \prod_i O_i(z_i) H^{0,1}(0) \rangle}{\langle H^{0,1}(\infty) H^{0,1}(0) \rangle} \tag{3.4.3}$$

where O_i are operators constructed out of the field ϕ itself and the prescription for calculating the left hand side is through the use of the mode expansion (3.4.2). The 2-point function of the boson in the presence of twist field can be easily calculated using the mode expansion (3.4.2),

$$\langle \phi(z, \bar{z}) \phi(w, \bar{w}) \rangle_t = log \left[\frac{\sqrt{z} + \sqrt{w}}{\sqrt{z} - \sqrt{w}} \right] + c.c. \tag{3.4.4}$$

We can use (3.2.4) and (3.4.4) to calculate the expectation value of the stress- energy tensor in the presence of the twist fields. A short computation gives,

$$\frac{\langle H^{0,1} | T(z) | H^{0,1} \rangle}{\langle H^{0,1} | H^{0,1} \rangle} = \frac{1}{16} \frac{1}{z^2} \tag{3.4.5}$$

This implies that the dimension of the twist field is $\frac{1}{16}$. The U(1) current is double valued around the twist field as shown by the following OPE,

$$\partial_z \phi(z) H^{0,1}(w) = \frac{\tau^{0,1}(w)}{(z-w)^{\frac{1}{2}}} + ... \tag{3.4.6}$$

where $\tau^{0,1}$ is a descendant field of dimension $\frac{1}{2} + \frac{1}{16}$. What we have here is a twisted $U(1)$ Kač-Moody algebra. The global $U(1)$ symmetry is broken and the charge neutrality condition of the correlation functions is not true any more. The states fall into representations of the twisted $U(1)$ Kač-Moody algebra. The twist fields, $H^{0,1}$, in particular are primary, that is they are hwvs of the algebra and they generate hw irreducible unitary representations. The descendant states in such representations are generated by acting on the hwv with the current modes, $a_{-n+1/2}$, $n=1,2,...$, (see (3.4.2)). For example the operator $\tau^{0,1}$ in (3.4.6) is obtained by acting $a_{-1/2}$ on the state $|H^{0,1}\rangle$. The primary fields of the conformal algebra have dimensions given by $\Delta_n = \frac{(2n+1)^2}{16}$, [7,44]. They are descendants of the primary fields $H^{0,1}$ in the twisted $U(1)$ verma module. To see this explicitly consider the partition function of the twisted $U(1)$ algebra†,

$$\chi_t(z) \equiv Tr[z^{L_0}] = z^{1/16}\prod_{n=1}^{\infty}\frac{1}{(1-z^{n-1/2})} \tag{3.4.7}$$

The respective partition function for the conformal algebra is,

$$\chi_c(z) = \prod_{n=1}^{\infty}\frac{1}{(1-z^{n})} \tag{3.4.8}$$

Their ratio will give us the primary conformal fields that are descendants in the twisted $U(1)$ algebra.

$$\frac{\chi_t(z)}{\chi_c(z)} = z^{1/16}\prod_{n=1}^{\infty}\frac{1-z^n}{1-z^{n-1/2}} = \sum_{n=0}^{\infty} z^{(2n+1)^2/16} \tag{3.4.9}$$

which proves our previous assertion. Such primary conformal fields can be constructed out of the the $|H^{0,1}\rangle$ by the action of certain lowering operators of the twisted $U(1)$ algebra. As an illustration the first few ones are given below.

$$|H_1\rangle = \sqrt{2}a_{-1/2}|H_0\rangle \ , \ |H_2\rangle = \frac{3}{\sqrt{2}}[a_{-3/2} - 2(a_{-1/2})^3]|H_0\rangle \tag{3.4.10}$$

Thus correlation functions of these fields can be computed from the correlation functions of $H^{0,1}$ in the presence of a $U(1)$ current. Something worth of noticing is that the dimensions of the primary states in the twisted sector do not depend on the radius of the orbifold.

In the untwisted sector one has to project out the states that are not invariant under $\phi \to -\phi$. Thus from the set of operators in (3.2.14) only the ones in (3.2.14a) survive.

Correlation functions of the operators (3.2.14a) can be easily computed in the presence of two twist fields using the operator formalism and the 2-point function (3.4.4),

$$\langle\prod_{i=1}^{N} V_{p_i}(z_i)\rangle_t = \prod_{i=1}^{N} 2^{-p_i^2} z_i^{-p_i^2/2} \prod_{i<j}^{N}\left[\frac{\sqrt{z_i}+\sqrt{z_j}}{\sqrt{z_i}-\sqrt{z_j}}\right]^{-p_ip_j} \tag{3.4.11}$$

and the same formula holds for the anti-holomorphic part.

† See Appendix 2.B.

The non-trivial correlation function in the orbifold model is the 4-point function of twist fields. This has been calculated in [45,46]. We quote here there result since it will be useful in the following‡. Let,

$$F_{\epsilon_1,\epsilon_2}(x,\bar{x}) = \lim_{z_\infty,\bar{z}_\infty\to\infty} |z_\infty|^{1/4}\langle H^0(z_\infty,\bar{z}_\infty)H^{\epsilon_1}(1)H^{\epsilon_1+\epsilon_2}(x,\bar{x})H^{\epsilon_2}(0)\rangle \tag{3.4.12}$$

where ϵ_1, ϵ_2 should be added modulo 2. Projective invariance was used to put three out of the four points at $0, 1, \infty$. Then,

$$F_{\epsilon_1,\epsilon_2}(x,\bar{x}) = |x(1-x)|^{-\frac{1}{12}}G_{\epsilon_1,\epsilon_2}(x,\bar{x}) \tag{3.4.13}$$

where,

$$G_{\epsilon_1,\epsilon_2}(x,\bar{x}) = |\vartheta_1'(\tau)|^{-\frac{2}{3}} \sum_{n\in 2Z+\epsilon_2}^{m\in Z} (-1)^{m\epsilon_1} exp\left[\frac{i\pi}{2}\left(\tau\left(\frac{m}{R}+\frac{nR}{2}\right)^2 - \bar{\tau}\left(\frac{m}{R}-\frac{nR}{2}\right)^2\right)\right] \tag{3.4.14}$$

and,

$$x = \left[\frac{\vartheta_3(\tau)}{\vartheta_4(\tau)}\right]^4 \tag{3.4.15}$$

We can use the 4-point function of the twist fields, (3.4.14) to obtain the appropriate OPE and OPE coefficients, [46,47].

$$[H^0]\otimes[H^0] = \sum_{n,m\in Z} C^{2n,2m}[V^+_{2n,2m}] + \sum_{n,m\in Z} C^{2n+1,2m}[V^+_{2n+1,2m}] \tag{3.4.16a}$$

$$[H^1]\otimes[H^1] = \sum_{n,m\in Z} C^{2n,2m}[V^+_{2n,2m}] - \sum_{m,n\in Z} C^{2n+1,2m}[V^+_{2n+1,2m}] \tag{3.4.16b}$$

$$[H^0]\otimes[H^1] = \sum_{m,n\in Z} C^{2n,2m+1}[V^+_{2n,2m+1}] \tag{3.4.16c}$$

where,

$$[C^{0,0}]^2 = 1\ ,\ [C^{n,m}]^2 = 2\times 8^{-(p^2_{m,n}+\bar{p}^2_{m,n})} \tag{3.4.17}$$

There is a D_4 invariance in the theory generated by,

$$(H^0, H^1, V^+_{m,n}) \to (-H^0, H^1, (-1)^n V^+_{m,n}) \tag{3.4.18a}$$

$$(H^0, H^1, V^+_{m,n}) \to (H^1, H^0, (-1)^m V^+_{m,n}) \tag{3.4.18b}$$

The invariance follows from the OPE, (3.4.16). All correlation functions must be invariant under

‡ In fact all partition functions and correlation functions of twisted and untwisted operators have been calculated on any genus Riemann surface, [47].

the transformations above which generate the dihedral group D_4.

The partition function of the orbifold model is the sum of the contributions coming from the untwisted and twisted sectors. The part coming from the untwisted sector is half the partition function of the corresponding torus model, (3.2.23), since we projected out half of the primary operators in (3.2.14). On the other hand the twisted part is independent of the radius R since as we pointed out before the critical dimensions and the structure of the representations of the twisted sector are independent of R. The twisted part will be determined when we discuss the multicritical point $R = \frac{1}{\sqrt{2}}$.

3.5 The Multi-Critical Point, $R=\frac{1}{\sqrt{2}}$

In section 3.2 we pointed out that the only other candidate in the set of torus models to be a multicritical point is the model with $R = \frac{1}{\sqrt{2}}$. Here we will show that this is indeed true.

