Structure of $N=2$ superconformally invariant unitary "minimal" theories: Operator algebra and correlation functions

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$N=2$ superconformal-invariant theories are studied and their general structure is analyzed. The geometry of $N=2$ complex superspace is developed as a tool to study the correlation functions of the theories above. The Ward identities of the global $N=2$ superconformal symmetry are solved, to restrict the form of correlation functions. Advantage is taken of the existence of the degenerate operators to derive the "fusion" rules for the unitary minimal systems with $\tilde{c}<1$. In particular, the closure of the operator algebra for such systems is shown. The $\tilde{c}=\frac{1}{3}$ minimal system is analyzed and its two-, three-, and four-point functions as well as its operator algebra are calculated explicitly.

I. INTRODUCTION

Recently there has been a lot of progress in understanding two-dimensional (2D) critical phenomena from the quantum-field-theory point of view.[1] A statistical system at the critical point is scale invariant and, in two dimensions, it has been proven that, under mild assumptions, it is invariant under the full conformal group which is infinite dimensional.[2] Belavin, Polyakov, and Zamolodchikov laid the foundation of 2D conformal field theory. Their techniques have been proven very powerful in analyzing the structure of conformal field theories, and in answering questions such as the following. What is the representation content of a given theory? What is its operator algebra? How can one calculate exactly correlation functions?

Conformal theory in two dimensions also attracted a lot of interest because it gives a nice natural framework in formulating (super)string theories, which are supposed to be candidates for a unified theory of nature.[3]

Conformal invariance puts severe constraints on the structure of a theory. By restricting the unitary representations of the conformal group, one can classify all possible anomalous dimensions that can appear in two-dimensional critical systems. In particular, for $c<1$ (c is the anomaly of the conformal algebra), there exists only a discrete list of values of c corresponding to theories which are unitary (i.e., theories that do not have negative-norm states in their Hilbert space[4]). The first few of these theories have been identified as well-known statistical systems, without continuous symmetries, the Ising model ($c=\frac{1}{2}$), the $\mathbf{Z}_3$ Potts model ($c=\frac{4}{5}$), the tricritical Ising model ($c=\frac{7}{10}$), and the tricritical $\mathbf{Z}_3$ Potts model ($c=\frac{6}{7}$). Knowledge of the structure of these theories allows one to understand better the universality classes of critical behavior.

This subset of conformal two-dimensional field theories contains some special representations (the so-called degenerate representations), the presence of which renders the theory exactly solvable, in the sense that all correlation functions satisfy linear differential equations, so that, in the worst case, it is up to a reasonable computer to evaluate them.

A natural extension of conformal invariance, inspired by attempts to construct fermionic string theories, has been its extension to include supersymmetry. Supersymmetry, so far, although having appealing theoretical advantages, has been elusive, despite a lot of both theoretical and experimental efforts to give some clues supporting its existence.

It was surprising that the first example of supersymmetry observed in nature came from a two-dimensional critical system, the tricritical Ising model,[5] realized experimentally by adsorbing helium on krypton-plated graphite.[6]

Superconformal invariance proves to be very fruitful also in the formulation of (super)string theories, where superconformal techniques are indeed valuable, making a lot of problems, which otherwise are difficult to tackle, almost elementary.[3,7,8]

A natural extension of $N=1$ superconformal invariance is to include an extended supersymmetry, thus giving rise to $N=2$ superconformal invariance. There has been considerable activity recently on this subject.[9–13] The Kač determinant has been calculated by several authors and it was used to study the irreducible unitary representations of the $N=2$ superconformal algebra.[10,12] The characters of the algebra have also been calculated.[13]

One might wonder about the utility of an extended superconformal algebra. There are various reasons explaining why there is such an interest.

First, the $N=2$ superconformal algebra is the gauge algebra of the U(1) string,[14] which, despite its phenomenological insignificance, is a good toy model to try techniques pertinent in string theories, both first and second quantized. From the point of view of superstring theories there is a need for $N=2$ superconformal invariance in the associated nonlinear σ model on a compact Ricci-flat manifold, to ensure $N=1$ supersymmetry in four dimensions.[15] We need supersymmetry to be exact after compactification and nonperturbatively stable. There is a serious hope that $N=2$ superconformal techniques will provide useful tools in disentangling various questions pertaining to the nature of different possible compactifications.

Another reason is that $N=2$ invariance might be relevant in some critical statistical systems. Such an example has been given in Ref. 16, where it was shown that the O(2) Gaussian model (XY model) is $N=2$ superconformal invariant for a specific value of the radius (see also Ref. 17). A specific radius corresponds to a specific point on the critical line in the phase diagram of the generalized Gaussian model.

In this paper we pursue our aim toward understanding the $N=2$ superconformal field theory. We develop some tools for doing complex analysis in $N=2$ superspace, and we use them to solve the Ward identities for the global $N=2$ symmetry, and find the general structure of the correlation functions. The two- and three-point functions in particular, are almost fixed. We derive the "fusion" rules for the degenerate representations of the $N=2$ algebra with $\tilde{c}<1$, and we prove the closure of the degenerate representation content of the unitary "minimal" theories. The simplest minimal system, with $\tilde{c}=\frac{1}{3}$, is analyzed in detail. We calculate the four-point function and we derive its operator algebra.

The structure of this paper is the following. In Sec. II we introduce $N=2$ supersymmetry and we develop the local analytic geometry of $N=2$ complex superspace. We characterize the global $N=2$ superconformal group $\mathrm{OSp}(2\,|\,2)$ and we give the explicit form of a general global supertransformation. In Sec. III we analyze the general structure of $N=2$ superconformal-invariant theories. Section IV is devoted to the analysis of the structure of the correlation functions, implied by global $N=2$ superconformal invariance, and in the derivation of "fusion" rules for the operator-product expansion of the minimal theories and in particular the unitary minimal models with $\tilde{c}<1$. In Sec. V we discuss the operator formalism in the Ramond sector and derive the corresponding fusion rules. In Sec. VI the simplest $N=2$ superconformal-invariant system with $\tilde{c}=\frac{1}{3}$ is analyzed. We compute its operator algebra and the four-point function. Section VII contains our conclusions. In Appendix A we give a justification of the "fusion" rules, through the construction of the unitary degenerate representations with $\tilde{c}<1$ using free fermions. In Appendix B we construct the primary fields of the $\tilde{c}=\frac{1}{3}$ theory as vertex operators in the Gaussian model, and we explicitly check some of the results of Secs. V and VI.

II. $N=2$ SUPERSYMMETRY AND THE ANALYTIC GEOMETRY OF $N=2$ COMPLEX SUPERSPACE

$N=2$ supersymmetry is a natural extension of $N=1$ supersymmetry. In this case we have two different supersymmetry generators (supercharges), as well as an O(2) [or U(1)] current which manifests the symmetry of the theory under an O(2) rotation of the two supersymmetries. The natural space to define the fields of the theory is $N=2$ superspace [or more precisely (2,0) superspace]. In a theory with (super)conformal invariance the left and right sectors of the theory completely decouple, so that the structure of the theory is that of a tensor product of the left and right sectors. From now on we will restrict ourselves to the left sector only, keeping in mind the previous remarks.

(2,0) superspace includes, apart from the complex analytic coordinate z, two other fermionic coordinates, θ and $\bar{\theta}$ corresponding to the two supersymmetries:

$$\theta^2=\bar{\theta}^2=\{\theta,\bar{\theta}\}=0 \; . \tag{2.1}$$

A point in superspace will be denoted by $\mathbf{z}\equiv(z,\theta,\bar{\theta})$.

A superfield is an analytic function in $\mathbf{z}$ defined through its power-series expansion in the fermionic coordinates:

$$\Phi(\mathbf{z})\equiv\phi(z)+\theta\bar{\psi}(z)+\bar{\theta}\psi(z)+\theta\bar{\theta}g(z) \; . \tag{2.2}$$

The two supersymmetry transformations can be written as

$$(z,\theta,\bar{\theta})\rightarrow(z-\epsilon\bar{\theta},\theta+\epsilon,\bar{\theta}) \; , \tag{2.3a}$$

$$(z,\theta,\bar{\theta})\rightarrow(z-\bar{\epsilon}\theta,\theta,\bar{\theta}+\bar{\epsilon}) \; , \tag{2.3b}$$

where $\epsilon,\bar{\epsilon}$ are anticommuting variables which are the parameters of the transformation. Under the two supersymmetry transformations [(2.3a),(2.3b)], a superfield transforms as

$$\Phi(z,\theta,\bar{\theta})\rightarrow\Phi(z-\epsilon\bar{\theta},\theta+\epsilon,\bar{\theta})=\phi(z-\epsilon\bar{\theta})+(\theta+\epsilon)\bar{\psi}(z-\epsilon\bar{\theta})+\bar{\theta}\psi(z-\epsilon\bar{\theta})+(\theta+\epsilon)\bar{\theta}g(z-\epsilon\bar{\theta})$$
$$=\phi(z)+\epsilon\bar{\psi}(z)+\bar{\theta}[\epsilon\partial_z\phi(z)-\epsilon g(z)+\psi(z)]+\theta\bar{\psi}(z)+\theta\bar{\theta}[g(z)+\epsilon\partial_z\bar{\psi}(z)] \; , \tag{2.4a}$$

$$\Phi(z,\theta,\bar{\theta})\rightarrow\Phi(z-\bar{\epsilon}\theta,\theta,\bar{\theta}+\epsilon)=\phi(z-\bar{\epsilon}\theta)+\theta\bar{\psi}(z-\bar{\epsilon}\theta)+(\bar{\theta}+\bar{\epsilon})\psi(z-\epsilon\theta)+\theta(\bar{\theta}+\bar{\epsilon})g(z-\bar{\epsilon}\theta)$$
$$=\phi(z)+\bar{\epsilon}\psi(z)+\theta[\bar{\psi}(z)+\bar{\epsilon}\partial_z\phi(z)+\bar{\epsilon}g(z)]+\bar{\theta}\psi(z)+\theta\bar{\theta}[g(z)-\bar{\epsilon}\partial_z\psi(z)] \tag{2.4b}$$

which implies the following transformation laws for the component fields:

$$
\begin{aligned}
&\delta_\epsilon\phi(z)=\epsilon\bar\psi(z), \quad \bar\delta_{\bar\epsilon}\phi(z)=\bar\epsilon\psi(z)\ ,\\
&\delta_\epsilon\psi(z)=\epsilon[\partial_z\phi(z)-g(z)], \quad \bar\delta_{\bar\epsilon}\psi(z)=0\ ,\\
&\delta_\epsilon\bar\psi(z)=0, \quad \bar\delta_{\bar\epsilon}\bar\psi(z)=\bar\epsilon[\partial_z\phi(z)+g(z)]\ ,\\
&\delta_\epsilon g(z)=\epsilon\partial_z\bar\psi(z), \quad \bar\delta_{\bar\epsilon}g(z)=-\bar\epsilon\partial_z\psi(z)\ .
\end{aligned}
\tag{2.5}
$$

It is easy to verify the global supersymmetry algebra:

$$[\delta_\epsilon,\bar\delta_{\bar\epsilon}]=2\epsilon\bar\epsilon\frac{\partial}{\partial z}, \quad [\delta_\epsilon,\delta_\epsilon]=[\bar\delta_{\bar\epsilon},\bar\delta_{\bar\epsilon}]=0\ . \tag{2.6}$$

The covariant derivatives in superspace are defined by

$$D\equiv\frac{\partial}{\partial\theta}+\bar\theta\frac{\partial}{\partial z}, \quad \bar D\equiv\frac{\partial}{\partial\bar\theta}+\theta\frac{\partial}{\partial z}\ , \tag{2.7a}$$

$$D^2=\bar D^2=0, \quad \{D,\bar D\}=2\frac{\partial}{\partial z}\ . \tag{2.7b}$$

We introduce here the notion of a chiral $N=2$ superfield, as a superfield satisfying one of the following conditions:

$$D\Phi(z)=0\Longrightarrow\Phi(z)=\phi(z)+2\bar\theta\psi(z)-\theta\bar\theta\partial_z\phi(z)\ , \tag{2.8a}$$

$$\bar D\bar\Phi(z)=0\Longrightarrow\Phi(z)=\bar\phi(z)+2\theta\bar\psi(z)+\theta\bar\theta\partial_z\bar\phi(z)\ . \tag{2.8b}$$

The Grassmann integration is defined through the usual standard rules:

$$\int d\theta\, d\bar\theta=\int d\theta\, d\bar\theta\,\theta=\int d\theta\, d\bar\theta\,\bar\theta=0, \quad \int d\theta\, d\bar\theta\,\bar\theta\theta=1\,. \tag{2.9}$$

If we call the generators of the two supersymmetries $G_{-1/2}$, $\bar G_{-1/2}$ then Eq. (2.6) is translated into

$$
\begin{aligned}
&\{G_{-1/2},G_{-1/2}\}=\{\bar G_{-1/2},\bar G_{-1/2}\}=0\ ,\\
&\{G_{-1/2},\bar G_{-1/2}\}=2L_{-1}\ ,
\end{aligned}
\tag{2.10}
$$

L_{-1} being the usual translation operator on the complex plane. The full superconformal symmetry is generated by the usual Virasoro generators L_n, the supersymmetry generators

$$
\begin{aligned}
&G_r\equiv\frac{2}{n+1}[L_n,G_{-1/2}]\ ,\\
&\bar G_r\equiv\frac{2}{n+1}[L_n,\bar G_{-1/2}], \quad r=n-\tfrac{1}{2}\ ,
\end{aligned}
\tag{2.11}
$$

and the U(1) current generators J_n, which implement the U(1) symmetry, under which the two supercurrents are in complex-conjugate representations. The full $N=2$ superconformal algebra then takes the form

$$
\begin{aligned}
&[L_m,L_n]=(m-n)L_{m+n}+\frac{\tilde c}{4}(m^3-m)\delta_{m+n,0}\ ,\\
&[L_m,G_r]=\left(\frac{m}{2}-r\right)G_{m+r},[L_m,\bar G_r]=\left(\frac{m}{2}-r\right)\bar G_{m+r}\ ,\\
&[J_m,J_n]=\tilde c m\delta_{m+n,0}, \quad [J_m,G_r]=G_{m+r},[J_m,\bar G_r]=\bar G_{m+r}\ ,\\
&\{G_r,G_s\}=\{\bar G_r,\bar G_s\}=0, \quad [L_m,J_n]=-nJ_{m+n}\ ,\\
&\{G_r,\bar G_s\}=2L_{r+s}+(r-s)J_{r+s}+\tilde c(r^2-\tfrac{1}{4})\delta_{r+s,0}\ .
\end{aligned}
\tag{2.12}
$$

It is the generating algebra of $N=2$ superanalytic transformations in $N=2$ superspace. We should, at this point, define what we mean by an extended superanalytic transformation. The most general coordinate transformation in $N=2$ superspace has the form ($f_0,f_2,g_1,\bar g_1,h_1,\bar h_1$ are commuting functions, whereas $f_1,\bar f_1,g_0,g_2,h_0,h_2$ are anticommuting ones)

$$
\begin{aligned}
&z'=f_0(z)+\theta f_1(z)+\bar\theta\bar f_1(z)+\theta\bar\theta f_2(z)\ ,\\
&\theta'=g_0(z)+\theta g_1(z)+\bar\theta\bar g_1(z)+\theta\bar\theta g_2(z)\ ,\\
&\bar\theta'=h_0(z)+\theta h_1(z)+\bar\theta\bar h_1(z)+\theta\bar\theta h_2(z)\ .
\end{aligned}
\tag{2.13}
$$

A natural definition for an extended superanalytic transformation is one under which the covariant derivatives transform homogeneously. Under (2.13) the covariant derivatives transform as

$$D=(D\theta')D'+(D\bar\theta')\frac{\partial}{\partial\bar\theta'}+[Dz'-(D\theta')\bar\theta']\frac{\partial}{\partial z'}\ , \tag{2.14a}$$

$$\bar D=(\bar D\bar\theta')\bar D'+(\bar D\theta')\frac{\partial}{\partial\theta'}+[\bar Dz'-(\bar D\bar\theta')\theta']\frac{\partial}{\partial z'}\ . \tag{2.14b}$$

Consequently the conditions for (2.13) to be a superanalytic transformation are

$$\bar D\theta'=D\bar\theta'=Dz'-(D\theta')\bar\theta'=\bar Dz'-(\bar D\bar\theta')\theta'=0\ . \tag{2.15}$$

[In fact, even if we demand that D transforms in general as $D=(D\theta')D+(D\bar\theta')\bar D'$ we end up at (2.15). There is a dual requirement, $D=(D\theta')\bar D'$ which gives conditions conjugate to (2.15).] Solving (2.15) we arrive at the most general form of an extended superanalytic transformation,

$$
\begin{aligned}
&z'=f_0(z)+\theta g_1(z)h_0(z)+\bar\theta\bar h_1(z)g_0(z)\\
&\qquad+\theta\bar\theta[g_0(z)h_0(z)]'\ ,\\
&\theta'=g_0(z)+\theta g_1(z)+\theta\bar\theta g_0'(z)\ ,\\
&\bar\theta'=h_0(z)+\bar\theta h_1(z)-\theta\bar\theta h_0'(z)\ ,
\end{aligned}
\tag{2.16}
$$

along with the supplementary condition

$$f_0'(z)=g_0'(z)h_0(z)-g_0(z)h_0'(z)+g_1(z)\bar h_1(z)\ , \tag{2.17}$$

where in (2.17) and on the left-hand side of (2.16) a prime means differentiation with respect to z.

In particular, the global supersymmetry transformations are special cases of (2.16) with $f_0(z)=z$, $g_0(z)=\epsilon$, $h_0(z)=0$, $g_1(z)=\bar h_1(z)=1$, and $f_0(z)=z$, $g_0(z)=0$, $h_0(z)=\bar\epsilon$, $g_1(z)=\bar h_1(z)=1$, respectively.

We define the two Abelian $N=2$ superdifferentials through their transformation properties under analytic superconformal transformations:

$$dz'\equiv(D\theta')dz, \quad d\bar z'\equiv(\bar D\bar\theta')d\bar z\ . \tag{2.18}$$

(The bar over $d\bar z$ should not be confused with the antiholomorphic coordinate $\bar z$; it denotes an independent Abelian differential.)

The superconformal tensor fields are defined by the condition that

$$\Phi(\mathbf{z})(d\mathbf{z})^{\Delta+Q/2}(d\bar{\mathbf{z}})^{\Delta-Q/2}$$

is an $N=2$ superconformal-invariant quantity, where Δ,Q are the dimensions and charge of the lowest component field. They are the primary superfields generating the highest weight irreducible representations of the $N=2$ superconformal algebra. Globally defined tensor superfields must have dimensions and charges which are integers or half-integers. They can be constructed as composite operations from locally defined fields. The Cauchy integral formulas can be extended in superspace.[18] The $N=2$ superanalytic transformations are generated by the energy-momentum superfield, which in component form can be written as

$$\mathbf{J}(z)\equiv J(z)+i\theta\bar{G}(z)+i\bar{\theta}G(z)+2\theta\bar{\theta}T(z) . \tag{2.19}$$

The Fourier modes of the generators are defined in the usual way

$$J(z)\equiv\sum_{n\in Z}\frac{J_n}{z^{n+1}},\quad T(z)\equiv\sum_{n\in Z}\frac{L_n}{z^{n+2}} ,$$
$$G(z)\equiv\sum_{n\in Z}\frac{G_{n-1/2}}{z^{n+1}},\quad \bar{G}(z)\equiv\sum_{n\in Z}\frac{\bar{G}_{n-1/2}}{z^{n+1}} . \tag{2.20}$$

These generators are represented in the space of superfield functions in the following way:

$$L_n\equiv -z^{n+1}\frac{\partial}{\partial z}-\frac{n+1}{2}z^n\left[\theta\frac{\partial}{\partial\theta}+\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right] ,$$
$$J_n\equiv z^n\left[\bar{\theta}\frac{\partial}{\partial\bar{\theta}}-\theta\frac{\partial}{\partial\theta}\right] ,$$
$$G_{n-1/2}\equiv z^n\left[\frac{\partial}{\partial\theta}-\bar{\theta}\frac{\partial}{\partial z}\right]+nz^{n-1}\theta\bar{\theta}\frac{\partial}{\partial\theta} ,$$
$$\bar{G}_{n-1/2}\equiv z^n\left[\frac{\partial}{\partial\bar{\theta}}-\theta\frac{\partial}{\partial z}\right]-nz^{n-1}\theta\bar{\theta}\frac{\partial}{\partial\bar{\theta}} . \tag{2.21}$$

It is straightforward to check that the generators in Eq. (2.21) satisfy the $N=2$ superconformal loop algebra [as in (2.14) with $\tilde{c}=0$], which is the algebra of $N=2$ superconformal transformations over S^1. The explicit representation (2.21) will be useful later on in this paper, to analyze the correlation functions of $N=2$ superconformal-invariant theories.

The energy-momentum tensor has an operator-product expansion with itself:

$$\mathbf{J}(\mathbf{z}_1)\mathbf{J}(\mathbf{z}_2)=\frac{\theta_{12}}{\mathbf{z}_{12}}D\mathbf{J}(\mathbf{z}_2)-\frac{\bar{\theta}_{12}}{\mathbf{z}_{12}}\bar{D}\mathbf{J}(\mathbf{z}_2)+2\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}^2}\mathbf{J}(\mathbf{z}_2)+2\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\mathbf{J}'(\mathbf{z}_2)+\frac{\tilde{c}}{\mathbf{z}_{12}^2} , \tag{2.22}$$

where the anomaly $\tilde{c}$ is normalized, so that a free scalar $N=2$ superfield has $\tilde{c}=1$. Equation (2.22) corresponds to a change of the energy-momentum tensor under a superconformal transformation

$$\delta_v\mathbf{J}(\mathbf{z})=[\partial_z v]\mathbf{J}(\mathbf{z})+v\partial_z\mathbf{J}(\mathbf{z})+\tfrac{1}{2}[\bar{D}v]D\mathbf{J}(\mathbf{z})+\tfrac{1}{2}[Dv]\bar{D}\mathbf{J}(\mathbf{z})+\frac{\tilde{c}}{4}\partial_z[\bar{D},D]v , \tag{2.23}$$

v being an infinitesimal $N=2$ superfield.

The change in the energy-momentum tensor under a finite superconformal transformation is given by

$$\mathbf{J}(\mathbf{z})=\mathbf{J}'(\mathbf{z}')[D\theta'][\bar{D}\bar{\theta}']+\frac{\tilde{c}}{2}S(\mathbf{z},\mathbf{z}') , \tag{2.24}$$

where the $N=2$ super-Schwarzian derivative is defined through

$$S(\mathbf{z},\mathbf{z}')\equiv\frac{\partial\bar{D}\bar{\theta}'}{\bar{D}\bar{\theta}'}-\frac{\partial D\theta'}{D\theta'}-2\frac{\partial\bar{\theta}'\partial\theta'}{(\bar{D}\bar{\theta}')(D\theta')} . \tag{2.25}$$

It satisfies the composition law

$$S(\mathbf{z}_1,\mathbf{z}_3)=S(\mathbf{z}_1,\mathbf{z}_2)+(D\theta_2)(\bar{D}\bar{\theta}_2)S(\mathbf{z}_2,\mathbf{z}_3) . \tag{2.26}$$

On the sphere for a vector field to be globally defined, it must have a vanishing "anomaly;" that is, under an infinitesimal transformation generated by it, the anomalous part in (2.23) must vanish: $\partial_z[\bar{D},D]v=0$, which gives an eight-parameter family of globally defined vector fields on the sphere

$$v(\mathbf{z})=v_{-1}+v_0z+v_1z^2+\theta(u_{-1/2}+u_{1/2}z)+\bar{\theta}(\bar{u}_{-1/2}+\bar{u}_{1/2}z)+q_0\theta\bar{\theta} . \tag{2.27}$$

These vector fields generate the global $N=2$ superconformal algebra, osp(2 | 2). [In fact they generate half of osp(2 | 2), its holomorphic part.] The global $N=2$ superconformal algebra is the maximal, finite-dimensional, subalgebra of the $N=2$ superconformal algebra. It contains the generators of the ordinary projective transformations, L_1,L_0,L_{-1}, the supercharges $G_{\pm1/2},\bar{G}_{\pm1/2}$, and the zero mode of the U(1) current. It is easy to check using (2.14) that this set of generators closes into itself, and it contains as a subalgebra, the $N=1$ superconformal algebra osp(2 | 1). Since the Schwarzian derivative transforms as in (2.26), the fact that it vanishes for infinitesimal global $N=2$ transformations continues to be true for finite transformations belonging to the identity component of the group.

The OSp(2 | 2) group transformations can be found either by exponentiating the generators of the algebra given in (2.21) or using the general form of superanalytic transformations (2.18), and some analyticity arguments.[19] Another way is to solve the equation $S(\mathbf{z},\mathbf{z}')=0$. There are three parameters associated with the subgroup SL(2,C), four supersymmetry parameters (Grassmann), ϵ_1, ϵ_2, $\bar{\epsilon}_1$, $\bar{\epsilon}_2$, and a parameter q associated with the zero mode of the U(1) current. The group transformations are

$$z'=\frac{az+b}{cz+d}+e^{q}\theta\frac{(1-\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\bar{\epsilon}_1 z+\bar{\epsilon}_2(1+\frac{1}{2}\epsilon_2\bar{\epsilon}_1)}{(cz+d)^2}+e^{-q}\bar{\theta}\frac{(1+\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\epsilon_1 z+(1-\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\epsilon_2}{(cz+d)^2}$$
$$+\frac{[2d\epsilon_1\bar{\epsilon}_1-2c(\bar{\epsilon}_1\epsilon_2+\bar{\epsilon}_2\epsilon_1)]z+d(\bar{\epsilon}_1\epsilon_2+\bar{\epsilon}_2\epsilon_1)-2c\bar{\epsilon}_2\epsilon_2}{(cz+d)^3}\ , \tag{2.28a}$$

$$\theta'=\frac{\epsilon_1 z+\epsilon_2}{cz+d}+e^{q}\theta\frac{1+\frac{1}{2}(\epsilon_2\bar{\epsilon}_1-\epsilon_1\bar{\epsilon}_2)+\frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2}{cz+d}+\theta\theta\frac{-\epsilon_1 d-\epsilon_2 c}{(cz+d)^2}\ , \tag{2.28b}$$

$$\bar{\theta}'=\frac{\bar{\epsilon}_1 z+\bar{\epsilon}}{cz+d}+e^{-q}\bar{\theta}\frac{1+\frac{1}{2}(\epsilon_2\bar{\epsilon}_1-\epsilon_1\bar{\epsilon}_2)+\frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2}{cz+d}+\theta\bar{\theta}\frac{\bar{\epsilon}_2 c-\bar{\epsilon}_1 d}{(cz+d)^2}\ . \tag{2.28c}$$

The $N=2$ superconformal vector field generates the group of $N=2$ superdiffeomorphisms on the circle, $\widehat{\mathrm{Diff}}(S^1)$. The Schwarzian derivative is the globally invariant generator of the second cohomology group of $\widehat{\mathrm{Diff}}(S^1)$. It generates a nontrivial transformation on the energy-momentum tensor viewed as a connection on moduli space.

As can be seen from (2.14), the subalgebra does not have an anomaly even if $\tilde{c}\neq 0$. This is of crucial importance in a superconformal theory as we will see later. It implies that all correlation functions are invariant under $\mathrm{OSp}(2\,|\,2)$, constraining, in such a way, their form. Along with some supplementary constraints on the correlation functions, present when the theory has degenerate representations, it helps to determine the correlation functions completely, rendering the theory exactly solvable.