The model has half the radius of the $SU(2)$ model. There is a well defined projection in order to get this model from the $SU(2)$ model. This is done by projecting onto even momentum states and adding extra states with half-integer winding number, (see (3.2.7)). Such states are created by the operators $V^{+}_{0,\frac{1}{2}}$. We would use the facts above because they are helpful to visualize the connection of this model with orbifold models.

Consider the $SU(2)$ model and introduce the operators θ_i which are $SU(2)$ elements and act on ϕ as,

$$\theta_1\phi\theta_1 = -\phi \ , \ \theta_2\phi\theta_2 = -\phi + \frac{\pi}{\sqrt{2}} \ , \ \theta_3\phi\theta_3 = \phi + \frac{\pi}{\sqrt{2}} \tag{3.5.1}$$

They satisfy,

$$\theta_i\theta_j = \delta^{ij} + \epsilon^{ijk}\theta_k \tag{3.5.2}$$

The projection operator that we mentioned before is $P_3 \equiv \frac{1}{2}(1 + \theta_3\bar{\theta}_3)$. Out of the nine (1,1) operators of the $SU(2)$ model only five survive the projection P_3. They are,

$$J^3\bar{J}^3 \ , \ J^a\bar{J}^b \ , \ a,b = 1,2 \tag{3.5.3}$$

The operators of the second set are equivalent because they are related by the $U(1)_L \otimes U(1)_R$ symmetry of the theory. On the other hand they generate perturbations which are inequivalent to those generated by $J^3\bar{J}^3$. Thus there exists a new continuous family of deformations of the model. The model is indeed multi-critical.

In order to see the nature of the new direction we notice that we could have used P_1 in order to project out unwanted states. This would have the effect of identifying ϕ with $-\phi$, as it is obvious from (3.5.1). So the model is in fact equivalent to the orbifold model at $R = \sqrt{2}$. The operator $J^3\bar{J}^3$ generates perturbations that move along the line of the orbifold models.

The aforementioned equivalence can be used to determine the twist partition function in the orbifold models as it was pointed out in the previous section. That is,

$$Z_t = Z(\frac{1}{\sqrt{2}}) - \frac{1}{2}Z(\sqrt{2}) = \frac{1}{2}\frac{|\vartheta_2\vartheta_3| + |\vartheta_2\vartheta_4| + |\vartheta_3\vartheta_4|}{|\eta|^2} \tag{3.5.4}$$

where $\vartheta_i = \vartheta_i(0|\tau)$. Thus,

$$Z_{orb}(R) = \frac{1}{2}Z(R) + Z_t \tag{3.5.5}$$

There is a notion of duality for the orbifold models too. In the untwisted part of the spectrum the transformation is the same as in the torus model, (3.2.24). In order to see what happens in the twisted sector it is easy to look at the self dual point $R = \sqrt{2}$. The twist fields $H^{0,1}$ in this case are equivalent to the vertex operators of the torus $R = \frac{1}{\sqrt{2}}$ model* $V^{+}_{0,\frac{1}{2}}$, $V^{-}_{0,\frac{1}{2}}$. Thus duality means shifting $\bar{\phi} \to \bar{\phi} + \frac{\pi}{\sqrt{2}}$, $\phi \to \phi$. The effect of such a transformation on the twist fields is,

$$H^0 \leftrightarrow \frac{1}{\sqrt{2}}(H^0 + H^1)\ ,\ H^1 \leftrightarrow \frac{1}{\sqrt{2}}(H^0 - H^1) \tag{3.5.6}$$

There are no (1,1) operators in the twisted sector of the orbifold models. Thus the analysis of the existence of multicritical points is the same as in the torus models.

3.6 The $c = 1$ N=2 Superconformal Model

It was already mentioned in the previous chapter that there is a minimal N=2 superconformal model with $c = 1$. In this section we will analyze it in more detail. In fact we will show that there are two distinct torus models with unbroken N=2 superconformal invariance and two distinct orbifold models where N=2 superconformal invariance is broken down to N=1 due to the presence of the twisted sector. As we showed in Appendix 2.C the model, (including its twisted sector), is equivalent to one of the minimal models with $N = 1$ Superconformal invariance. In Appendix 2.E we showed that the model with unbroken $N = 2$ supersymmetry† is in fact the torus model with $R = \frac{1}{\sqrt{3}}$ or $R = \frac{2}{\sqrt{3}}$. In this section we are going to discuss the full model including the twisted sector too. We will use the formalism of N=1 supersymmetry which in this case gives a more economical way of describing things, especially in the twisted case.

In most of our discussion we will focus on the left sector only. At the end we will discuss the combination of left and right sectors.

⋆ In fact after some manipulations of equation (3.4.14,15) one arrives at the power-like behavior present in the correlation functions of vertex operators

† That is the untwisted sectors.

In the NS sector of the model the dimensions of the primary N=1 superfields are, 1, $\frac{1}{6}$ and $\frac{1}{16}$. The last one will come from the twisted sector of the orbifold model.

Let's start from the untwisted model.

To find the superpartner of $T(z)$, we have to find an operator with $\Delta = \frac{3}{2}$. There are two candidates, $V_{\sqrt{3}}(z)$ and $V_{-\sqrt{3}}(z)$, as well as any linear combination of the two, which has the correct dimension. For reasons that will be explained below, the correct form is:

$$G(z) \equiv \frac{i}{\sqrt{3}} \left[: e^{i\sqrt{3}\phi(z)} : - : e^{-i\sqrt{3}\phi(z)} : \right] \tag{3.6.1}$$

Then the $N = 1$ superconformal algebra closes correctly:

$$G(z)G(w) = \frac{2}{3} \frac{1}{(z-w)^3} + \frac{2T(w)}{(z-w)} \tag{3.6.2}$$

The primary operators in the NS sector are generated by primary superfields acting on the NS vacuum. A superfield is a function in superspace:‡ $\Phi(\mathbf{z}) = g(z) + \theta\psi(z)$. If Δ is the dimension of the bosonic component $g(z)$, then the corresponding dimension for the fermionic partner $\psi(z)$ is $\Delta + \frac{1}{2}$. A primary superfield operator is defined through the following OPE with the super-stress-energy tensor:

$$\begin{aligned} G(z)g(w) &= \frac{\psi(w)}{(z-w)} \quad , \quad T(z)g(w) = \frac{\Delta g(w)}{(z-w)^2} + \frac{\partial_w g(w)}{(z-w)} \\ G(z)\psi(w) &= \frac{\partial_w g(w)}{(z-w)} + 2\frac{g(w)}{(z-w)^2} \,, T(z)\psi(w) = \left(\Delta + \frac{1}{2}\right) \frac{\psi(w)}{(z-w)^2} + \frac{\partial_w \psi(w)}{(z-w)} \end{aligned} \tag{3.6.3}$$

The obvious candidate for the $\Delta = 1$ primary operator is the $U(1)$ current of the system, $g_1(z) \equiv i\partial_z\phi(z)$. It has an O.P.E. with G(z):

$$G(z)g_1(w) = -i\frac{: e^{i\sqrt{3}\phi(w)} : + : e^{-i\sqrt{3}\phi(w)}}{(z-w)} \tag{3.6.4}$$

From (3.6.3) we can infer that the superpartner of $g_1(z)$ is $\psi_1(z) = -i[V_{\sqrt{3}}(z) + V_{-\sqrt{3}}(z)]$. It is an easy exercise to check that the rest of the relations (3.6.3) are satisfied.

‡ z and θ are the coordinates in superspace denoted collectively with $\mathbf{z}$.

$G(z)$ and $\psi_1(z)$ are the two supercharges of the corresponding $N=2$ minimal theory which, along with $T(z)$ and $g_1(z)$, complete the $N=2$ super-stress-energy tensor multiplet, [6,33]. As far as the other representations are concerned they can be built from the N=1 representations without adding new fields in the super-multiplet. As it was shown in Appendix 2.E, for N=2 representations degenerate at level 1/2 one of the fermionic components vanishes identically while the second bosonic component is the derivative of the first one. Thus the N=2 super-multiplets contain the same number of degrees of freedom as the N=1 super-multiplets. Using the remarks above the N=2 structure can be easily reconstructed from the N=1 structure.

There are two $\Delta = \frac{1}{6}$ operators with opposite $U(1)$ charge: $g^{\pm}_{\frac{1}{6}}(z) \equiv e^{\pm\frac{i}{\sqrt{3}}\phi(z)}$. We can calculate the superpartners of $g^{\pm}_{\frac{1}{6}}(z)$: $\psi^{\pm}_{1/6} = \mp\frac{i}{\sqrt{3}}$ $: e^{\mp\frac{2i}{\sqrt{3}}\phi(z)}$. Thus apart from the $\Delta = \frac{1}{16}$ operator, the above exhaust the set of primary operators in the NS sector. The identifications above imply the following operator algebra:

$$[1] \otimes [1] = [0] \quad , \quad [1] \otimes \left[\frac{1}{6}\right] = \left[\frac{1}{6}\right] \tag{3.6.5a}$$

$$\left[\frac{1}{6}\right] \otimes \left[\frac{1}{6}\right] = [0] \oplus [1] \oplus \left[\frac{1}{6} + \frac{1}{2}\right] \tag{3.6.5b}$$

which is in accord with the known "fusion rules," [48].