III. THE GENERAL STRUCTURE OF $N=2$ SUPERCONFORMAL THEORIES

An $N=2$ superconformal field theory is a field theory invariant under the $N=2$ superanalytic transformations described in the previous section, which form the $N=2$ superconformal group. The infinitesimal transformations are generated by an infinitesimal local superfield $v(\mathbf{z})$:

$$v(\mathbf{z})\equiv v_0(z)+\theta v_1(z)+\bar{\theta}\bar{v}_1(z)+\theta\bar{\theta}v_2(z)\ , \tag{3.1}$$

$$z'=z+v(\mathbf{z})+\tfrac{1}{2}[(\bar{D}v)\theta+(Dv)\bar{\theta}]\ ,$$
$$\theta'=\theta+\tfrac{1}{2}\bar{D}v,\quad \bar{\theta}'=\bar{\theta}+\tfrac{1}{2}Dv\ . \tag{3.2}$$

The function $v_1,\bar{v}_1$ are Grassmann functions anticommuting among themselves and with $\theta,\bar{\theta}$, whereas v_0,v_2 are the usual meromorphic functions. The superconformal transformations are generated by the superenergy-momentum tensor, see (2.19). Using the Cauchy formulas of the previous section we can write the change of a local superfield under a superconformal transformation as

$$\delta_v\Phi(\mathbf{z})=-\frac{1}{4\pi i}\oint_{C_z}d\mathbf{z}'\,v(\mathbf{z}')\mathbf{J}(\mathbf{z}')\Phi(\mathbf{z})\ , \tag{3.3}$$

where the contour C_z surrounds the point z in the complex plane.

The variation (3.3) is determined by the singularities of the operator-product expansion (OPE), of the energy-momentum tensor with the superfield. In particular a superfield function transforms under an infinitesimal transformation as

$$\delta_v\Phi=v\partial\Phi+\tfrac{1}{2}Dv\,\bar{D}\Phi+\tfrac{1}{2}\bar{D}v\,D\Phi\ . \tag{3.4}$$

It is usually convenient to use radial quantizations going (through a superanalytic transformation), from the cylinder to the plane: $(\ln z, z^{-1/2}\theta, z^{-1/2}\bar{\theta})\leftrightarrow(\tau+i\sigma,\theta,\bar{\theta})$.

The fermionic fields on the cylinder can have two possible boundary conditions: periodic or antiperiodic. On the plane, this is translated to G, $\bar{G}(ze^{2\pi i})=\pm G$, $\bar{G}(z)$, the corresponding subspaces of the full Hilbert spaces being the Neveu-Schwarz (NS) and Ramond (R) sectors. In the NS sector $G(ze^{2\pi i})=G(z)$ whereas in the R sector $G(ze^{2\pi i})=-G(z)$; that is, the fermionic fields are double valued on the plane.

The operator-product expansion for the energy-momentum tensor was given in (2.22). The germs that appear in (2.22) are the most general terms that are allowed in a Euclidean $N=2$ supersymmetric quantum field theory, satisfying the standard constructive-field-theory axioms. The proof of Ref. 2 can be extended easily in our case, to guarantee (2.22) provided the theory has scale invariance and global $N=2$ supersymmetry. Using the mode expansions (2.20) we can derive (2.14) from (2.22). The energy-momentum tensor must be a Hermitian operator, implying some Hermiticity conditions among its components:

$$L_n^\dagger=L_{-n},\quad J_n^\dagger=J_{-n},\quad G_r^\dagger=\bar{G}_{-\eta},\quad \bar{G}_r^\dagger=G_{-r}\ . \tag{3.5}$$

We define the in vacuum $|0\rangle$ of the theory at time $\tau=-\infty$ ($z=0$), to be $\mathrm{OSp}(2\,|\,2)$ invariant. This means that it is annihilated by L_n, $n\geq -1$, J_n, $n\geq 0$, $G_r,\bar{G}_r$, $r\geq -\frac{1}{2}$ (NS sector) or $G_n,\bar{G}_n$, $n\geq 0$ in the R sector. In the same way the out vacuum is defined at $z\to\infty$. The vacuum state belongs to the NS sector and it is the ground state of the theory. The unitary irreducible representations of the $N=2$ superconformal algebra are generated from highest-weight vectors (HWV's), by the action of the lowering operators of the algebra, $L_n,J_n,G_r,\bar{G}_r,n,r<0$.

In the NS sector the HWV's are generated by the action of primary conformal superfields on the vacuum state. Their defining relations are their transformation properties under superconformal transformations encoded in their OPE with the energy-momentum tensor:

$$\mathbf{J}(\mathbf{z}_1)\Phi(\mathbf{z}_2)=2\Delta\frac{\theta_{12}\bar\theta_{12}}{\mathbf{z}_{12}^2}\Phi(\mathbf{z}_2)+2\frac{\theta_{12}\bar\theta_{12}}{\mathbf{z}_{12}}\Phi'(\mathbf{z}_2)+\frac{\theta_{12}}{\mathbf{z}_{12}}D\Phi(\mathbf{z}_2)-\frac{\bar\theta_{12}}{\mathbf{z}_{12}}\bar D\Phi(\mathbf{z}_2)+\frac{Q\Phi(\mathbf{z}_2)}{\mathbf{z}_{12}} . \quad (3.6)$$

Using (3.3) and (3.6) we can derive the transformation law for a primary superfield operator:

$$\delta_v\Phi(\mathbf{z})=\Delta(\partial_z v)\Phi(\mathbf{z})+v\partial_z\Phi(\mathbf{z})+\tfrac{1}{2}[\bar D v]D\Phi(\mathbf{z})+\tfrac{1}{2}[Dv]\bar D\Phi(\mathbf{z})-\frac{Q}{4}\{[D,\bar D]v\}\Phi(\mathbf{z}) . \quad (3.7)$$

Under a finite transformation $\Phi(\mathbf{z})$ transforms as

$$\Phi(\mathbf{z})=\Phi(\mathbf{z}')[D\theta']^{\Delta+Q/2}[\bar D\bar\theta']^{\Delta-Q/2} , \quad (3.8)$$

where (Δ,Q) are its dimension and U(1) charge. The HWV in the NS sector are characterized by their eigenvalues under the zero modes of the algebra:

$$L_0\,|\,\Phi\rangle=\Delta\,|\,\Phi\rangle,\quad J_0\,|\,\Phi\rangle=Q\,|\,\Phi\rangle . \quad (3.9)$$

Being HW states they must be annihilated by the raising operators of the algebra:

$$L_n\,|\,\Phi\rangle=J_n\,|\,\Phi\rangle=G_n\,|\,\Phi\rangle=\bar G_n\,|\,\Phi\rangle,\quad n>0 . \quad (3.10)$$

The OPE (3.6) can be written also as commutation relations which will be useful later on:

$$\begin{aligned}
[L_n,\Phi(\mathbf{z})]&=z^{n+1}\frac{\partial}{\partial z}\Phi(z)+(n+1)z^n\left[\Delta+\frac{1}{2}\left[\theta\frac{\partial}{\partial\theta}+\bar\theta\frac{\partial}{\partial\bar\theta}\right]\right]\Phi(\mathbf{z})+\frac{Q}{2}n(n+1)z^{n-1}\theta\bar\theta\Phi(\mathbf{z}) ,\\
[J_n,\Phi(\mathbf{z})]&=z^n\left[Q+\bar\theta\frac{\partial}{\partial\bar\theta}-\theta\frac{\partial}{\partial\theta}\right]\Phi(\mathbf{z})+2n\Delta z^{n-1}\theta\bar\theta\Phi(\mathbf{z}) ,\\
[G_r,\Phi(\mathbf{z})]&=z^{r+1/2}\left[\frac{\partial}{\partial\bar\theta}-\theta\frac{\partial}{\partial z}\right]\Phi(\mathbf{z})-(r+\tfrac{1}{2})z^{r-1/2}\left[(2\Delta+Q)\theta+\theta\bar\theta\frac{\partial}{\partial\bar\theta}\right]\Phi(\mathbf{z}) ,\\
[\bar G_r,\Phi(\mathbf{z})]&=z^{r+1/2}\left[\frac{\partial}{\partial\theta}-\bar\theta\frac{\partial}{\partial z}\right]\Phi(\mathbf{z})-(r+\tfrac{1}{2})z^{r-1/2}\left[(2\Delta-Q)\bar\theta-\theta\bar\theta\frac{\partial}{\partial\theta}\right]\Phi(\mathbf{z}) .
\end{aligned}\quad (3.11)$$

In the R sector the zero modes are L_0, J_0, and $\bar G_0, G_0$, their eigenvalues characterizing HWV's. There are two kinds of HWV's, $|\,\Delta,Q\mp\frac{1}{2}\rangle_\pm$,[10]

$$\begin{aligned}
L_0\,|\,\Delta,Q\mp\tfrac{1}{2}\rangle_\pm&=\Delta\,|\,\Delta,Q\mp\tfrac{1}{2}\rangle_\pm ,\\
J_0\,|\,\Delta,Q\mp\tfrac{1}{2}\rangle_\pm&=(Q\mp\tfrac{1}{2})\,|\,\Delta,Q\mp\tfrac{1}{2}\rangle_\pm ,
\end{aligned}\quad (3.12)$$

which satisfy an additional HWV condition with respect to the supercharges:

$$G_0\,|\,\Delta,Q+\tfrac{1}{2}\rangle_-=0,\quad \bar G_0\,|\,\Delta,Q-\tfrac{1}{2}\rangle_+=0 . \quad (3.13)$$

Consequently there are two kinds of representations: $R^\pm$. The two representations are isomorphic under charge conjugation ($G_n\leftrightarrow\bar G_n$, $J_n\to -J_n$).

From now on we will restrict ourselves to one of them, say R^+, our statements being valid for R^- as well.

In the R sector, the ground state is not unique. There are two ground states degenerate in energy (i.e., having the same dimension): $|\,\Theta^+\rangle$ and $G_0\,|\,\Theta^+\rangle\equiv|\,\Theta^-\rangle$. They are generated from the vacuum $|\,0\rangle$ (which belongs to the NS sector), by primary fields $\Theta^\pm(\mathbf{z})$, much like the spin fields of the $N=1$ superconformal theories. The spin fields have double-valued OPE with the energy-momentum tensor: for example,

$$G(z)\Theta^\pm(\omega)=\tfrac{1}{2}\alpha_\pm\frac{\Theta^\mp(\omega)}{(z-\omega)^{3/2}} , \quad (3.14)$$

where $\alpha_+=1$, $\alpha_-=\Delta-\tilde c/8$. This happens in order for the spin field to be able to change the boundary conditions of the fermionic parts of the superfields. We can view the spin fields as opening and closing cuts on the cylinder. The states in the R sector are generated by ordinary conformal superfields acting on the Ramond ground states. The generators of global $N=2$ supersymmetry transformations in the R sector are $G_0,\bar G_0$.

Unbroken $N=2$ supersymmetry is implied by the existence of a ground state which is annihilated by the global $N=2$ supersymmetry generators. The state $|\,\Theta^+\rangle$ is annihilated by $\bar G_0$ because of Eq. (3.13). Applying $\{G_0,\bar G_0\}$ to it we obtain

$$\begin{aligned}
\{G_0,\bar G_0\}\,|\,\Theta^+\rangle&=\bar G_0 G_0\,|\,\Theta^+\rangle=(2L_0-\tilde c/4)\,|\,\Theta^+\rangle\\
&=(2\Delta-\tilde c/4)\,|\,\Theta^+\rangle .
\end{aligned}\quad (3.15)$$

Consequently, in order for G_0 to annihilate $|\,\Theta^+\rangle$, its dimension must be $\Delta_+=\tilde c/8$. The operator $\{G_0,\bar G_0\}$ is a Hermitian positive operator; thus, any dimension in the R sector has to be $\geq\tilde c/8$. This is the reason that the vacuum $|\,0\rangle$, the lowest-energy state must belong to the NS sector. In the same way $\bar G_0\,|\,\Theta^-\rangle=0$ implies $\Delta_-=\tilde c/8$. Therefore, the existence of a state in the R sector with $\Delta=\tilde c/8$ implies unbroken $N=2$ supersymmetry on the cylinder. On the other hand, if such a state does not exist in the theory, one supersymmetry out of the two is broken.

So far we have been discussing the two sectors of the

$N=2$ superconformal theory that parallel the situation in ordinary $N=1$ superconformal theories. In the $N=2$ case though, unlike the $N=1$, there is another sector present in general due to the fact that $N=2$ superfields contain two fermionic components, so there is also the possibility of choosing periodic boundary conditions for one of them, and antiperiodic for the other one. This can be seen easier if we write the algebra (2.14) in an O(2) basis:

$$G_r^1 \equiv \frac{G_r+\bar{G}_r}{\sqrt{2}}, \quad G_r^2=\frac{G_r-\bar{G}_r}{i\sqrt{2}} . \tag{3.16}$$

In this basis the algebra (2.14) becomes

$$\begin{aligned}
&[L_m,L_n]=(m-n)L_{m+n}+\frac{\tilde{c}}{4}(m^3-m)\delta_{m+n,0} , \\
&[L_m,G_r^i]=\left[\frac{m}{2}-r\right]G^i_{m+r}, \quad [L_m,J_n]=-nJ_{m+n} , \\
&[J_m,J_n]=\tilde{c}m\,\delta_{m+n,0}, \quad [J_m,G_r^i]=i\epsilon^{ij}G^j_{m+r} , \\
&\{G_r^i,G_s^j\}=2\delta^{ij}L_{r+s}+i\epsilon^{ij}(r-s)J_{r+s}+\tilde{c}(r^2-\tfrac{1}{4})\delta^{ij}\delta_{r+s,0} .
\end{aligned} \tag{3.17}$$

The twisted (T) $N=2$ algebra is defined by choosing integer modes for G_m^1, L_m, and half-integer modes for G_r^2,J_r choices, compatible with the commutation relations (3.17). In the O(2) basis the energy-momentum tensor becomes

$$\mathbf{J}(\mathbf{z})\equiv J(z)+\epsilon^{ij}\theta^i G^j(z)+\epsilon^{ij}\theta^i\theta^j T(z) , \tag{3.18}$$

where θ^i is an O(2) doublet of Grassmann coordinates. A twisted superfield

$$\Phi(\mathbf{z})\equiv\phi(z)+\epsilon^{ij}\theta^i\psi^j(z)(z)+\tfrac{1}{2}\epsilon^{ij}\theta^i\theta^j g(z) \tag{3.19}$$

has antiperiodic boundary conditions for $\phi(z)$ and $\psi^2(z)$ and periodic boundary conditions for $g(z)$ and $\psi^1(z)$, on the cylinder, that is ϕ and ψ^1 are $\mathbf{Z}_2$ twisted. Again here, G_0^1 is a Hermitian operator. Its square, acting on a primary state must give positive eigenvalues, which implies that all of the dimensions in the T sector satisfy $\Delta\geq\tilde{c}/8$. In particular, it implies that if there is a state with $\Delta=\tilde{c}/8$ this is then the ground state, and it is doubly degenerate since this state $|H^+\rangle$ and $|H^-\rangle \equiv G_0^1|H^+\rangle$, have the same energies. One of the two supersymmetries, namely, the one generated by G_0^1, is then unbroken, since G_0^1 annihilates the ground states:

$$\begin{aligned}
(G_0^1)^2|H^+\rangle&=\tfrac{1}{2}\{G_0^1,G_0^1\}|H^+\rangle \\
&=(L_0-\tilde{c}/8)H^+\rangle=0 , \\
G_0^1|H^-\rangle&=(G_0^1)^2|H^+\rangle=0 .
\end{aligned} \tag{3.20}$$

The global supersymmetry generated by $G_{-1/2}{}^2$ is broken since $G_{-1/2}{}^2$ fails to annihilate the ground states. This is obvious since in order for $G_{1/2}{}^2$ to annihilate a primary state, its dimension has to be zero, and as we argued above, stages with zero dimension do not exist in the T sector. Thus in the T sector we have at most a remnant $N=1$ supersymmetry. The ground states are generated from the NS vacuum by the "twist" fields $H^{\pm}(\mathbf{z})$, the presence of which induces cuts on the complex plane such that $\phi(z)$ and $\psi^1(z)$ are double valued around the point where the twist field lies. In the T sector there is a parity operator $(-1)^F$ which commutes with L_m,J_m and anticommutes with G_m^i. In particular,

$$(-1)^F|H^+\rangle=|H^+\rangle, \quad (-1)^F|H^-\rangle=-|H^-\rangle . \tag{3.21}$$

In the R sector the two-spin fields are nonlocal with respect to each other. Their operator-product expansion contains square-root singularities in the complex plane which induce nonlocality when we project to Euclidean space. The same is true in the T sector. In order to obtain a local theory we must suitably project out one-fermion parity, the same way as in the $N=1$ case.

Unitarity, as in $N=0,1$ conformal theories, puts severe constraints in the representation content of an $N=2$ superconformal theory. A basic tool for studying unitarity is the Kač determinant. It has been derived in Refs. 10–12. We will include the main results, since they will be useful later on, in this paper.

In the NS sector HWV's are labeled by their dimension h and charge q. Any secondary state is then characterized by its level (eigenvalue of L_0-h) and relative charge (eigenvalue of J_0-q). The Kač determinant at level n and relative charge m is given by

$$\det M^{\rm NS}_{n,m}(\tilde{c},h,q)=\prod_{\substack{1\leq rs\leq 2n\\ s,\text{even}}}[f^{\rm NS}_{r,s}]^{P_{\rm NS}(n-rs/2,m)}\prod_{k\in\mathbf{Z}+1/2}[q_k^{\rm NS}]^{\tilde{P}_{\rm NS}(n-|k|,m-\text{sgn}(k);k)} , \tag{3.22}$$

$$F^{\rm NS}_{r,s}=2(\tilde{c}-1)h-q^2-\frac{(\tilde{c}-1)^2}{4}+\frac{[(\tilde{c}-1)r+s]^2}{4}, \quad r\in\mathbf{Z}^+, \quad s\in 2\mathbf{Z}^+ , \tag{3.23a}$$

$$g_k^{\rm NS}=2h-2kq+(\tilde{c}-1)(k^2-\tfrac{1}{4}), k\in\mathbf{Z}+\tfrac{1}{2} , \tag{3.23b}$$

while the NS partition functions are defined by

$$\sum_{n,m}P_{\rm NS}(n,m)z^n w^m =\prod_{k=1}^{\infty}\frac{(1+z^{k-1/2}w)(1+z^{k-1/2}w^{-1})}{(1-z^k)^2} , \tag{3.24a}$$

$$\sum_{n,m}\tilde{P}_{\rm NS}(n,m;k)z^n w^m =(1+z^{|k|}w^{\text{sgn}(k)})^{-1}\sum_{n,m}P_{\rm NS}(n,m)z^n w^m , \tag{3.24b}$$

where z,w are formal complex variables. Equation (3.22) implies that whenever there is a vanishing of $f_{r,s}^{\rm NS}$, there exists a HWV at level $rs/2$ and relative charge zero. When $g_k^{\rm NS}=0$, there is a HWV at level $|k|$ and relative charges sgn(k). For $\tilde{c}<1$ unitary representations exist only for the discrete series of values for $\tilde{c}$:

$$\tilde{c}=1-\frac{2}{m},\quad m=2,3,4,\ldots\ . \tag{3.25}$$

Their dimension and charges are given by (we employ the notation of Ref. 10)

$$h=\frac{4jk-1}{4m},\quad q=\frac{j-k}{m},\quad j,k\in\mathbf{Z}+\tfrac{1}{2}\ ,$$
$$0<j,k,j+k\le m-1\ . \tag{3.26}$$

Representations belonging to this class ($\rm NS_0$) are degenerate. For $\tilde{c}\ge 1$ there are two other classes of unitary representations.

(a) $\rm NS_2$. These representations are characterized by the condition

$$g_n^{\rm NS}=0,\quad g_{n+{\rm sgn}(n)}^{\rm NS}<0\ ,$$
$$f_{1,2}^{\rm NS}>\rho\quad \text{for some } n\in\mathbf{Z}+\tfrac{1}{2}\ . \tag{3.27}$$

They are also degenerate.

(b) $\rm NS_3$. The representation in this class are characterized by $g_n^{\rm NS}\ge 0$, $\forall n\in\mathbf{Z}+\frac{1}{2}$. They contain degenerate representations when $g_n^{\rm NS}=0$ for some $n\in\mathbf{Z}+\frac{1}{2}$.

In the $R^{\pm}$ sector HWV's are characterized by their dimension h and charge $q\pm\frac{1}{2}$. They satisfy also the supplementary HWV conditions (3.13). The Kač determinant at level n and relative charge m is

$$\det M_{n,m}^{R}(\tilde{c},h,q)=\prod_{\substack{1\le rs\le 2n\\ s,\text{even}}}[f_{r,s}^{R}]^{P_R(n-rs/2,m)}\prod_{k\in\mathbf{Z}}[g_k^{R}]^{\tilde{P}_R(n-|k|,m-{\rm sgn}(k);k)}\ , \tag{3.28}$$

$$f_{r,s}^{R}=2(\tilde{c}-1)(h-\tilde{c}/8)-q^2+\frac{[(\tilde{c}-1)r+s]^2}{4},\quad r\in\mathbf{Z}^+,\ s\in 2\mathbf{Z}^+\ , \tag{3.29a}$$

$$g_k^{R}=2h-2kq+(\tilde{c}-1)(k^2-\tfrac{1}{4})-\tfrac{1}{4},\quad k\in\mathbf{Z}\ , \tag{3.29b}$$

where the Ramond partition functions are defined by

$$\sum_{n,m}P_R(n,m)z^n w^m=(w^{1/2}+w^{-1/2})\prod_{k=1}^{\infty}\frac{(1+z^k w)(1+z^k w^{-1})}{(1-z^k)^2}\ , \tag{3.30a}$$

$$\sum_{n,m}\tilde{P}_R(n,m;k)z^n w^m=(1+z^{|k|}w^{{\rm sgn}(k)})^{-1}\sum_{n,m}P_R(n,m)z^n w^m\ , \tag{3.30b}$$

sgn(k)$=1$, $k>0$, -1 for $k<0$, and sgn(0)$=1$ (-1) for the R^+ (R^-) algebra, respectively.

A vanishing of $f_{r,s}^{R}$ signals the existence of a HWV at level $rs/2$ and relative charge $-\frac{1}{2}$. When $g_k^R=0$ there is a HWV at level $|k|$ and relative charge sgn(k) $-\frac{1}{2}$sgn(0). For $\tilde{c}<1$ unitary representations exist only for the values given in (3.25), the respective dimensions and (1) charges being

$$h=\frac{\tilde{c}}{8}+\frac{jk}{m},\quad q={\rm sgn}(0)\frac{j-k}{m}\ ,$$
$$j,k\in\mathbf{Z},\quad 0=\le j-1,\quad k,j+k\le m-1\ . \tag{3.31}$$

This class of representations ($R_0^{\pm}$) contains only degenerate representations. For $\tilde{c}\le 1$, there are again two classes of unitary representations.

(a) $R_2^{\pm}$. The representations in this class are characterized by

$$g_n^R=0,\quad g_{n+{\rm sgn}(n)}^R<0,\quad f_{1,2}^R\ge 0\ , \tag{3.32}$$

for some $n\in\mathbf{Z}$. They are also degenerate.

(b) $R_3^{\pm}$. These representations are characterized by $g_n^R\ge 0$, $\forall n\in\mathbf{Z}$.

They contain a subset of degenerate representations corresponding to $g_n^R=0$ for some $n\in\mathbf{Z}$.

Finally in the T sector the HWV are characterized by their dimension h and fermion parity $(-1)^F$. Each level contains two equal subspaces of opposite fermion parity. The Kač determinant for the T algebra is

$$\det M_{+0}^{T}=1,\quad \det M_{-0}^{T}=h-\tilde{c}/8\ , \tag{3.33a}$$

$$\det M_{\pm n}^{T}(\tilde{c},h)=(h-\tilde{c}/8)^{P_T(n)/2}\prod_{\substack{1\le rs\le 2n\\ s,\text{odd}}}[f_{r,s}^{T}]^{P_T(n-rs/2)}\ , \tag{3.33b}$$

$$f_{r,s}^{T}=2(\tilde{c}-1)(h-\tilde{c}/8)+\frac{[(\tilde{c}-1)r+s]^2}{4},\quad s\in 2\mathbf{Z}^+-1\ , \tag{3.34}$$

$$\sum_n P_T(n)z^n=\prod_{k=1}^{\infty}\frac{(1+z^k)(1+z^{k-1/2})}{(1-z^k)(1-z^{k-1/2})}\ . \tag{3.35}$$

For $\tilde{c}<1$ unitary representations exist again for the values (3.25), all being degenerate, with dimensions given by

$$h=\frac{\tilde{c}}{8}+\frac{(m-2r)^2}{16m}\ ,$$
$$m=2,3,\ldots,\quad r\in\mathbf{Z},\quad 1\le r\le m/2\ . \tag{3.36}$$

For $\tilde{c}\ge 1$, we have another class of unitary representations, T_2, with $h\ge\tilde{c}/8$. None of them are degenerate.

The degenerate representations belonging to the class $\rm NS_0$, $R_0^{\pm}$, T_0, have been proven to be unitary through an explicit unitary construction of their Hilbert space.[20] The question about the unitary of $R_2^{\pm}$ and NS_2 is still open.

IV. CORRELATION FUNCTIONS AND OPERATOR ALGEBRA OF THE UNITARY DEGENERATE REPRESENTATIONS. NS SECTOR

As mentioned earlier in this work, the invariance of the vacuum under the global $N=2$ superconformal group OSp(2 | 2) turns out to be very useful towards the evaluation of the correlation functions. From now on we restrict ourselves to the NS sector. Similar techniques, however, apply to the $R^{\pm}$ sector as described in the next section.

Using the commutations relations (3.11), derived in the previous section, we can write the Ward identities for global superconformal invariance. Their derivation is obvious. For example, L_{-1} annihilates the in vacuum. But we can move it to the left using (3.11), so we end up with a differential equation for the correlation function. Thus the n-point function

$$F_n \equiv \langle 0 \mid \Phi_1(\mathbf{z}_1)\Phi_2(z_2)\Phi_3(\mathbf{z}_3)\cdots\Phi_n(\mathbf{z}_n)\mid 0\rangle \tag{4.1}$$

satisfies the Ward identities

$$\begin{aligned}
L_{-1}&: \left[\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\right]F_n=0 ,\\
L_0&: \sum_{i=1}^{n}\left[z_i\frac{\partial}{\partial z_i}+\Delta_i+\frac{1}{2}\left[\theta_i\frac{\partial}{\partial\theta_i}+\bar\theta_i\frac{\partial}{\partial\bar\theta_i}\right]\right]F_n=0 ,\\
L_{+1}&: \sum_{i=1}^{n}\left\{z_i^2\frac{\partial}{\partial z_i}+2z_i\left[\Delta_i+z_i\left[\theta_i\frac{\partial}{\partial\theta_i}+\bar\theta_i\frac{\partial}{\partial\bar\theta_i}\right]\right]+Q_i\theta_i\bar\theta_i\right\}F_n=0 ,\\
J_0&: \sum_{i=1}^{n}\left[Q_i+\bar\theta_i\frac{\partial}{\partial\bar\theta_i}-\theta_i\frac{\partial}{\partial\theta_i}\right]F_n=0 ,\\
G_{-1/2},\bar G_{-1/2}&: \sum_{i=1}^{n}\left[\frac{\partial}{\partial\bar\theta_i}-\bar\theta_i\frac{\partial}{\partial z_i}\right]F_n=\sum_{i=1}^{n}\left[\frac{\partial}{\partial\theta_i}-\bar\theta_i\frac{\partial}{\partial z_i}\right]F_n=0 ,\\
G_{1/2}&: \sum_{i=1}^{n}\left[z_i\left[\frac{\partial}{\partial\bar\theta_i}-\theta_i\frac{\partial}{\partial z_i}\right]-(2\Delta_i+Q_i)\theta_i-\theta_i\bar\theta_i\frac{\partial}{\partial\bar\theta_i}\right]F_n=0 ,\\
\bar G_{1/2}&: \sum_{i=1}^{n}\left[z_i\left[\frac{\partial}{\partial\theta_i}-\bar\theta_i\frac{\partial}{\partial z_i}\right]-(2\Delta_i-Q_i)\bar\theta_i+\theta_i\bar\theta_i\frac{\partial}{\partial\theta_i}\right]F_n=0 ,
\end{aligned} \tag{4.2}$$

where Δ_i, Q_i are dimensions and charges of the various fields appearing in the correlation function (4.1).