In the Ramond sector the two ground states are generated from the NS vacuum by the corresponding spin field operators, $\Theta(z)$ and $\bar{\Theta}(z)$ of dimension $\Delta = \frac{1}{24}$, $G_0|\,\Theta\rangle = |\,\bar{\Theta}\rangle$.

One of them, $|\,\bar{\Theta}\rangle$ is degenerate at level zero and thus decouples. Correspondingly, $G(z)\Theta(w) \sim O[(z-w)^{\frac{1}{2}}]$. $\Theta(z)$ can be also represented as a vertex operator: $\Theta^{\pm}(z) =: e^{\pm\frac{i}{2\sqrt{3}}\phi(z)}$. We can explicitly compute:

$$G(z)\Theta^{\pm}(w) = \mp\frac{i}{\sqrt{3}} \quad \frac{: e^{\mp i\frac{5}{2\sqrt{3}}\phi(w)} :}{(z-w)^{\frac{1}{2}}} \tag{3.6.6}$$

As expected, $\Theta^{\pm}(z)$ create cuts in the complex plane around which the fermionic components of the superfields are double valued. The Ramond primary operators are generated from the Ramond ground state by the action of superfield operators.

The operator of dimension $\Delta = \frac{3}{8}$ in the R-sector can be represented by $g^{\pm}_{\frac{3}{8}}(z) = : e^{\pm i\frac{\sqrt{3}}{2}\phi(z)}:$. It is generated by the superfield operator $\Phi_{\frac{1}{6}}(\mathbf{z})$ acting on the R-vacuum. We can explicitly verify the following O.P.E.:

$$\left[0+\frac{3}{2}\right]\otimes\left[\frac{1}{24}\right]_{+}=\left[\frac{1}{24}+1\right]_{+},\quad \left[0+\frac{3}{2}\right]\otimes\left[\frac{3}{8}\right]_{-}=\left[\frac{3}{8}\right]_{+}$$
$$\left[\frac{1}{6}\right]_{+}\otimes\left[\frac{1}{24}\right]_{+}=\left[\frac{3}{8}\right]_{+},\quad \left[\frac{1}{6}\right]_{+}\otimes\left[\frac{1}{24}\right]_{-}=\left[\frac{1}{24}\right]_{-} \tag{3.6.7}$$
$$\left[\frac{1}{6}+\frac{1}{2}\right]_{+}\otimes\left[\frac{1}{24}\right]_{+}=\left[\frac{3}{8}\right]_{-},\quad \left[\frac{1}{6}+\frac{1}{2}\right]_{+}\otimes\left[\frac{1}{24}\right]_{-}=\left[\frac{1}{24}+1\right]_{+}$$

By replacing $+\leftrightarrow-$, (12) remains valid.

The operators constructed so far correspond to all the operators of the NS and R sectors of the corresponding $N=2$ model.

At this point we should be more careful concerning putting together the left and right parts in order to have a well defined model. We can rewrite the partition function (3.2.23) using the N=2 characters derived in section 2.3.

$$Z(\frac{1}{\sqrt{3}})=\frac{1}{2}\left[|\chi^{NS}_{0+}|^2+|\chi^{NS}_{0-}|^2+2|\chi^{NS}_{\frac{1}{6}+}|^2+2|\chi^{NS}_{\frac{1}{6}-}|^2+4|\chi^{R}_{\frac{3}{8}+}|^2+2|\chi^{R}_{\frac{1}{24}+}|^2+2|\chi^{R}_{\frac{1}{24}-}|^2\right] \tag{3.6.8}$$

The explicit expression for the characters is,

$$\chi^{NS}_{0\pm}(\tau)=\frac{1}{\eta(\tau)}\sum_{n\in Z}(\pm1)^n q^{\frac{3n^2}{2}} \tag{3.6.9a}$$

$$\chi^{NS}_{\frac{1}{6}\pm}(\tau)=\frac{1}{\eta(\tau)}\sum_{n\in Z}(\pm1)^n q^{\frac{3}{2}(n+\frac{1}{3})^2} \tag{3.6.9b}$$

$$\chi^{R}_{\frac{3}{8}\pm}(\tau)=\frac{1}{2\eta(\tau)}\sum_{n\in Z}(\pm1)^n q^{\frac{3}{2}(n+\frac{1}{2})^2} \tag{3.6.9c}$$

$$\chi^{R}_{\frac{1}{24}\pm}(\tau)=\frac{1}{\eta(\tau)}\sum_{n\in Z}(\pm1)^n q^{\frac{3}{2}(n+\frac{1}{6})^2} \tag{3.6.9d}$$

where as usual $q=e^{2\pi i\tau}$ ⋆.

⋆ In fact $\chi^{R}_{\frac{3}{8}-}=0$ and this is the reason that it does not appear in (3.6.8).

The characters with a minus subscript in (3.6.9) are defined with an insertion of the fermion number operator, $(-1)^F$ in the trace. Thus (3.6.8) implies that all the operators in the theory have even fermion number. Using the character decomposition formulas from Appendix 2.C we can write the partition function using N=1 characters. The model contains the following N=1 representations, $(0,0)^{NS}$, $(\frac{1}{6},\frac{1}{6})^{NS}$, $(1,1)^{NS}$, $(\frac{1}{24},\frac{1}{24})^{R}$, $(\frac{3}{8},\frac{3}{8})^{R}$ of spin zero and $(0,1)^{NS}$, $(1,0)^{NS}$ of spin ± 1. The N=2 supersymmetry is unbroken.

There is another torus model with unbroken N=2 superconformal invariance at $R=\frac{2}{\sqrt{3}}$. The local operator analysis is the same as above. The only difference is in the way left and right part are sewn together. In fact it is easy to show that $Z(\frac{2}{\sqrt{3}})$ can be written as in (3.6.8) with the only difference that the $\chi^{R}_{\frac{1}{24}-}$ term comes with a minus sign in front. This has a consequence that the Ramond ground state $(\frac{1}{24},\frac{1}{24})$ is projected out †.

Let's now discuss superconformal invariance in the orbifold models. As anticipated the two models with N=1 unbroken invariance have $R=\frac{1}{\sqrt{3}}$ and $R=\frac{2}{\sqrt{3}}$. We will start from the $R=\frac{1}{\sqrt{3}}$ model. As in the torus case the other model will be obtained by switching the appropriate sign in its partition function, (see (3.5.5)). In Appendix 2.C it was shown that the single operator of the twisted sector of the $\tilde{c}=\frac{1}{3}$, $N=2$ model with $\Delta=\frac{1}{16}$ decomposes into the $\Delta=\frac{1}{16}$ operator in the NS sector of the $N=1$ system. Since the operator in the T-sector twists one of the two bosonic components of the $N=2$ superfields, it is natural to expect that a candidate for the $\Delta=\frac{1}{16}$ operator is the "twist" field $H^{0,1}(z)$, which twists the scalar field $\phi(z)$.

Thus we identify the dimension $\frac{1}{16}$ operator in the NS sector with one of the twist fields, H^0. Using (3.4.16) one can ascertain that from the primary operators in the NS sector, only $g_1(z)$ and $G(z)$ have vanishing three-point functions with two twist fields. The three-point function of three twist fields is automatically zero due to the D_4 symmetry of the orbifold model.

Thus we have the following O.P.E. ‡:

$$\left[\frac{1}{16}\right] \otimes \left[\frac{1}{16}\right] = [0] \oplus \left[1+\frac{1}{2}\right] \oplus \left[\frac{1}{6}\right] \oplus \left[\frac{1}{6}+\frac{1}{2}\right] \tag{3.6.10}$$

The superpartner of H^0 is given by:

$$G(z)H^0(w) = \frac{\tau^0(w)}{(z-w)} \quad , \quad \Delta_{\tau^0} = \frac{1}{16}+\frac{1}{2} \tag{3.6.11}$$

Let's now investigate the operator product $[1]\otimes[\frac{1}{16}]$. Due to twist conservation, the only families that are allowed to appear are $[\frac{1}{16}]$ and $[\frac{1}{16}+\frac{1}{2}]$.

† Part of the representation it generates is still present
‡ We use N=1 representations.

Since the expectation value of $g_1(z)$ in the presence of two twist fields is zero, $[\frac{1}{16}]$ is not present in the operator product. To investigate the appearance of $[\frac{1}{16}+\frac{1}{2}]$ we must find $\langle 0|H^0(\infty)i\partial_z\phi\tau^0(0)|0\rangle$. To evaluate this three-point function, we first compute:

$$F(z,w) \equiv \frac{\langle 0 \mid H^0(\infty)i\partial_z\phi(z)G(w)H^0(0)\mid 0\rangle}{\langle 0 \mid H^0(\infty)H^0(0)\mid 0\rangle} = \frac{i\sqrt{3}}{4}\,\frac{z^{\frac{1}{2}}}{(z-w)\cdot w} \tag{3.6.12}$$

Now, if we let $w \to 0$, we can find $\langle 0 \mid H^0(\infty)i\partial_z\phi(z)\tau^0(0)\mid 0\rangle$ as the residue of the $\frac{1}{w}$ pole. This gives:

$$\frac{\langle 0 \mid H^0(\infty)i\partial_z\phi(z)\tau^0(0)\mid 0\rangle}{\langle 0 \mid H^0(\infty)H^0(0)\mid\rangle} = -\frac{i\sqrt{3}}{4}\quad z^{-\frac{3}{2}} \neq 0 \tag{3.6.13}$$

Consequently $[1]\otimes[\frac{1}{16}] = [\frac{1}{16}+\frac{1}{2}]$. The only remaining O.P.E. to compute in the NS sector is $[\frac{1}{6}]\otimes[\frac{1}{16}]$. Again, conservation of twist implies that only the families $[\frac{1}{16}]$ and $[\frac{1}{16}+\frac{1}{2}]$ can appear in the operator product. Doing an analogous computation as above we find indeed:

$$\left[\frac{1}{6}\right] \otimes \left[\frac{1}{16}\right] = \left[\frac{1}{16}\right] \oplus \left[\frac{1}{16}+\frac{1}{2}\right] \tag{3.6.14}$$

in accord with [48]. Now the picture of the NS sector is complete.

In the Ramond sector the twisted states will be generate by the action of H^0 on the Ramond vacuum. Indeed using (3.4.16) and the explicit form of the Ramond vacuum it is easy to find that the dimension $\frac{1}{16}$ oparator in the Ramond sector is in fact H^1 whereas the dimension $\frac{9}{16}$ operator is τ^1. This is also supported by equation (2.C.5). Using the identifications above and (3.4.16) we can verify the fusion rules which are already known, [48],

$$\left[\frac{1}{16}\right]^R \otimes \left[\frac{1}{16}\right]^R = [0]^{NS} \oplus \left[\frac{1}{6}\right]^{NS}\ ,\ \left[\frac{1}{16}\right]^R \otimes \left[\frac{3}{8}\right]^R = \left[\frac{1}{16}\right]^{NS} \tag{3.6.15a}$$

$$\left[\frac{9}{16}\right]^R \otimes \left[\frac{1}{16}\right]^R = [1]^{NS} \oplus \left[\frac{1}{6}\right]^{NS}\ ,\ \left[\frac{9}{16}\right]^R \otimes \left[\frac{9}{16}\right]^R = [0]^{NS} \oplus \left[\frac{1}{6}\right]^{NS} \tag{3.6.15b}$$

$$\left[\frac{1}{16}\right]^R \otimes \left[\frac{1}{24}\right]^R = \left[\frac{1}{16}\right]^{NS}\ ,\ \left[\frac{9}{16}\right]^R \otimes \left[\frac{1}{24}\right]^R = \left[\frac{1}{16}\right]^{NS} \tag{3.6.15c}$$

What remains to be done is the description of both left and right sectors of the model. This can be done by looking at its partition function. As it was already shown in section 3.4 the partition function of the model can be written as,

$$Z_{orb}(\frac{1}{\sqrt{3}}) = \frac{1}{2}Z(\frac{1}{\sqrt{3}}) + Z_t \tag{3.6.16}$$

where an explicit expression for Z_t was given in (3.5.4) and the torus part was already shown to be written in terms of the N=2 characters. What remains to show is how to write the twisted part in terms of the N=2 twisted or N=1 characters. In terms of the N=1 characters we can easily verify that,

$$Z_t = \frac{1}{2}|\chi^{NS}_{0+} - \chi^{NS}_{1+}|^2 + |\chi^{NS}_{\frac{1}{16}+}|^2 + |\chi^{NS}_{\frac{1}{16}-}|^2 \tag{3.6.17}$$

where,

$$\chi^{NS}_{0\pm} - \chi^{NS}_{1\pm} = \frac{1}{\eta(\tau)}\sum_{n\in Z}(-1)^n q^{n^2} \tag{3.6.18a}$$

$$\chi^{NS}_{\frac{1}{16}\pm} = \frac{1}{\eta(\tau)}\sum_{n\in Z}(\pm 1)^n q^{(n+\frac{1}{4})^2} \tag{3.6.18b}$$

Thus the extra representations, $(\frac{1}{16},\frac{1}{16})^{NS}$, $(\frac{1}{16},\frac{1}{16})^{R}$, $(\frac{9}{16},\frac{9}{16})^{R}$ appear which have all spin zero.

3.7 The Bosonic Representation of the Critical Ising Model

In this section we are going to discuss a certain bosonization of the critical Ising model, [8]. We include this in the present chapter for reasons that will become obvious in the next section.

The critical Ising model is the first member of the conformal discreet series with $c = \frac{1}{2}$. It is the continuum limit of a massless free Majorana fermion. The operator content is, $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{16},\frac{1}{16})$. The operator of dimension $\frac{1}{2}$ is the fermion whereas the operator of dimension $\frac{1}{16}$ is the spin field around which the fermion is double-valued.

One might wonder how can, a free scalar field with $c = 1$ be equivalent to an Ising fermion $(c = \frac{1}{2})$? The answer is that the stress-energy tensor of the bosonic model does not have the standard quadratic form. In particular, as we shall see, it is far from obvious that the scalar field is free.

The construction that we are going to describe is at the operator level. We will consider a scalar field with radius $R = 1$. We will assume that the two point function is the free one:*

$$\langle 0|\phi(z)\phi(w)|0\rangle = -\ln(z-w). \tag{3.7.1}$$

Consider the most general operator of dimension two. It is a linear combination of $:\partial_z\phi\partial_z\phi:$, $\partial_z^2\phi$, $V_{\pm 2}(z)$, $\partial_z\phi V_{\pm\sqrt{2}}(z)$. Thus we will consider a general linear combination of the operators

* This is indeed an assumption since the stress-energy tensor is not the free one.

above. If we impose (1.1.21), then there are two distinct possibilities: The first is $T(z) = -\frac{1}{2} : \partial_z\phi\partial_z\phi : +\beta\partial_z^2\phi$, which has been known already from the work of ref. [49]. The second is:

$$T(z) = -\frac{1}{4} : \partial_z\phi\partial_z\phi : +\beta V_2(z) + \bar{\beta}V_{-2}(z). \tag{3.7.2}$$

with $\beta\bar{\beta} = \frac{1}{16}$, and $\beta, \bar{\beta}$ are otherwise arbitrary. In this case a direct computation shows that $c = \frac{1}{2}$! From now on we will focus on the second case.

The value of the central charge hints that somehow the theory described by (3.7.2) is the Ising model. Let's investigate, what are the primary operators in this theory.

Recall that a primary operator $\Phi(z)$, of dimension Δ, satisfies the following O.P.E.

$$T(z)\Phi(w) = \Delta\frac{\Phi(w)}{(z-w)^2} + \frac{\partial_w\Phi(w)}{(z-w)} + \cdots \tag{3.7.3}$$

It is easy to show that derivatives of ϕ cannot be primary operators. But what about vertex operators? Since $V_a \otimes V_b \simeq V_{a+b}$, only $V_{\pm 1}$ have a chance of being primary. In fact, by imposing (4), we can deduce that $\psi(z) = kV_1(z) + \bar{k}V_{-1}(z)$ is primary if and only if $4\beta\bar{k} = k$, and its dimension is $\Delta = \frac{1}{2}$. The dimension suggests that this operator represents the fermion of the Ising model. There is another operator that we have to look for, the spin field (order and disorder operator), with $\Delta = \frac{1}{16}$. In the standard free scalar theory there is an operator, (in fact two, $H^{0,1}(z)$), of dimension $\frac{1}{16}$, the "twist fields" of the boson. We need though to compute again the dimension of these operators using the new form of the stress-energy tensor, (3.7.2). A straightforward calculation gives, $\Delta_{H^0} = \frac{1}{32} + \frac{\beta+\bar{\beta}}{16}$, $\Delta_{H^1} = \frac{1}{32} - \frac{\beta+\bar{\beta}}{16}$. Thus in order for one of $H^{0,1}$ to have the correct dimension, $\beta + \bar{\beta} = \frac{1}{2}$, or $\beta + \bar{\beta} = -\frac{1}{2}$ which fixes them completely, $\beta = \bar{\beta} = \frac{1}{4}$ or $\beta = \bar{\beta} = -\frac{1}{4}$. We will focus on the first possibility, and we will comment on the second later on. Thus $\Delta_{H^0} = \frac{1}{16}$ and $\Delta_{H^1} = 0$. The operator H^1 seems to decouple.