A superfluid operator in terms of components has the form

$$\Phi(\mathbf{z})\equiv\phi(z)+\theta\bar\psi(z)+\bar\theta\psi(z)+\theta\bar\theta g(z) . \tag{4.3}$$

The two-point function is completely fixed by the Ward identities, up to an irrelevant normalization constant:

$$\langle 0\mid\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\mid 0\rangle=\mathbf{z}_{12}^{-(\Delta_1+\Delta_2)}\exp\left[Q_2\frac{\theta_{12}\bar\theta_{12}}{\mathbf{z}_{12}}\right]\delta_{Q_1+Q_2,0}\delta_{\Delta_1,\Delta_2} . \tag{4.4}$$

It is a function of the supersymmetry invariant distances in superspace, $\mathbf{z}_{12}=z_1-z_2-\theta_1\bar\theta_2-\bar\theta_1\theta_2$, $\theta_{12}=\theta_1-\theta_2$, $\bar\theta_{12}=\bar\theta_1-\bar\theta_2$. The three-point function depends on nine independent variables $(z_i,\theta_i,\bar\theta_i)$. Since OSp(2 | 2) has eight generators we can fix at most eight of them, so there must be a unique combination invariant under OSp(2 | 2). This is a commuting combination which turns out to be nilpotent:

$$\hat R=\frac{\theta_{12}\bar\theta_{12}}{\mathbf{z}_{12}}-\frac{\theta_{13}\bar\theta_{13}}{\mathbf{z}_{13}}+\frac{\theta_{23}\bar\theta_{23}}{\mathbf{z}_{23}},\quad \hat R^2=0 . \tag{4.5}$$

So, for any particular solution of the Ward identities, we can obtain the general solution by multiplying it with $(1+\alpha\hat R)$, α being an arbitrary commuting constant. Solving the Ward identities for the three-point function we obtain

$$\langle 0\mid\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(\mathbf{z}_3)\mid 0\rangle=C\left[\prod_{i<j}^{3}\mathbf{z}_{ij}^{-\Delta_{ij}}\right]\exp\left[\sum_{i<j}^{3}A_{ij}\frac{\theta_{ij}\bar\theta_{ij}}{\mathbf{z}_{ij}}\right]\delta_{Q_1+Q_2+Q_3,0} , \tag{4.6}$$

where the constants

$$A_{ij}=-A_{ji},\quad \sum_{\substack{j=1\\ j\neq i}}^{3} A_{ij}=-Q_i\ . \tag{4.7}$$

It is easily seen from (4.7) that the equations defining the constants A_{ij} are not fixing all of them because of the charge neutrality condition, for the correlation function. In particular, if A_{ij} is some solution of (4.7) then $A_{12}+\alpha$, $A_{31}+\alpha$, $A_{23}+\alpha$, is also a solution. Of course this is expected. It corresponds to multiplying the three-point function by the OSp(2 | 2) invariant $(1+\alpha\hat{R})$. For the three-point function to be nonzero, the OPE of the operators Φ_1,Φ_2 must contain the family Φ_3. Then the normalization constant C of the three-point function is the Glebsch-Gordan coefficient for the decomposition $[\Phi_1]\otimes[\Phi_2]\rightarrow[\Phi_3]$. In the $N=2$ case, like the $N=1$, there is another operator-product coefficient to be determined, namely, one of the A_{ij}, due to the existence of the OSp(2 | 2) invariant $\hat{R}$.

In general OSp(2 | 2) invariance constrains the n-point function to have the form

$$\langle 0\,|\,\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2),\ldots,\Phi_n(\mathbf{z}_n)\,|\,0\rangle \sim \prod_{i<j}^{n}[\mathbf{z}_{ij}^{-\Delta_{ij}}]\exp\left[\sum_{i<j}^{n} A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{\mathbf{z}_{ij}}\right]F_n(x_1,x_2,\ldots,x_{3n-8})\delta_{\sum_{i=1}^{n}Q_i,0}\ , \tag{4.8}$$

$$A_{ij}=-A_{ji},\quad \Delta_{ij}=\Delta_{ji},\quad \sum_{\substack{j=1\\ i\neq j}}^{n} A_{ij}=-Q_i,\quad \sum_{\substack{i=1\\ j\neq i}}^{n}\Delta_{ij}=2\Delta_i\ , \tag{4.9}$$

where x_i, $i=1,2,\ldots,3n-8$ are the combinations of the coordinates, with dimension zero, invariant under OSp(2 | 2). They are functions of the invariant distances, $\mathbf{z}_{ij},\theta_{ij},\bar{\theta}_{ij}$. All the nontrivial information about the theory is encoded in the functions F_n. In most cases they are determined by the specific details of the theory. In certain cases though, that we will discuss below, they can be evaluated, just by knowing the representation content of the theory. So let us focus on these interesting situations where there are more constraints on the form of the correlation functions.

Consider a HW unitary irreducible representation of the $N=2$ superconformal algebra. It is generated by a HWV, $|\Delta,Q\rangle$, the primary state, satisfying the usual HWV conditions. The full representation is obtained from $|\Delta,Q\rangle$ by applying the lowering operations of the algebra. In some special situations it may turn out that one of the secondary states satisfies the HWV conditions. That means that the representation generated by $|\Delta,Q\rangle$ is not irreducible, but there is another representation (the one generated by the secondary vector) embedded in it. The secondary HWV, $|\chi\rangle$, has the interesting property that it is null [i.e., $(\chi\,|\,\chi)=0$] and orthogonal to any other state in the Hilbert space. We may thus consistently set $|\chi\rangle$ to be equal to zero, a condition that decouples all its family from the correlation function of the theory. In fact this condition will generate constraints on the correlation functions, of the primary state $|\Delta,Q\rangle$. To see how such constraints arise we have to remember that $|\chi\rangle$ is given by some operator $\hat{O}$, constructed out of the lowering operators of the algebra, acting on $|\Delta,Q\rangle$; thus,

$$\begin{aligned}0&\equiv\langle 0\,|\,\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\cdots\Phi_{n-1}(\mathbf{z}_{n-1})\,|\,\chi\rangle\\ &=\langle 0\,|\,\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\cdots\Phi_{n-1}(\mathbf{z}_{n-1})\hat{O}\,|\,\Delta,Q\rangle\ .\end{aligned} \tag{4.10}$$

Moving the operator $\hat{O}$ to the left using the commutation relations (3.11) we end up with a superdifferential equation for the correlation function. Solving these equations we can determine all the correlation functions that the degenerate family is participating in. An interesting set of theories are those that contain only degenerate representations, the so-called "minimal" theories. They contain a discrete (in general, infinite) set of primary operators. A subset of them are unitary and their content has presented in Sec. III according to the results of Refs. 10–12. They contain a finite set of primary fields. Such theories are exactly solvable in the previous sense. Unitary minimal theories are known for systems realizing conformal,[1,21] or superconformal invariance.[22,23] The "fusion" rules in the $N=0$ case were derived in Ref. 1. In the $N=1$ case they were partially derived in Ref. 22 and in full generality in Ref. 24.

A necessary and sufficient condition for the existence of such systems is the closure of the operator algebra of a set of unitary degenerate representations. In fact we will show that the operator algebra of the unitary degenerate representations of the $N=2$ superconformal algebra, with $\tilde{c}<1$, does close. We will derive also the "fusion" rules for the operator algebra.

Consider the OPE of two primary operators:

$$\Phi_1(\mathbf{z})\Phi_2(0)=\sum_i\Phi_i(0)\mathbf{z}^{\Delta_1+\Delta_2-\Delta_3}\ , \tag{4.11}$$

where the notation in the right-hand side of (4.11) is symbolic, meaning the product can be written as a sum of primary operators and/or their descendants, and the $(z,\theta,\bar{\theta})$ dependence can be easily substituted back. What we want to know is which irreducible representation can appear in the operator product of two given representations. There is a simple criterion for representations which are not allowed, and this is the vanishing of the appropriate three-point function.

The strategy is to use the superdifferential equations stemming from the degeneracy of the representations to derive selection rules for the operator-product algebra. Let us consider a concrete example. Take a representation which has a null vector at the first level. Such a representation is, for example, one with $\Delta=(m-2)/2m$, $Q=-(m-2)/m$, when $\tilde{c}=1-2/m$, $m=2,3,\ldots$. The null vector at level one is given by

$$|\chi_1^0\rangle=[(Q-1)L_{-1}-(2\Delta+1)J_{-1}+G_{-1/2}\bar{G}_{-1/2}]\,|\,\Delta,Q\rangle \ . \tag{4.12}$$

It is easy to verify, using the commutation relations (2.14), that $|\chi_1^0\rangle$ satisfies all the HWV conditions. Consider now the n-point function where this state is participating. We have mentioned already that such a correlation function is identically zero:

$$\begin{aligned}0&\equiv\langle 0\,|\,\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\cdots\Phi_n(\mathbf{z}_n)\,|\,\chi_1^0\rangle\\&=\langle 0\,|\,\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\cdots\hat{O}\Phi(0)\,|\,0\rangle \ .\end{aligned} \tag{4.13}$$

Commuting $\hat{O}$ through to the left we arrive at the following superdifferential equation:

$$\Bigg\{(1-Q)\sum_{n=1}^{n}\frac{\partial}{\partial z_i}+(2\Delta+1)\sum_{i=1}^{n}\left[\frac{Q_i}{z_i}+\frac{1}{z_i}\left[\bar\theta_i\frac{\partial}{\partial\bar\theta_i}-\theta_i\frac{\partial}{\partial\theta_i}\right]-\frac{2\Delta_i}{z_i^2}\theta_i\bar\theta_i\right]+\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial}{\partial\bar\theta_i}-\theta_i\frac{\partial}{\partial z_i}\right]\left[\frac{\partial}{\partial\theta_j}-\bar\theta_i\frac{\partial}{\partial z_i}\right]\Bigg\}\langle 0\,|\,\Phi(\mathbf{z}_1)\cdots\Phi_n(\mathbf{z}_n)\Phi(0)\,|\,0\rangle=0 \ . \tag{4.14}$$

We will specialize (4.14) to the three-point function $\langle 0\,|\,\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(\mathbf{z}_3)\,|\,0\rangle$, where Φ_3 is the degenerate operator mentioned above.

Doing a translation and two global supersymmetry transformations [we have the freedom to do that, thanks to the OSp(2 | 2) invariance of the correlation function], we can write the three-point function in the form $\langle 0\,|\,\Phi_1(\tilde{\mathbf{z}}_1)\Phi_2(\mathbf{z}_2)\Phi_3(0)\,|\,0\rangle$, where

$$\begin{aligned}\tilde{\mathbf{z}}_1&\equiv(z_1-z_3-\theta_1\bar\theta_3-\bar\theta_1\theta_3,\theta_1-\theta_3,\bar\theta_1-\bar\theta_3) \ ,\\ \tilde{\mathbf{z}}_2&\equiv(z_2-z_3-\theta_2\bar\theta_3-\bar\theta_2\theta_3,\theta_2-\theta_3,\bar\theta_2-\bar\theta_3) \ .\end{aligned} \tag{4.15}$$

Using the form of the three-point function found earlier, in (4.14) we arrive at the following set of conditions for the dimension Δ_{ij} and the constants A_{ij}:

$$\begin{aligned}&\Delta_{13}(Q_3-1)+Q_1(2\Delta_3+1)+A_{13}+\Delta_{13}=0 \ ,\\&(\Delta_{12}-A_{12})(A_{13}+\Delta_{13})=0 \ ,\\&(\Delta_{12}+A_{12})(A_{13}+\Delta_{13})=0 \ ,\\&(Q_3-1)A_{13}-2\Delta_1(2\Delta_3+1)\\&\qquad+(\Delta_{13}-A_{13}+1)(\Delta_{13}+A_{13})=0 \ ,\\&(2\Delta_3+1)(\Delta_{12}-A_{12})+(\Delta_{13}-A_{13})(\Delta_{23}+A_{23})=0 \ ,\\&(2\Delta_3+1)(A_{12}+\Delta_{12})+(\Delta_{23}-A_{23})(\Delta_{13}+A_{13})=0 \ .\end{aligned} \tag{4.16}$$

The state mentioned above happens to be also degenerate at level $\frac{1}{2}$ and relative charge -1, the null vector being

$$|\chi_{1/2}^-\rangle=\bar{G}_{-1/2}\,|\,\Delta,Q\rangle \ . \tag{4.17}$$

In the same way we derive another equation:

$$\sum_{i-1}^{n}\left[\frac{\partial}{\partial\theta_i}-\bar\theta_i\frac{\partial}{\partial\theta_i}\right]\langle 0\,|\,\Phi_1(\mathbf{z}_1)\cdots\Phi_n(\mathbf{z}_n)\Phi(0)\,|\,0\rangle=0 \tag{4.18}$$

which for the three-point function in particular implies

$$A_{13}=-\Delta_{13}, \quad A_{23}=-\Delta_{23} \ . \tag{4.19}$$

Solving (4.16) and (4.19) we obtain

$$2\Delta_1=Q_1, \quad 2\Delta_2=Q_2, \quad \Delta_1=\Delta_3-\Delta_2 \ . \tag{4.20}$$

Consequently in the operator product of Φ_2, with $2\Delta_2=Q_2$, and Φ_3, only fields with $2\Delta_1=Q_1$, and $\Delta_1=\Delta_3-\Delta_2$ can appear.