Then,

$$T(z) = -\frac{1}{4} : \partial_z\phi\partial_z\phi : +\frac{1}{4}(V_2(z) + V_{-2}(z)) \equiv -\frac{1}{4} : \partial_z\phi\partial_z\phi : +\frac{1}{2} : \cos(2\phi) : \tag{3.7.4a}$$

$$\psi(z) = \frac{1}{\sqrt{2}}(V_1(z) + V_{-1}(z)) \equiv \sqrt{2} : \cos\phi :, \langle 0|\psi(z)\psi(w)|0\rangle = \frac{1}{z-w} \tag{3.7.4b}$$

The next step is to verify the operator algebra of the Ising model:

$$\left[\frac{1}{2}\right] \otimes \left[\frac{1}{2}\right] = [0],\ \left[\frac{1}{2}\right] \otimes \left[\frac{1}{16}\right] = \left[\frac{1}{16}\right],\ \left[\frac{1}{16}\right] \otimes \left[\frac{1}{16}\right] = [0] \oplus \left[\frac{1}{2}\right] \tag{3.7.5}$$

That this is indeed true can be seen from (3.4.16). For example,

$$\langle 0|H^0(z_1)\psi(z_2)H^0(z_3)|0\rangle = \frac{1}{\sqrt{2}}z_{13}^{-\frac{1}{8}}\left[\frac{z_{13}}{z_{12}z_{23}}\right]^{\frac{1}{2}} \tag{3.7.6}$$

The dihedral symmetry, (D_4), of the bosonic system translates into the $\mathbf{Z}_2$ symmetry of the

Ising model and its dual $\widetilde{\mathbf{Z}}_2$.

Next we calculate the 4-point functions in the bosonic theory. The following two are very simple to calculate:

$$\langle 0|\psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)|0\rangle = \frac{1}{z_{14}z_{23}}\left[\frac{x^2-x+1}{x}\right] \tag{3.7.7}$$

$$\frac{\langle 0|\psi(z_1)\psi(z_2)H^0(z_3)H^0(z_4)|0\rangle}{\langle 0|H^0(z_3)H^0(z_4)|0\rangle} = \frac{1}{2}\frac{z_{34}}{z_{14}z_{23}}\frac{x-2}{x}\sqrt{1-x} \tag{3.7.8}$$

where,

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{3.7.9}$$

The correlation function of four twist fields is the most non-trivial test. We can use equation (3.4.14). After some tedious algebra we can express the formula (3.4.14) for $R=1$ explicitly in terms of x. The result is,

$$\begin{aligned}\langle 0|H^0(z_1,\bar{z}_1)H^0(z_2,\bar{z}_2)H^0(z_3,\bar{z}_3)H^0(z_4,\bar{z}_4)|0 rangle = \\ = \frac{1}{2}|z_{12}z_{34}|^{-\frac{1}{4}}|x(1-x)|^{-\frac{1}{4}}G(x,\bar{x})\end{aligned} \tag{3.7.10}$$

$$G(x,\bar{x}) = \sqrt{1-\sqrt{x}}\sqrt{1-\sqrt{\bar{x}}} + \sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{\bar{x}}}$$

(3.7.7), (3.7.8) and (3.7.10) coincide with the correlation functions of the Ising model.

As a final check we compute the partition function of the bosonic theory on a strip with periodic boundary conditions (that is, on a torus).

The method relies on computing $\langle T\rangle$ and integrating with respect to the modulus of the torus, τ, to obtain the partition function.

In order to compute $\langle T\rangle$ in the bosonic theory we need the propagator for the scalar field on the torus. We will employ the results on chiral bosonization, [50]. The path integral over the torus contains also a sum over the instanton sectors. Thus we split the scalar field ϕ into a classical, (instanton) part and a quantum part, $\phi = \phi_{cl} + \phi_{qu}$. Then

$$\langle 0|\phi_{qu}(z)\phi_{qu}(w)|0\rangle = -\ln E(z,w),\ E(z,w) = \frac{\vartheta_1(z,w|\tau)}{\vartheta_1'(0|\tau)}, \tag{3.7.11}$$

where ϑ_1 is the standard ϑ-function on the torus, [42].

$$\langle T(z)\rangle = -\frac{1}{4}\lim_{w\to z}\left\{\langle\partial_z\phi(z)\partial_w\phi(w)\rangle + \frac{1}{(z-w)^2}\right\} \tag{3.7.12}$$

since the expectation value of $V_{\pm 2}(z)$ vanishes. A straightforward computation gives $\langle T(z)\rangle = \frac{e_\nu}{4}$, where

$$e_\nu = -4\pi i \frac{\partial}{\partial\tau} \ln\left[\frac{\vartheta_{\nu+1}(0|\tau)}{\eta(\tau)}\right], \ \nu = 1,2,3 \tag{3.7.13}$$

and ν labels the periodicity properties of the fermion operator and $\eta(\tau)$ is the Dedekind η-function. (In the bosonic theory, this is generated by an appropriate sum over instanton sectors, [50].) $\nu = 1,2,3$ corresponds to (P,AP), (AP,AP), (AP,P) boundary conditions.

Integrating with respect to τ we obtain

$$Z_\nu \propto \left[\frac{\theta_{\nu+1}(0|\tau)}{\eta(\tau)}\right]^{\frac{1}{2}} \tag{3.7.14}$$

Thus the partition function of the bosonic theory is given by the sum over the various sectors,

$$Z_{tot} = \sum_{\nu=1}^{3} \mathbf{Z}_\nu(\tau)\bar{\mathbf{Z}}_\nu(\bar\tau) \tag{3.7.15}$$

which is equal to the partition function of the Ising model*:

$$Z_{Ising} = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{16}}|^2 \tag{3.7.16}$$

where χ_0, $\chi_{\frac{1}{2}}$, $\chi_{\frac{1}{16}}$ are the appropriate characters of the Virasoro algebra for $c = \frac{1}{2}$.

We mentioned previously that there is another set of values for β, $\bar\beta$ in (3.7.2) so that we have a consistent spectrum. The stress energy tensor in this case is†,

$$T_-(z) = -\frac{1}{4} : \partial_z\phi\partial_z\phi : -\frac{1}{2} : cos(2\phi) : \tag{3.7.17}$$

In this case the primary operator of dimension $\frac{1}{2}$ is,

$$\psi_-(z) = \frac{i}{\sqrt{2}}\left(V_1(z) - V_{-1}(z)\right) \tag{3.7.18}$$

The twist field H^0 has dimension zero under T_- whereas the dimension of H^1 is now $\frac{1}{16}$. Combining this with the fact that,

$$T_+(z) + T_-(z) = -\frac{1}{2} : \partial_z\phi\partial_z\phi : \tag{3.7.19}$$

we realize that in fact what we have done is we split the c=1 orbifold model at $R = 1$ in a direct product of two Ising models. Till now it was known that this is true, and we could construct the

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$\star$ In Appendix 3.A we will show that this equivalence persists on an arbitrary compact Riemann surface.

$\dagger$ We will use $T_+(z)$ to denote the stress energy tensor in (3.7.4a) fermions asl vertex operators, as in (3.7.4b,18), and the "spin" fields that twisted both fermions,

$$\Sigma = \sqrt{2} : cos \left[\frac{1}{2}(\phi - \bar{\phi})\right] \tag{3.7.20}$$

of dimension $\frac{1}{8}$. Obviously Σ is the product of the spin fields of the two independent Ising models. But we could not tell what the individual spin fields looked like. The construction above in fact answers this question. From (3.4.14) we learn,

$$H^0(z)H^1(w) = \Sigma(w) + \cdots \tag{3.7.21}$$

Another non-trivial check that points in the same direction is the fact that the 4-point function of two H^0 and two H^1 calculated from (3.4.14) in fact factorizes[‡] in a product of 2-point functions,

$$\langle 0|H^0(z_1,\bar{z}_1)H^0(z_2,\bar{z}_2)H^1(z_3,\bar{z}_3)H^1(z_4,\bar{z}_4)|0\rangle_{R=1} = |z_{12}|^{-\frac{1}{4}}|z_{34}|^{-\frac{1}{4}} \tag{3.7.22}$$

Finally the partition function of the orbifold model at $R = 1$ is easily shown to be the square of the partition function of the Ising model,

$$\left[Z_{Ising}\right]^2 = \frac{1}{2}Z(1) + Z_t \tag{3.7.23}$$

We can see the construction above in another way using the vertex operator representation of Kač-Moody algebras and the construction of the Ising model as a G/H CFT. This construction seems to be generalized to all G/H CFTs and will be discussed in Appendix 2.B.

3.8 The Critical Ashkin-Teller Model and CFT

In this section we are going to discuss the critical structure of the Ashkin-Teller, (A-T), model and describe the usefulness of the previous sections in understanding this critical behavior.

The A-T model is defined as the system of two Ising spins coupled with a 4-spin interaction, [51]. The Hamiltonian is,

$$H = -\sum_{<ij>} \left[g_2(s_i s_j + t_i t_j) + g_4(s_i s_j t_i t_j)\right] \tag{3.8.1}$$

where the spins take the values ± 1 and are positioned on the sites of a 2-d lattice and the interaction is a nearest-neighbor interaction. When $g_4 = 0$ then the model is equivalent to decoupled Ising models.

‡ After some tedious calculation

The most rich phase diagram is obtained in the transfer matrix approach, [52]. This involves the highly anisotropic hamiltonian lattice. In this approach we obtain the A-T quantum chain with a quantum Hamiltonian,

$$\hat{H} = -\frac{1}{2(1+\epsilon)} \sum_{i=1}^{N} \left[(\sigma_i + \sigma_i^\dagger + \epsilon \sigma_i^2) + \lambda(\Gamma_i \Gamma_{i+1}^\dagger + \Gamma_i^\dagger \Gamma_{i+1} + \epsilon \Gamma_i^2 \Gamma_{i+1}^2) \right] \tag{3.8.2}$$

where the chain has N sites, ϵ is a coupling constant, λ plays the role of the inverse temperature and

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.8.3}$$

The generic symmetry of the model is D_4. It is generated by the transformations,

$$\tilde{\Gamma}_i^m = M^{mn} \Gamma_i^n \ , \ m, n = 1, 2, 3 \tag{3.8.4}$$

where the eight matrices M^{mn} are given by Σ^l and $\Sigma^l C$,

$$\Sigma^l = \begin{pmatrix} e^{\frac{i\pi l}{2}} & 0 & 0 \\ 0 & e^{\frac{2i\pi l}{2}} & 0 \\ 0 & 0 & e^{\frac{3i\pi l}{2}} \end{pmatrix} \ , \ l = 0, 1, 2, 3 \ , \ C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.8.5}$$

There are eight different spectra of the quantum hamiltonian (3.8.2) generated by boundary conditions related to the various elements of the symmetry group D_4. For most of our discussion we will focus on periodic boundary conditions.

The phase diagram of the A-T quantum chain is known§, [52]. There a continuous line of critical points with $c = 1$ and continuously varying critical exponents, $\lambda = 1$, $-1 < \epsilon \leq 1$. This line terminates in a Z_4 model and then splits into two lines of critical points belonging to the Ising universality class. At the other end of the $c = 1$ line which terminates in the Kostelritz-Thouless point, start two lines that define a "critical fan", that is a whole area of criticality. There is a paramagnetic region, (I), which is disordered and the expectation values of $s + t$ and st are zero. Region III is fully ordered and $s + t$ has an expectation value. Region II is partially ordered. Here $s + t$ has zero expectation value unlike the expectation value of st which is non-zero. There is an anti-ferromagnetic frozen region, (IV), in which the system can be divided in two sub-lattices that behave differently. In one of them $s + t$ and its conjugate operator have eigenvalues respectively 1 and -1. In the other both eigenvalues are -1. In this region the expectation value of $s + t$ vanishes. For most of the rest we will focus on the c=1 line as well as the two Ising lines.

§ For convenience and to fix notation it is presented in fig 17.

On the c=1 critical line the model reduces to a free gaussian model with Hamiltonian, [53],

$$H = \frac{\kappa}{2} \sum_{<ij>} (\phi_i - \phi_j)^2 \tag{3.8.6}$$

where ϕ_i is a periodic scalar variable with period 2π. The continuum action is,

$$S = \frac{\kappa}{2} \int d\tau d\sigma \phi(\partial_\tau^2 + \partial_\sigma^2)\phi \tag{3.8.7}$$

where κ is given in terms of the coupling constant ¶,

$$\kappa = \frac{2}{\pi}(1 - \frac{1}{\pi} arccos(\epsilon)) \tag{3.8.8}$$

If we rescale the field ϕ so that the 2-point function is given by (3.2.3) then ϕ is periodic modulo $\frac{2\pi}{\sqrt{\pi\kappa}}$. Thus the radius of the torus is given by,

$$R(\epsilon) = \left[2(1 - \frac{1}{\pi} arccos(\epsilon))\right]^{-\frac{1}{2}} \tag{3.8.9}$$

It varies between the limits, $\frac{1}{\sqrt{2}} \leq R < \infty$. It is known that at the point $\epsilon = 0$ the two Ising spins decouple. Then the partition function has to be the square of the Ising partition function. This in fact along with modular invariance fixes the partition function of the A-T model to be the orbifold partition function (3.5.5).

Thus the whole discussion of the previous sections applies to this critical line.

The line $-\frac{1}{\sqrt{2}} \leq \epsilon \leq 1$ is a critical line corresponding to $\frac{1}{\sqrt{2}} \leq R \leq \sqrt{2}$. It contains the following well known points. The 4-state Potts model at $\epsilon = 1$, the Z_4 parafermionic model at $\epsilon = \frac{1}{\sqrt{2}}$, two decoupled Ising models at $\epsilon = 0$, an N=1 superconformal model* at $\epsilon = -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$, and the Kosterlitz-Thouless model at $\epsilon = -\frac{1}{\sqrt{2}}$.

This line continues into the critical fan, $-1 < \epsilon < -\frac{1}{\sqrt{2}}$, or $\sqrt{2} < R < \infty$. In there appear the dual models of the above with one exception. There is the N=1 superconformal model at $\epsilon = -\frac{\sqrt{3}}{2}$, two decoupled Ising models at $\epsilon = -\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$, the Z_4 parafermionic model at $\epsilon = -\frac{\sqrt{3}+1}{2\sqrt{2}}$, the 4-state Potts model at $\epsilon = -\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}}$, and a N=1 superconformal model** at $\epsilon = -\frac{1}{2}\sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{2}}}$. At the end of this line, $\epsilon = -1$ the gaussian analysis breaks down. There is a first-order phase transition there. This model could be described as an "anti-ferromagnetic" 4-state Potts model.

¶ This is known by maping the A-T model to the six-vertex model, [54].

* That is the twisted N=2 model.

** This is distinct from the previous two.

For the whole critical line above we know, (thanks to the previous sections), the exact partition function, the spectrum and the correlation functions. In particular we showed that there are operators coming from the twisted sector of the scalar theory with critical indices which are constant along the line. The most relevant one corresponds to the $\left[\frac{1}{16}, \frac{1}{16}\right]$ family coresponding to the leading magnetic exponent $x_H = \frac{1}{8}$. Our analysis predicts the existence and value of the second magnetic exponent $\tilde{x}_H = \frac{9}{8}$ which corresponds to the presence of the family $\left[\frac{9}{16}, \frac{9}{16}\right]$ in the spectrum.

It remains to describe the two Ising lines in terms of the scalar theory. In fact the renormalization group analysis of [52] derived the form of the Hamiltonian for those lines. Up to normalization it coincides with the stress-energy tensors, $T_\pm$ we presented in section 3.7. Thus the "bosonization" described in 3.7 in fact gives the correct mapping between the bosonic variables of the A-T model and the nature of the Ising critical lines.

The critical points of the A-T model have phenomenological importance since they seem to describe the superfluid-to-normal transition of He^4 films, [55] and possibly the critical behavior in planar magnets, [56] and liquid crystals, [57].

3.9 Conclusions and Prospects

In this chapter we analyzed in detail CFTs with $c = 1$. We subsequently used them to analyze the critical behavior of the quantum A-T chain. We were able to explain all important critical lines and calculate the critical partition functions, the spectrum and the correlation functions. All the calculations above are *exact*.

There is a potentially complete classification of $c = 1$ CFTs, [58]. A large part of them appear in the critical A-T model. One of course would like to solve more complicated critical systems and to find new ones. Such hopes seem well founded in the context of CFT. As it was already mentioned in the introduction, there are new statistical models that were found by knowing their critical points. On the other hand their critical points were found using CFT techniques. The models in question are the G/H RSOS models which turn out to be integrable even outside the critical regime. In Appendix 3.B we point out that such critical points will appear in multi-component scalar models which are not in general free.

It is conceivable that a lot of progress will be achieved in the coming years along these lines.