As was mentioned in the previous section, the unitary irreducible representations in the NS sector with $\tilde{c}<1$, exist when $\tilde{c}=1-2/m$, $m\in\mathbf{Z}^+-\{1\}$ and their dimensions and charges are given by (3.26). It can be shown that for the family (j,k), there are three independent null HWV embedded in it, one at relative charge zero and level $m-(j+k)$, another at relative charge 1 and level k, and another one at relative charge -1 and level j (for more details see Ref. 13). Consequently the correlation functions of (j,k) satisfy three superdifferential equations of orders $j,k,m-(j+k)$ simultaneously. The existence of three null vectors renders the $N=2$ case qualitatively different from the $N=0,1$ cases.

The "fusion" rules coming from the consideration of the two charged null vectors at levels j_1,k_1 of the family (j_1,k_1) are (in Appendix A we present another heuristic justification of the fusion rules based on the unitary construction of these representations using free fermions)

$$(j_1,k_1)\otimes(j_2,k_2)=\sum_{n=1/2-k_2}^{j_2-1/2}(j_1+n,n-j_2+k_1+k_2), \quad j_1+k_1\geq j_2+k_2 \ , \tag{4.21a}$$

$$(j_1,k_1)\otimes(j_2,k_2)=\sum_{n=1/2-k_1}^{j_1-1/2}(j_2+n,n-j_1+k_1+k_2), \quad j_1+k_1\leq j_2+k_2 \ . \tag{4.21b}$$

The strategy to derive the fusion rules in their general form, Eqs. (4.21), is parallel to the one used before in the $N=0,1$ cases. The operator families $(\frac{3}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{3}{2})$ are the shifting up and down operators and the following relations are easily checked using the superdifferential equations

$$(\tfrac{3}{2},\tfrac{1}{2})\otimes(j,k)=(j,k-1)\oplus(j+1,k)\ , \tag{4.22a}$$

$$(\tfrac{1}{2},\tfrac{3}{2})\otimes(j,k)=(j-1,k)\oplus(j,k+1)\ . \tag{4.22b}$$

Then (4.21) is proven by taking various products of the shifting operators and using the commutativity and associativity properties of the OPE. [We have independently checked (4.21) up to level $\frac{5}{2}$ using the relevant superdifferential equations.]

As mentioned above, the family (j,k) is also degenerate at relative charge zero and level $m-(j+k)$. The extra conditions from this new null HWV have the effect of truncating the sums in (4.21) into the bounds, $0<j$, $k,j+k\leq m-1$, where $(j_1,k_1)\otimes(j_2,k_2)\sim(j,k)$. This truncation phenomenon is known already to happen in the analogous minimal theories of the $N=0,1$ algebras. Thus it is consistent to built $N=2$ unitary minimal systems, with $\tilde{c}<1$, where there is a finite number of representations, all degenerate, and all the correlation functions calculable.

We present the two explicit examples of the operator algebra of the first two nontrivial theories with $\tilde{c}=\frac{1}{3}$ $(m=3)$, $\tilde{c}=\frac{1}{2}$ $(m=4)$. In the $\tilde{c}=\frac{1}{3}$ theory the representation content of the NS sector is shown in Fig. 1. The operator algebra is

$$\begin{aligned}&(\tfrac{1}{6},\pm\tfrac{1}{3})\otimes(0,0)\sim(0,0)\ ,\\&(\tfrac{1}{6},\pm\tfrac{1}{3})\otimes(\tfrac{1}{6},\mp\tfrac{1}{3})\sim(0,0)\ .\end{aligned} \tag{4.22c}$$

This system, is somewhat special and it will be analyzed in more detail in the next section.

The $\tilde{c}=\frac{1}{2}$ system has the representation content shown in Fig. 2. Its fusion rules are

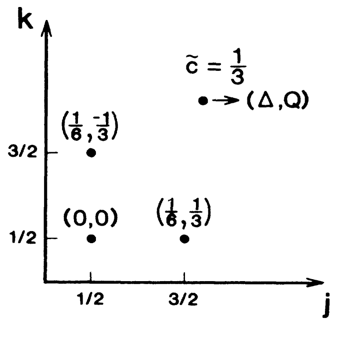

FIG. 1. Operator content of the $\tilde{c}=\frac{1}{3}$ minimal system (NS sector) (the first entry is the dimension of an operator and the second its change). Circles represent operators belonging to the R sector. The solid circles indicate the $N=2$ subsystem of $\tilde{c}=\frac{1}{3}$.

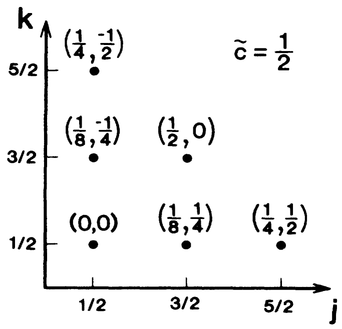

FIG. 2. Operator content of the $\tilde{c}=\frac{1}{2}$ minimal system (NS sector). Notation same as in Fig. 1.

$$\begin{aligned}&(\tfrac{1}{8},\pm\tfrac{1}{4})\otimes(\tfrac{1}{8},\mp\tfrac{1}{4})\sim(0,0)\ ,\\&(\tfrac{1}{8},\pm\tfrac{1}{4})\otimes(\tfrac{1}{4},\mp\tfrac{1}{2})\sim(\tfrac{1}{8},\mp\tfrac{1}{4})\ ,\\&(\tfrac{1}{8},\pm\tfrac{1}{4})\otimes(\tfrac{1}{2},0)\sim(\tfrac{1}{8},\pm\tfrac{1}{4})\ ,\\&(\tfrac{1}{4},\pm\tfrac{1}{2})\otimes(\tfrac{1}{8},\mp\tfrac{1}{4})\sim(\tfrac{1}{8},\mp\tfrac{1}{4})\ ,\\&(\tfrac{1}{4},\pm\tfrac{1}{2})\otimes(\tfrac{1}{2},0)\sim(\tfrac{1}{4},\pm\tfrac{1}{2})\ ,\\&(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0)\sim(\tfrac{1}{2},0)\oplus(0,0)\ .\end{aligned} \tag{4.23}$$

We should remind the reader that the "fusion" rules we have derived give the maximum possible set of operators that can appear in an operator-product expansion. To determine exactly which of them contribute and to evaluate their Glebsch-Gordan coefficients one has to evaluate the four-point function. This is what we will do for the $\tilde{c}=\frac{1}{3}$ system in the next section.

It is worth saying a few words here about the situation for other values of the central charge. We consider again the set of primary fields with dimensions and U(1) charges given by

$$\Delta=\tfrac{1}{2}(jk-\tfrac{1}{4})(1-\tilde{c}),\quad Q=\tfrac{1}{2}(j-k)(1-\tilde{c}),\quad j,k\in\mathbf{Z}+\tfrac{1}{2}\ . \tag{4.24}$$

For any $\tilde{c}$ there are charged null vectors one at level k and relative charge $\mathrm{sgn}(k)$ and another one at level j and relative charge $-\mathrm{sgn}(j)$. If $\tilde{c}$ is irrational there are no other null vectors. But if $\tilde{c}$ is rational, then it can be written in the form

$$\tilde{c}=1-\frac{2p}{q},\quad p,q\in\mathbf{Z}\ , \tag{4.25}$$

and there is an infinity of null vectors including also neutral ones.[13]

The derivation of (4.21) used only the conditions stemming from the charged null vectors; thus, it remains true for every $\tilde{c}$. When $\tilde{c}=1-2p/q$ then the existence of the neutral null vectors implies the truncation of the infinite set (4.24) to the finite subset $0\leq j$, $k,j+k\leq q/p-1$. The latter case is of interest because otherwise operators with negative dimensions occur in the theory. Thus $N=2$ minimal theories exist for $\tilde{c}=1-2p/q$, $p,q\in\mathbf{Z}$ and contain a finite set of representations closed under OPE.

V. THE OPERATOR FORMALISM IN THE RAMOND SECTOR

In Sec. III we gave a brief description of the Ramond sector and its ground states. We will continue this discussion and develop in a parallel way the structure that we outlined in Sec. IV for the NS sector.

The ground state that preserves $N=2$ supersymmetry has $\Delta=\tilde{c}/8$. The rest of the primary states are generated from the ground state by the action of NS superfield operators. Since primary operators are labeled also by their charge there is a nontrivial question to answer: What is the charge of the ground state? To find a reasonable answer we will use a piece of knowledge pertaining to the $N=2$ superconformal algebras.

The algebra given in (3.17) can be consistently defined by a general choice of the periodicity properties of the two supercharges:[25]

$$G(z)=e^{2\pi i a}G(e^{2\pi i}z),\quad \bar{G}(z)=e^{-2\pi i a}\bar{G}(e^{2\pi i}z)\ ,\qquad 0\le a\le 1\ , \tag{5.1}$$

$a=0,1$ correspond to $R(\mp)$ boundary conditions whereas $a=\frac{1}{2}$ to NS. There is an automorphism of the algebra,[25] that relates algebras with different a's given by

$$J_n^a=J_n-a\tilde{c}\delta_{n,0},\quad G_{n+a}^a=G_n\ , \tag{5.2a}$$

$$L_n^a-L_n-aJ_n+\frac{a^2}{2}\tilde{c}\delta_{n,0},\quad \bar{G}_{n-a}^a=\bar{G}_n\ . \tag{5.2b}$$

This shows that algebras with different a's are in fact equivalent. In particular, the NS algebra is equivalent to the $R^{\pm}$ algebra. The relations (5.2) imply a relation among dimensions and charges of NS and $R^{\pm}$ representations:

$$\Delta_{R^+}=\Delta_{\rm NS}-\tfrac{1}{2}Q_{\rm NS}+\tfrac{1}{8}\tilde{c},\quad Q_{R^+}=Q_{\rm NS}-\tfrac{1}{2}\tilde{c}\ , \tag{5.3a}$$

$$\Delta_{R^-}=\Delta_{\rm NS}+\tfrac{1}{2}Q_{\rm NS}+\tfrac{1}{8}\tilde{c},\quad Q_{R^-}=Q_{\rm NS}+\tfrac{1}{2}\tilde{c}\ . \tag{5.3b}$$

From now on we will focus on the R^+ sector. (The R^- sector is obtained by charge conjugation.)

It is natural to consider as the in ground state $|R_+\rangle$ of the R sector the state which is the image of the unit operator in the NS sector under the automorphism (5.2). It has dimension $\Delta=\tilde{c}/8$ and charge $Q=-\tilde{c}/2$, as it can be seen from (5.3a). The out ground state $\langle R_-|$ then must have charge $Q=\tilde{c}/2$. The states $|R_+\rangle$ and $|R_-\rangle$ being HWV's of the R^+ algebra satisfy, among others, the following HWV conditions:

$$\bar{G}_0|R_+\rangle=\bar{G}_0|R_-\rangle=0\ . \tag{5.4}$$

The representations corresponding to $|R_+\rangle$ and $|R_-\rangle$ are also degenerate. By looking at the Kač determinant in the R sector (3.29) we can easily verify that $|R_+\rangle$ is degenerate at level zero, relative charge 1 as well as level one, relative charge -1, while $|R_-\rangle$ is degenerate at level zero and relative charge 1 (for special values of $\tilde{c}$ there are additional degeneracies). The vanishing conditions for the null vectors mentioned above are

$$G_0|R_+\rangle=\bar{G}_{-1}|R_+\rangle=G_0|R_-\rangle=0\ . \tag{5.5}$$

We define the correlation functions in the R sector as

$$F_n(\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_n)\equiv\frac{\langle R_-|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\cdots\Phi_n(\mathbf{z}_n)|R_+\rangle}{\langle R_-|R_+\rangle}\ , \tag{5.6}$$

where $\Phi_i(\mathbf{z}_i)$ is a NS superfield. Then the correlation functions (5.6) satisfy Ward identities due to (5.4) and (5.5) which parallel the global $N=2$ Ward identities in the NS sector:

$$\sum_{i=1}^{n}\left[Q_i+\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}-\theta_i\frac{\partial}{\partial\theta_i}\right]F_n\langle\mathbf{z}_1,\ldots,\mathbf{z}_n)=0\ , \tag{5.7a}$$

$$\sum_{i=1}^{n}\left[z_i\frac{\partial}{\partial z_i}+\Delta_i+\frac{1}{2}\left[\theta_i\frac{\partial}{\partial\theta_i}+\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\right]\right]F_n(\mathbf{z}_1,\ldots,\mathbf{z}_n)=0\ , \tag{5.7b}$$

$$\sum_{i=1}^{n}\left[\sqrt{z_i}\left[\frac{\partial}{\partial\bar{\theta}_i}-\theta_i\frac{\partial}{\partial z_i}\right]-\frac{1}{2\sqrt{z_i}}\left[(2\Delta_i+Q_i)\theta_i+\theta_i\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}\right]\right]F_n(\mathbf{z}_1,\ldots,\mathbf{z}_n)=0\ , \tag{5.7c}$$

$$\sum_{i=1}^{n}\left[\sqrt{z_i}\left[\frac{\partial}{\partial\theta_i}-\bar{\theta}_i\frac{\partial}{\partial z_i}\right]-\frac{1}{2\sqrt{z_i}}\left[(2\Delta_i-Q_i)\bar{\theta}_i-\theta_i\bar{\theta}_i\frac{\partial}{\partial\theta_i}\right]\right]F_n(\mathbf{z}_1,\ldots,\mathbf{z}_n)=0\ , \tag{5.7d}$$

$$\sum_{i=1}^{n}\left[\frac{1}{\sqrt{z_i}}\left[\frac{\partial}{\partial\theta_i}-\bar{\theta}_i\frac{\partial}{\partial z_i}\right]+\frac{1}{2z_i\sqrt{z_i}}\left[(2\Delta_i-Q_i)\bar{\theta}_i-\theta_i\bar{\theta}_i\frac{\partial}{\partial\theta_i}\right]\right]F_n(\mathbf{z}_1,\ldots,\mathbf{z}_n)=0\ , \tag{5.7e}$$

where Δ_i, Q_i are the (NS) dimensions and charges of Φ_i. Equation (5.7) can be used to constrain the form of the correlation functions in the R sector. We will work out as an example the constraints on the two-point function:

$$F_2(\mathbf{z}_1,\mathbf{z}_2)\equiv\frac{\langle R_-|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)|R_+\rangle}{\langle R_-|R_+\rangle}\ . \tag{5.8}$$