APPENDIX 3.A

The Bosonized Ising Model at Higher Genus

In this appendix we prove the equivalence of the bosonic and fermionic versions of the Ising model on an arbitrary compact Riemann surface.

We will show that the two theories possess the same partition function on any compact Riemann surface. To achieve that we will show that the expectation value of the stress-energy tensor is the same in both theories and thus their partition function are the same up to trivial constant.[⋆]

Let's first compute $\langle T\rangle$ in the fermionic case. The two-point function of the fermion on a compact Riemann surface of genus $g \geq 2$ is given by the Szego kernel[†]

$$\langle 0|\Psi(z)\Psi(w)|0\rangle = \frac{\theta\left[{a \atop b}\right](\int_w^z \nu)}{\theta\left[{a \atop b}\right](0)} \cdot \frac{1}{E(z,w)} \equiv P\left[{a \atop b}\right](z,w) \qquad (3.A.1)$$

where the pair (a,b), (a, b are g-dimensional vectors whose components are either 0 or $\frac{1}{2}$), specifies an arbitrary even spin-structure on the surface, $E(z,w)$ is the prime form, and ν^i, $i = 1, 2, ..., g$, is a basis of holomorphic one-forms.

$$\langle T(z)\rangle_F = -\frac{1}{2}\lim_{w\to z}\left\{\langle\Psi(z)\partial_w\Psi(w)\rangle - \frac{1}{(z-w)^2}\right\} \qquad (3.A.2)$$

The Szego kernel satisfies the following identity, [59],

$$\left[P\left[{a \atop b}\right](z,w)\right]^2 = \omega(z,w) + \sum_{i,j=1}^{g} A_{ij}\nu^i(z)\nu^j(w) \qquad (3.A.3)$$

$$A_{ij} \equiv \frac{\partial^2 \ln\theta\left[{a \atop b}\right]}{\partial z_i \partial z_j}[0],\ \omega(z,w) = \frac{\partial^2}{\partial z \partial w}\ln E(z,w). \qquad (3.A.4)$$

We need also the short distance expansion of the prime form.

$$E(z,w) = (z-w) - \frac{(z-w)^3}{12}S(w) + O[(z-w)^5] \qquad (3.A.5)$$

where S is the projective connection, [59]. Using (3.A.3), (3.A.4), (3.A.5) we can easily show

⋆ We will only discuss even spin structures where there are no zero modes for the fermion.

† For notation and more details see ref. [59].

that

$$\langle T(z)\rangle_F = \frac{1}{4}\sum_{i,j=1}^{g} A_{ij}\nu^i(z)\nu^j(z) - \frac{S(z)}{24} \tag{3.A.6}$$

Note that $\langle T(z)\rangle_F$ depends on z, since translation invariance is not a symmetry of the correlation functions for $g > 1$.

The corresponding calculation in the bosonic model proceeds along the same lines.

$$\langle T(z)\rangle_B = -\frac{1}{4}\lim_{w\to z}\left\{\langle\partial_z\phi(z)\partial_w\phi(w)\rangle + \frac{1}{(z-w)^2}\right\} \tag{3.A.7}$$

$$\phi(z) \equiv \sum_{i=1}^{g} p^i \int_{P_o}^{z} \nu^i + \phi_{qu}(z)$$

where the winding number p^i takes the appropriate values, $p^i = 2m + a^i$, $m \in \mathbf{Z}$, P_0 is an arbitrary point on the surface and

$$\langle\partial_z\phi_{qu}(z)\partial_w\phi_{qu}(w)\rangle = -\partial_z\partial_w lnE(z,w) = -\omega(z,w) \tag{3.A.8}$$

The sum over instanton sectors is weighed by the holomorphic instanton action, $S_m \equiv \frac{1}{2}(m+a)^i\Omega_{ij}(m+a)^j + 2\pi i b^i(m+a)^i$, where Ω_{ij} is the period matrix of the surface. The instanton sum contributes a factor $-2\frac{\partial}{\partial\Omega_{ij}}\ln\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](0)\nu^i(z)\nu^i(z)$ which by the heat equation satisfied by the θ-functions is equal to $-\sum_{i,j=1}^{g} A_{ij}\nu^i(z)\nu^j(z)$. Thus $\langle T(z)\rangle_F = \langle T(z)\rangle_B$ which completes the proof.

APPENDIX 3.B

The Ising Bosonization as a G/H Construction

In this appendix we will show that the bosonized version of the Ising model presented in section 3.7 can be understood through its G/H construction.

Let's consider the tensor product of two $SU(2)$ Kač-Moody algebras at level one. The currents satisfy the following algebra,

$$J_i^a(z)J_i^b(w) = i\epsilon^{abc}\frac{J_i^c(w)}{(z-w)} + \frac{1}{2}\frac{\delta^{ab}}{(z-w)^2} + \cdots \ , \ i = 1,2 \tag{3.B.1}$$

and the two algebras for $i = 1,2$ commute. The stress-energy tensor of this theory is of the Sugawara form and the central charge is $c = 2$,

$$T_G(z) = \frac{1}{3}\sum_{i=1,}^{2}\sum_{a=1}^{3} : J_i^a(z)J_i^a(z) : \tag{3.B.2}$$

Let's consider the diagonal $SU(2)$ subalgebra, generated by $J_1^a + J_2^a$. This is an $SU(2)$ Kač-Moody algebra at level two. There is an associated stress-energy tensor with it with central charge $c = \frac{3}{2}$,

$$T_H(z) = \frac{1}{4}\sum_{a=1}^{3} : (J_1^a(z) + J_2^a(z))(J_1^a(z) + J_2^a(z)) : \tag{3.B.3}$$

If we form the difference $T_G(z) - T_H(z)$ we can show that it is a stress-energy tensor with $c = \frac{1}{2}$ and that it commutes with T_H. Thus we can write $SU(2)_{k=1} \otimes SU(2)_{k=1} = SU(2)_{k=2} \otimes M_{G/H}$, [60]. This expresses the fact that the initial CFT can be written as a direct product of two other CFTs. The piece $M_{G/H}$ generated by $T_{G/H} = T_G - T_H$ is the critical Ising model as it is suggested by the value of the central charge.

We will now use the fact that the $SU(2)$ Kač-Moody algebra at level one can be constructed out of free boson of radius $R = \sqrt{2}^{\star}$. We will need two such bosons, ϕ_1 and ϕ_2 in order to make the product. We normalize as usual†,

$$\langle 0|\phi_i(z)\phi_j(w)|0\rangle = -\delta^{ij} log(z-w) \tag{3.B.44}$$

⋆ As it was shown in section 3.3.

† In this appendix we will deal only with holomorphic aspects

Then the $SU(2)\otimes SU(2)$ currents are[‡],

$$J_i^3(z)=\frac{i}{\sqrt{2}}\partial_z\phi_i(z)\ ,\ J_i^{\pm}(z)\equiv\frac{1}{\sqrt{2}}\left(J_i^1(z)\pm iJ_i^2(z)\right)=\frac{1}{\sqrt{2}}e^{\pm i\sqrt{2}\phi_i(z)} \qquad (3.B.5)$$

The Sugawara stress-energy tensor is just the sum of the stress-energy tensors of the two bosons,

$$T_G(z)=-\frac{1}{2}\partial_z\phi_1\partial_z\phi_1-\frac{1}{2}\partial_z\phi_2\partial_z\phi_2 \qquad (3.B.6)$$

The currents of the diagonal subalgebra can be written in terms of the bosons,

$$J_H^3=\frac{i}{\sqrt{2}}(\partial_z\phi_1+\partial_z\phi_2)\ ,\ J_H^{\pm}=\frac{1}{\sqrt{2}}\left[e^{\pm i\sqrt{2}\phi_1}+e^{\pm i\sqrt{2}\phi_2}\right] \qquad (3.B.7)$$

Now we can use can use (3.B.3) in order to calculate the stress-energy tensor for the subalgebra. The result is,

$$T_H(z)=-\frac{1}{8}(\partial_z\phi_1+\partial_z\phi_2)^2-\frac{1}{4}[(\partial_z\phi_1)^2+(\partial_z\phi_2)^2]+\frac{1}{4}\left[e^{i\sqrt{2}(\phi_1-\phi_2)}+e^{-i\sqrt{2}(\phi_1-\phi_2)}\right] \qquad (3.B.8)$$

We will use another basis for the bosons in such a way that the formulas become more transparent. Define[§],

$$\rho_1=\frac{1}{\sqrt{2}}(\phi_1+\phi_2)\ ,\ \rho_2=\frac{1}{\sqrt{2}}(\phi_1-\phi_2) \qquad (3.B.9)$$

The 2-point functions are still diagonal,

$$\langle 0|\rho_i(z)\rho_j(w)|0\rangle=-\delta^{ij}log(z-w) \qquad (3.B.10)$$

In this basis, $T_H(z)$ becomes,

$$T_H(z)=-\frac{1}{2}(\partial_z\rho_1)^2-\frac{1}{4}(\partial_z\rho_2)^2+\frac{1}{4}\left[e^{2i\rho_2}+e^{-2i\rho_2}\right]=-\frac{1}{2}(\partial_z\rho_1)^2+T_+ \qquad (3.B.11)$$

and,

$$T_{G/H}(z)\equiv T_G(z)-T_H(z)=-\frac{1}{4}(\partial_z\rho_2)^2-\frac{1}{4}\left[e^{2i\rho_2}+e^{-2i\rho_2}\right]=T_- \qquad (3.B.12)$$

which is exactly the expression we used for the Ising model in section 3.7. It is instructive to write also the $SU(2)_{k=2}$ currents in the new basis[¶],

$$J_H^3=i\partial_z\rho_1\ ,\ J_H^{\pm}=\frac{1}{\sqrt{2}}e^{\pm i\rho_1}\left[e^{i\rho_2}+e^{-i\rho_2}\right]=e^{\pm i\rho_1}\psi_+ \qquad (3.B.13)$$

It is known that the level two $SU(2)$ algebra can be constructed out of three free fermions. We can combine two of them to make a boson. Then (3.B.13) and (3.B.11) is written in terms of

‡ We will omit normal ordering symbols. All operators should be taken to be normal ordered.

§ The radius of the new bosons becomes now $R=1$.

¶ The notations, $T_{\pm}$, $\psi_{\pm}$ are the same as in section 3.7.

the boson ρ_1 and the fermion in its bosonized form in terms of ρ_2. The same remarks aply in decomposing the $SU(2)_1 \otimes SU(2)_1$ representations into representations of $SU(2)_2 \otimes Ising$.

The construction above suggests that this could be done for any pair (G,H) by starting from level one algebras that have a bosonic form which is well understood*. We will present an example by constructing the level N $SU(2)$ algebra and thus the associated Z_N parafermion theory, [61].

The level N $SU(2)$ algebra will be constructed as the diagonal subgroup of the product of N level one $SU(2)$ algebras. We will need N free scalar fields ϕ_i of radius $R = \sqrt{2}$ in order to construct N copies of $SU(2)_1$. We will normalize them as usual,

$$\langle 0|\phi_i(z)\phi_j(w)|0\rangle = -\delta^{ij} log(z-w)\ ,\ i,j = 1,2,\cdots,N \tag{3.B.14}$$

The individual $SU(2)_1$ currents are as in (3.B.5). The currents of the diagonal $SU(2)_N$ become,

$$J^3_N = \frac{i}{\sqrt{2}}\sum_{i=1}^{N}\partial_z\phi_i\ ,\ J^{\pm}_N = \frac{1}{\sqrt{2}}\sum_{i=1}^{N} e^{\pm i\sqrt{2}\phi_i} \tag{3.B.15}$$

while the Sugawara stress-energy tensor is,

$$T_N(z) = \frac{1}{N+2}\left[-\frac{1}{2}\left(\sum_{i=1}^{N}\partial_z\phi_i\right)^2 - \sum_{i=1}^{N}(\partial_z\phi_i)^2 + \sum_{i<j}^{N}\left(e^{i\sqrt{2}(\phi_i-\phi_j)} + e^{-i\sqrt{2}(\phi_i-\phi_j)}\right)\right] \tag{3.B.16}$$

We must now make a transformation on our basic variables ϕ_i to make things more transparent. Let's define,

$$\phi_i = \frac{\Phi}{\sqrt{N}} + \sqrt{2}\vec{\mu}^i \cdot \vec{\rho} \tag{3.B.17}$$

where $\vec{\rho}$ is an (N-1)-dimensional vector of scalar fields. The $\vec{\mu}^i$ are the weights of the fundamental representation of $SU(N)$. They are (N-1)-dimensional vectors and there are N of them. They are normalized as follows,

$$\vec{\mu}_i \cdot \vec{\mu}_j = -\frac{1}{2N} + \frac{1}{2}\delta^{ij} \tag{3.B.18}$$

The roots of $SU(N)$ are $\vec{\alpha}_{ij} = \vec{\mu}^i - \vec{\mu}^j$. They are normalized to $\vec{\alpha}\cdot\vec{\alpha} = 1$. The $\vec{\alpha}_i = \vec{\mu}^i - \vec{\mu}^{i+1}$

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* There are indications that one could construct theories in this formalism that are of a wider variety than G/H theories.

are the simple roots. The new basis for the bosons defined by (3.B.17) is still orthonormal,

$$\langle 0|\rho_i(z)\rho_j(w)|0\rangle = -\delta^{ij} log(z-w) \ , \ \langle 0||\Phi(z)\Phi(w)|0\rangle = -log(z-w) \tag{3.B.19}$$

The $SU(2)$ currents now become,

$$J_N^3 = i\sqrt{\frac{N}{2}}\partial_z\Phi \ , \ J_N^{\pm} = \frac{1}{\sqrt{2}}e^{\pm i\sqrt{\frac{2}{N}}\Phi}\left[\sum_{i=1}^{N} e^{\pm 2i\vec{\mu}^i\cdot\vec{\rho}}\right] \tag{3.B.20}$$

whereas the stress-energy tensor is,

$$T_N(z) = -\frac{1}{2}(\partial_z\Phi)^2 + T_N^{par}(z) \tag{3.B.21a}$$

$$T_N^{par}(z) = -\frac{1}{N+2}\left[\partial_z\vec{\rho}\cdot\partial_z\vec{\rho} + \sum_{\vec{\alpha}} e^{2i\vec{\alpha}\cdot\vec{\rho}}\right] \tag{3.B.21b}$$

The sum is over all roots of $SU(N)$**. From (3.B.20) one can easily identify the Z_N parafermion operators***.

$$\psi_1(z) = \frac{1}{\sqrt{N}}\left[\sum_{i=1}^{N} e^{i\sqrt{2}\vec{\mu}^i\cdot\vec{\rho}}\right] \tag{3.B.22}$$

One then using the OPE among vertex operators can construct explicitly the whole parafermion algebra****,

$$\psi_k(z)\psi_{k'}(w) = C_{k,k'}(z-w)^{-\frac{2kk'}{N}}\left[\psi_{k+k'}(w) + O[(z-w)]\right] \tag{3.B.23}$$

with,

$$\psi_k(z) = \begin{pmatrix} N \\ k \end{pmatrix}^{-\frac{1}{2}} \sum_{i_1<i_2<\cdots<i_k}^{N} e^{i\sqrt{2}(\vec{\mu}^{i_1}+\cdots+\vec{\mu}^{i_N})\cdot\vec{\rho}} \tag{3.B.24}$$

and

$$C_{k,k'}^2 = \frac{(k+k')!(N-k)!(N-k')!}{(k!k'!(N-k-k')!N!} \tag{3.B.25}$$

The spin fields of the parafermionic theory are twist fields for the vertex operators $\psi_{\pm 1}$. The

** The reason for the appearence of the roots of $SU(N)$ is the fact that the $SU(2)_N$ parafermions can also be constructed as the coset space $SU(N)_1 \otimes SU(N)_1/SU(N)_2$.

*** T_N^{par} is the stress-energy tensor for the Z_N parafermion theory.

**** The indices k, k', $k+k'$ are always understood modulo N.

fact that σ_k twists $\psi_{\pm 1}$ by $\pm\frac{k}{N}$ completely determines the correlation function,

$$\frac{\langle 0|\sigma_k^\dagger(\infty)\psi_1(z)\psi_{-1}(w)\sigma_k(0)|0\rangle}{\langle 0|\sigma_k^\dagger(\infty)\sigma_k(0)|0\rangle} = \frac{z^{-1+\frac{k}{N}}w^{-\frac{k}{N}}}{(z-w)^{2-\frac{2}{N}}}\left[\left(1-\frac{k}{N}\right)z+\frac{k}{N}w\right] \qquad (3.B.26)$$

Using,

$$\psi_{-1}(z)\psi_1(w) = (z-w)^{-2+\frac{2}{N}}\left[\mathbf{1}+\frac{N+2}{N}\frac{T_N^{par}(w)}{(z-w)^2}+\cdots\right] \qquad (3.B.27)$$

we find the dimension of the spin field σ_k to be $\Delta_k = \frac{k(N-k)}{2N(N+2)}$. Unfortunately we do not have a complete description of these fields in the bosonic language but is conceivable that their correlation functions can be computed with techniques similar to those of ref. [46].

The construction above can be done for any G/H CFT by starting first from level one algebras. It is obvious that the bosonic coordinates are compactified on lattices that are direct products of root lattices of Lie algebras.