Equation (5.7a) implies that $Q_1+Q_2=0$ and

$$F_2(\mathbf{z}_1,\mathbf{z}_2)=f_0(z_1,z_2)+\theta_1\bar\theta_1 f_1(z_1,z_2)+\theta_2\bar\theta_2 f_2(z_1,z_2)+\theta_1\bar\theta_2 f_3(z_1,z_2)+\theta_1\theta_2 f_4(z_1,z_2)+\theta_1\bar\theta_1\theta_2\bar\theta_2 g(z_1,z_2)\ . \qquad (5.9)$$

We define the variables, $u=\sqrt{z_1/z_2}, v=\sqrt{z_1 z_2}$ in order to split the dimensional dependence. Equation (5.7b) implies

$$f_0(u,v)=\frac{f_0(u)}{v^{2\Delta}},\quad g(u,v)=\frac{g(u)}{v^{2\Delta+2}}\ ,\qquad f_i(u,v)=\frac{f_i(u)}{v^{2\Delta+1}},\quad i=1,2,3,4\ . \qquad (5.10)$$

The rest of the equations are solved by

$$f_0(u)=\frac{u^{2\Delta-Q_1}}{(u^2-1)^{2\Delta}},\quad f_1(u)=\frac{\Delta-(\Delta+Q_1)u^2}{u(u^2-1)}f_0(u)\ , \qquad (5.11a)$$

$$f_4(u)=\frac{2(\Delta-Q_1)}{u^2-1}f_0(u)\ ,\qquad f_2(u)=-\frac{\Delta u^2+Q_1-\Delta}{u^2-1}uf_0(u)\ ,\qquad f_3(u)=2\frac{2\Delta u^2-(2\Delta-Q_1)}{u^2-1}f_0(u)\ , \qquad (5.11b)$$

$$g(u)=2\Delta\left[\frac{(2\Delta+1)u^4}{(u^2-1)^2}-\frac{(2\Delta-Q_1+1)u^2}{u^2-1}+\frac{\Delta-Q_1}{2}\right]f_0(u)\ . \qquad (5.11c)$$

The two-point function is asymmetric due to the asymmetry in the charge assignments of the in and out ground states. The two-point function with $|R_+\rangle\leftrightarrow|R_-\rangle$ is given by (5.11) with the following substitutions made: $f_0\to f_0$, $g\to g$, $f_1\to -f_1$, $f_2\to -f_2$, $f_3\leftrightarrow f_4$, $Q_1\to -Q_1$. In a similar way higher correlation functions can be constrained by (5.7).

Let us now discuss the fusion rules in the R sector. It is important to note that the automorphism (5.2) preserves the structure of the Kač determinant [the relations (5.3) have of course to be taken into account]. Consequently it preserves the form of the fusion rules derived in the NS sector. Consider the set of HWV's of the R^+ algebra, $|\Delta,Q-\frac{1}{2}\rangle$ with dimensions and charges given by

$$\Delta=\frac{\tilde c}{8}+\frac{jk}{2}(1-\tilde c),\quad Q=\frac{j-k}{2}(1-\tilde c),\quad j,k\in\mathbf{Z}\ . \qquad (5.12)$$

Using (5.3) we can establish the correspondence

$$\mathrm{NS}\ni(j,k)\leftrightarrow(j+\tfrac{1}{2},k-\tfrac{1}{2})\in R^+,\quad j,k\in\mathbf{Z}^+ +\tfrac{1}{2} \qquad (5.13)$$

which along with (4.21) implies the following fusion rules in the R^+ sector:

$$(j_1,k_1)\otimes(j_2,k_2)=\sum_{n=-k_2}^{j_2-1}(j_1+n,n+k_1+k_2-j_2+1),\quad j_1+k_1\geq j_2+k_2\ , \qquad (5.14a)$$

$$(j_1,k_1)\otimes(j_2,k_2)=\sum_{n=-k_1}^{j_1-1}(j_2+n,n+k_1+k_2-j_1+1),\quad j_1+k_1\leq j_2+k_2\ , \qquad (5.14b)$$

where j_1, j_2, k_1, k_2 are integers.

In the case $\tilde c=1-2p/q$, p,q being integers, the existence of the neutral null HWV's implies again the truncation of (5.14) in the interval $0\leq j-1$, $k,j+k\leq q/p-1$. The unitary minimal models ($p=1$, $q=m$, $m=2,3,\ldots$) also belong to this class. What we have discussed so far for the R sector is illustrated explicitly in Appendix B for the $\tilde c=\frac{1}{3}$ model.

VI. THE $\tilde c=\frac{1}{3}$, $N=2$ SUPERCONFORMAL THEORY

This theory has the simplest operator content compared to the other unitary minimal $N=2$ theories. It is also the only member of the $N=2$ discrete series which has the same central element with a member of the $N=1$ discrete series. The authors of Ref. 11 identified some operators in the NS and R sector of the $N=2$, $\tilde c=\frac{1}{3}$ system with corresponding operators in the $\hat c=\frac{2}{3}$, $N=1$ system. In Ref. 13, the rest of the operators of the $N=2$ system were identified and a rigorous proof of the decomposition was given using character formulas. The correspondence is as follows (subscripts indicate $N=1,2$). The unit operator $(0)_2$ decomposes into $(0)_1$ and a primary field $(1)_1$ in the NS sector. The energy-momentum tensor and one of the supercharges are contained in $(0)_1$, whereas the other supercharge and the U(1) current are contained in $(1)_1$. The NS representation $(h,q)=(\frac{1}{6},\pm\frac{1}{3})$ decomposes into $(\frac{1}{6})_1$ in the NS sector. In the R sector, $(\frac{3}{8},0)_2$ decomposes into $(\frac{3}{8})_1$ in the R sector of the $N=1$ system whereas $(\frac{1}{24},\pm\frac{1}{3},\pm\frac{2}{3})$ decomposes into $(\frac{1}{24})_1$ in the R sector. Finally $(\frac{1}{16})_2$ in the twisted sector of the $N=2$ theory decomposes into $(\frac{1}{16})_1$ in the NS sector of the $N=1$ theory. Thus the $\tilde c=\frac{1}{3}$, $N=2$ system is a subsector of the $\hat c=\frac{2}{3},N=1$ system, as is shown in Fig. 3.

The general discussion of the previous section can be specialized in this situation. The model contains the unit (superfield) operator and a conjugate pair of primary operators, representing the $\Delta=\frac{1}{6}$, $Q=\pm\frac{1}{3}$ states of the model. We will denote by $\Phi_\pm$ and Φ_0 the corresponding superfield operators. The two-point function is

$$\langle 0 \mid \Phi_+(\mathbf{z}_1)\Phi_-(\mathbf{z}_2) \mid 0 \rangle = \mathbf{z}_{12}{}^{-1/2} \exp\left[-\frac{1}{3}\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} \right], \tag{6.1}$$

where we suppressed the antianalytic part and we have chosen a particular convenient normalization for the two-point function. The only three-point function which is nonzero is $\langle 0 \mid \Phi_0(\mathbf{z}_1)\Phi_+(\mathbf{z}_2)\Phi_-(\mathbf{z}_3) \mid 0\rangle$. It is fixed up to a normalization constant by the OSp(2 | 2) invariance and the extra differential equations that it is satisfying due to the fact that it contains degenerate fields:

$$\langle 0 \mid \Phi_0(\mathbf{z}_1)\Phi_+(\mathbf{z}_2)\Phi_-(\mathbf{z}_3) \mid 0 \rangle = C\mathbf{z}_{23}{}^{-1/3} \exp\left[-\frac{1}{3}\frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}} \right]. \tag{6.2}$$

It implies the following operator-product expansions for the component fields

$$\Phi_\pm(\mathbf{z}) \equiv \phi_\pm(z) + \theta\bar{\psi}_\pm(z) + \bar{\theta}\psi_\pm(z) + \theta\bar{\theta}g_\pm(z),$$

$$\phi_+\phi_- \sim J, \quad \phi_+ g_- \sim -\tfrac{1}{3}J, \quad \phi_- g_+ \sim -\tfrac{1}{3}J, \tag{6.3a}$$

$$\bar{\psi}_+\psi_- \sim \tfrac{1}{3}J, \quad \psi_+\bar{\psi}_- \sim \tfrac{1}{3}J, \quad g_+ g_- \sim \tfrac{4}{9}J \tag{6.3b}$$

which are determined up to an overall normalization constant. The first nontrivial correlation function is the four-point function. Its evaluation enables us to fix the Glebsch-Gordan coefficient in the OPE in (6.3).

There are two ways to evaluate the four-point function. One is to solve the superdifferential equations that it satisfies due to degeneracy of the operators contained in it. The other is to use the Feigin-Fuks construction. The only nontrivial four-point function is $\langle 0 \mid \Phi_-(\mathbf{z}_1)\Phi_+(\mathbf{z}_2)\Phi_-(\mathbf{z}_3)\Phi_+(\mathbf{z}_4) \mid 0\rangle$. The operator $\Phi_+(\mathbf{z})$ is degenerate at level 1, relative charge zero, at level $\frac{1}{2}$, relative charge one, and at level $\frac{3}{2}$, relative charge -1. The relevant superdifferential equations for the four-point function $F_4(\tilde{\mathbf{z}}_1,\tilde{\mathbf{z}}_2,\tilde{\mathbf{z}}_3)$ are

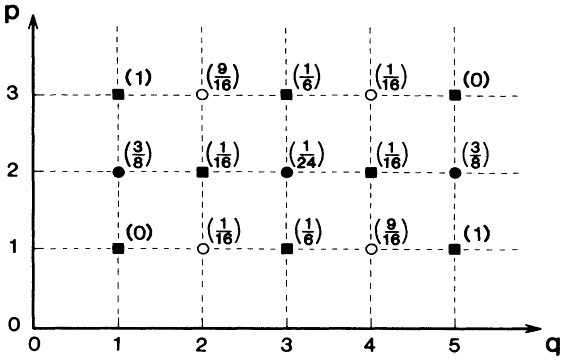

FIG. 3. Operator content of the $\hat{c} = \frac{2}{3}$, $N=1$ minimal system. Here squares represent operators belonging to the NS sector. Other notation same as in Fig. 1.

$$\left[\sum_{i=1}^{3} G^i_{1/2} \right] F_4(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3) = 0, \tag{6.4a}$$

$$\sum_{i=1}^{3} (L^i_1 - J^i_1) F_4(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3) = 0, \tag{6.4b}$$

$$\left[\sum_{i=1}^{3} \bar{G}^i_{3/2} - \sum_{i,j=1}^{3} (J^i_1 + L^i_1)\bar{G}^j_{1/2} \right] F_4(\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3) = 0, \tag{6.4c}$$

where the relevant differential operators can be read from (3.11) and we have simplified (6.4b) using (6.4a). The variables $\mathbf{z}_i$ are the shifted variables we mentioned in the last section.

Global $N=2$ superconformal invariance constrains the four-point function to be

$$F_4(bfz_1,\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_4) = C(\mathbf{z}_{12}\mathbf{z}_{34})^{-1/3} \exp\left[\frac{1}{3}\left[\frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}} - \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}} \right] \right] G_4(x_1,x_2,x_3,x_4), \tag{6.5}$$

where x_i, $i=1,2,3,4$, are the four independent combinations of the coordinates invariant under the OSp(2 | 2) group. The obvious (dependent) invariants are

$$\begin{aligned}
x_1 &= \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}} - \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}}, \\
x_2 &= \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} + \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}} - \frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}}, \\
x_3 &= \frac{\theta_{13}\bar{\theta}_{13}}{\mathbf{z}_{13}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}} - \frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}}, \\
y_1 &= \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} + \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}} - \frac{\theta_{13}\bar{\theta}_{13}}{\mathbf{z}_{13}}, \\
y_2 &= \frac{\mathbf{z}_{14}\mathbf{z}_{23}}{\mathbf{z}_{12}\mathbf{z}_{34}}, \quad y_3 = \frac{\mathbf{z}_{13}\mathbf{z}_{24}}{\mathbf{z}_{12}\mathbf{z}_{34}}.
\end{aligned} \tag{6.6}$$

Since $y_1 = x_1 + x_2 - x_3$, y_1 can be deleted. We have also the additional relations

$$x_1{}^2 = x_2{}^2 = x_3{}^2 = y_1{}^2 = 0, \quad x_1x_2 = (x_1+x_3)x_2, \tag{6.7a}$$

$$x_1x_2x_3 = 0,$$

$$(y_2 - y_3 + 1)^2 = 2y_2x_1x_2, \quad x_2x_3 = y_2x_1x_3, \tag{6.7b}$$

$$x_1x_2 = y_3x_1x_3.$$

The relations above imply that in fact x_1, x_2, x_3, and $x_4 \equiv y_2$ are independent invariants. Solving Eqs. (6.4) we arrive at a four-point function of the form

$$G_4(x_1,x_2,x_3,x_4)=C\left[\frac{x_4+1}{x_4}\right]^{1/3}\times\exp\left[\frac{1}{3(x_4+1)}(y-x_1+x_4x_2)\right], \tag{6.8}$$

where $y\equiv y_2-y_3+1$.

The four-point function (6.8) is powerlike, something to be expected since the primary fields of the $\tilde{c}=\frac{1}{3}$ theory can be constructed as vertex operators of a single $c=1$ scalar field (see Appendix B). We have performed the same calculations using the vertex operator method.[20,26] We find the same result as in (6.8) (the hypergeometric function obtained through this method truncate to polynomials of the first degree giving a powerlike four-point function). It is difficult though in this method to obtain the result as a super meromorphic function in $N=2$ superspace.

By factorizing over two-point functions we can find that $C=1$. This implies that the OPE coefficient in (6.3) is in fact unity. The full construction of the four-point function, including its antiholomorphic part does not involve any subtleties related to monodromy invariance (locality in the Euclidean domain). We simply have to multiply the holomorphic and antiholomorphic pieces which have the same form. Knowledge of the four-point function (6.8) is enough to determine any n-point function using the OPE coefficient for the degenerate operators.

Closing this section we shall remark again that the $\tilde{c}=\frac{1}{3},N=2$ superconformal theory describes a particular point of the Gaussian model for a specific value of the radius (see Ref. 16 and Appendix B).

VII. CONCLUSIONS AND PROSPECTS

In this paper we analyzed the general structure of $N=2$ superconformal-invariant theories. We developed the local analytic geometry of (2,0) complex superspace and we constructed the global $N=2$ superconformal group. The Ward identifies for the global $N=2$ superconformal symmetry were solved, which provided a partial determination of the correlation functions. In particular, the two-point functions are determined up to an irrelevant normalization, whereas the three-point function is determined up to two OPE coefficients.

We then specialized to unitary minimal theories with $\tilde{c}<1$. We derived extra superdifferential equations, satisfied by all the correlation functions of the degenerate operators. Solving these equations for the three-point function we derived the "fusion" rules for the $N=2$ unitary minimal systems. In particular, we showed that the operator algebra of the unitary degenerate representations with $\tilde{c}<1$ closes, which in turn guarantees the consistency of $N=2$ superconformal unitary minimal theories with $\tilde{c}<1$. We analyzed in particular the simplest such system, that is the one with $\tilde{c}=\frac{1}{3}$. This system has been shown to be a subsector of the $\hat{c}=\frac{2}{3}$, $N=1$ superconformal minimal system.[11,13] We calculated its four-point function by solving the relevant superdifferential equation and we thus determined its operator algebra. The $\tilde{c}=\frac{1}{3}$ system is realized at some special point in the Gaussian model.[16] We think that it is interesting and important to search for critical systems which realize the $N=2$ superconformal symmetry since their structure seems to be very exciting.

Of course much more needs to be done in the context of $N=2$ superconformally invariant theories. A unitary construction of the degenerate representations with $\tilde{c}\geq 1$ is still missing. Their operator algebra needs to be found. (Work in that direction is in progress.)

This class of representations is very important since they occur in $N=2$ nonlinear σ models on compact Ricci-flat manifolds, arising in superstring compactification. The existence of four-dimensional supersymmetry relies heavily on the $N=2$ superconformal invariance of the respective σ model.[15] $N=2$ superconformal methods may turn out to be important tools in understanding superstring compactification and low-energy superstring phenomenology.

Note added. During the completion of this work we received Ref. 27 where the analytic geometry of extended super Riemann surfaces was developed. We do agree with the results of Ref. 27 concerning the local geometry (since we have not dealt with global aspects). After the completion of this work we received Refs. 9, 17, 18, 24, and 28, where some related issues have been discussed.

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APPENDIX A

In this appendix we present a heuristic justification of the "fusion rules" obtained for the NS sector in Sec. IV, and we extend it to derive also the fusion rules for the R and T sectors of the $N=2$ algebra. We make use of the unitary construction of these representations using free fermions.[20]

In order to achieve this goal, n SU(2) doublets with U(1) charge zero and an SU(2) singlet with U(1) charge ± 1 are used. They are HW irreducible representation of the SU(2)$\times$U(1) affine algebra. The central charge of the $N=2$ algebra constructed this way turns out to be $\tilde{c}=1-2/(n+2)$ and the U(1) current is given by $J^3(z)-I(z)$, where $I(z)$ is the U(1) generator of the SU(2)$\times$U(1) affine algebra. The HWV are constructed out of the modes $\psi^i_{-1/2}$ of the fermions acting on the vacuum, and we can multiply at most n of them so that the SU(2) isospin of the generated states can take the values $l=0,\frac{1}{2},1,\ldots,n/2$ while the third component takes the usual values $-l\leq l_3\leq l$. The dimensions and U(1) charges of the corresponding HWV's are

$$\Delta=\frac{l(l+1)-l_3{}^2}{n+2},\quad Q=\frac{2l_3}{n+2}\ . \tag{A1}$$

The above are HW irreducible representations of the affine algebra and they are all integrable. Their operator-product rules are the same as in the SU(2) Lie-algebra case:

$$(l_1,m_1)\otimes(l_2,m_2)=\sum_{l=|l_1-l_2|}^{l'}(l,m_1+m_2)\ , \qquad (A2)$$

where $l'=\min(n/2,l_1+l_2)$, and the upper truncation is due to the integrability requirements, see, for example, Ref. 29.

Now in order to make contact with the parametrization used in Secs. III and IV we set $m=n+2$, $j=l+l_3+\frac{1}{2}$, $k=l-l_3+\frac{1}{2}$. Then (A2) reduces to the "fusion rules" (4.21). This "derivation" can be applied to the R sector as well giving the same fusion rules as in the NS sector.

In the T sector a similar construction can be made[20] using twisted fermions generating a twisted SU(2)×U(1) affine algebra. Only J^2 has integer modes and the dimensions of the HWV of the $N=2$ algebra are given by

$$\Delta=\frac{\tilde{c}}{8}+\frac{j_2{}^2}{4(n+2)} \qquad (A3)$$

in terms of the eigenvalues j_2 of J_2, $n/2\le j_2\le n/2$.

To make contact with our notation in Sec. III we have to identify $2j_2=m-2r$. The fusion rules for the T sector then are

$$[r_1]\otimes[r_2]=[r]\ , \qquad (A4)$$

where $r=r_1+r_2$, if $r_1+r_2\le m/2$ or $r=r_1+r_2-m/2$ if $r_1+r_2>m/2$.

There is no rigorous proof that the above constructions do indeed generate the HWV of the degenerate representations. In that sense the derivation above is heuristic.

APPENDIX B

In this appendix we construct the components of the primary superfields of the $\tilde{c}=\frac{1}{3}$, $N=2$ superconformal system (NS sector) using a single $c=1$ scalar field. We use these operators to give an alternative calculation of the four-point function (6.8) which was computed in the main body of this paper.

We consider a scalar field $\phi(z)$ with a two-point function given by

$$\langle 0\,|\,\phi(z)\phi(w)\,|\,0\rangle=-\ln(z-w)\ . \qquad (B1)$$

We define the standard energy-momentum tensor $T(z)=-\frac{1}{2}\!:\partial_z\phi\partial_z\phi\!:$ satisfying

$$T(z)T(w)=\frac{1/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_w T(w)}{(z-w)}+\text{nonsingular terms}\ . \qquad (B2)$$

A vertex operator defined by $V_a(z)\equiv\,:e^{ia\phi(z)}:$ has dimension $\Delta_a=a^2/2$:

$$T(z)V_a(w)=\frac{a^2}{2}\frac{V_a(w)}{(z-w)^2}+\frac{\partial_w V_a(w)}{z-w}+\text{nonsingular}\ . \qquad (B3)$$

In this system the $N=2$ superconformal algebra is realized by $T(z)$ and[16]

$$J(z)\equiv\frac{i}{\sqrt{3}}\partial_z\phi(z),\quad G(z)\equiv\sqrt{2/3}\,:e^{i\sqrt{3}\phi(z)}:\ ,$$
$$\bar{G}(z)=\sqrt{2/3}\,:e^{-i\sqrt{3}\phi(z)}:\ . \qquad (B4)$$

We can evaluate operator-product expansions of vertex operators using the familiar formula

$$V_a(z)V_b(w)=(z-w)^{ab}:e^{ia\phi(z)+ib\phi(w)}: \qquad (B5)$$

by expanding the second exponential around $z=w$ and keeping the singular terms. Since

$$J(z)V_a(w)=\frac{a}{\sqrt{3}}\frac{V_a(w)}{z-w} \qquad (B6)$$

we can easily establish that $T(z)$, $G(z)$, $\bar{G}(z)$, and $J(z)$ satisfy the standard $N=2$ superconformal algebra (2.14) or (2.22) with $\tilde{c}=\frac{1}{3}$.

Candidates for the lowest components of the primary superfields $\phi_\pm(z)$ with dimension $\frac{1}{6}$ and charge $\pm\frac{1}{3}$ are the vertex operators

$$\phi_+(z)\equiv\,:e^{(i/\sqrt{3})\phi(z)}:\ ,$$
$$\phi_-(z)=\,:e^{-(i/\sqrt{3})\phi(z)}: \qquad (B7)$$

which by (B3) and (B6) have the correct dimension and U(1) charge. We have now to find the superpartners of $\phi_\pm$. Using the relations (3.6) in component form we have

$$G(z)\phi_\pm(w)=\frac{\psi_\pm(w)}{z-w}+\text{nonsingular}\ , \qquad (B8a)$$

$$\bar{G}(z)\phi_\pm(w)=\frac{\bar{\psi}_\pm(w)}{z-w}+\text{nonsingular}\ . \qquad (B8b)$$

Applying (B8a) and (B8b) to (B7) we find

$$\psi_+(z)=0,\quad \bar{\psi}_+(z)=\sqrt{2/3}\,:e^{(2i/\sqrt{3})\phi(z)}:\ ,$$
$$\psi_-(z)=\sqrt{2/3}\,:e^{(2i/\sqrt{3})\phi(z)}:,\quad \bar{\psi}_-(z)=0\ . \qquad (B9)$$

Using then

$$G(z)\psi_\pm(w)=0,\quad \bar{G}(z)\bar{\psi}_\pm(w)=0\ ,$$
$$G(z)\bar{\psi}_\pm(w)=(2\Delta+Q)\frac{\phi_\pm(w)}{(z-w)^2}+\frac{\partial_w\phi_\pm(w)}{z-w}+\frac{g_\pm(w)}{z-w}\ , \qquad (B10)$$

$$\bar{G}(z)\psi_\pm(w)=(2\Delta-Q)\frac{\phi_\pm(w)}{(z-w)^2}+\frac{\partial_w\phi_\pm(w)}{z-w}-\frac{g_\pm(w)}{z-w}\ ,$$

we find that they are satisfied if $g_+(z)=\partial_z\phi_+(z)$ and $g_-(z)=-\partial_z\phi_-(z)$.

The fact that one of the fermionic components is zero and the fourth component is a descendant of the first component explains the group-theoretic result[13] that the family ($\Delta=\frac{1}{6}$, $Q=\pm\frac{1}{3}$) decomposes to the $N=1$ family with $\Delta=\frac{1}{6}$ and half the apparent degrees of freedom.

This means, using our definition (2.8), that $\phi_\pm$ are chiral primary operators of opposite chirality. In fact, looking at (3.11) we can establish that any primary superfield, degenerate at $n_0=\pm\frac{1}{2}$, is chiral in the sense of (2.8a) and (2.8b) and thus contains half the apparent degrees of freedom.

Computing correlation functions of Φ_+ and Φ_- are now trivial. Using

$$\langle 0|\,V_{a_1}(z_1)V_{a_2}(z_2)\cdots V_{a_n}(z_n)\,|0\rangle = \prod_{i<j}^{n}(z_{ij})^{a_i a_j}\delta\left[\sum_i a_i\right]. \tag{B11}$$

We can evaluate the different components of (6.8). Such a correlation is nonzero only if $\sum_i a_i=0$; otherwise IR divergences force it to vanish. Such a calculation has been performed for the four-point function and as expected it agrees with the result (6.8).

Let us also illustrate the situation in the R^+ sector of the model. We have two operators of dimension $\frac{1}{24}$ and charge $\pm\frac{1}{6}$ and two operators of dimension $\frac{3}{8}$ and charge $\pm\frac{1}{2}$. The ground states can be represented by the $\Delta=\frac{1}{24}$ vertex operators:

$$R_-(z)\equiv :e^{(i/2\sqrt{3})\phi(z)}:,\quad R_+(z)\equiv :e^{-(i/2\sqrt{3})\phi(z)}: . \tag{B12}$$

The operator of dimension $\frac{3}{8}$ is represented by $:e^{\pm(i\sqrt{3}/2)\phi(z)}:$. It is easy to see that it is generated from the R vacuum by the action of the $\Delta=\frac{1}{6}$ operators of the NS sector due to the following OPE:

$$:e^{(i/\sqrt{3})\phi(z)}::e^{(i/2\sqrt{3})\phi(w)}: =(z-w)^{1/6}[:e^{(i\sqrt{3}/2)\phi(w)}:+O(z-w)]\ , \tag{B13a}$$

$$:e^{-(i/\sqrt{3})\phi(z)}::e^{-(i/2\sqrt{3})\phi(w)}: =(z-w)^{1/6}[:e^{(i\sqrt{3}/2)\phi(w)}:+O(z-w)]\ . \tag{B13b}$$

The two-point function

$$F_2(\mathbf{z}_1,\mathbf{z}_2)\equiv\frac{\langle R_-(\infty)\Phi^+_{1/6}(\mathbf{z}_1)\Phi^-_{1/6}(\mathbf{z}_2)R_+(0)\rangle}{\langle R_-(\infty)R_+(0)\rangle} \tag{B14}$$

can be computed using vertex operator techniques, giving the result derived in Sec. V.

[1]A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).

[2]M. Lüscher and G. Mack (unpublished); I. T. Todorov, Bulg. J. Phys. **12**, 1 (1985).

[3]D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. **B271**, 93 (1986).

[4]D. Friedan, Z. Qiu, and S. Shenker, in *Vertex Operators in Mathematics and Physics,* edited by J. Lepowsky (Springer, New York, 1984); EFI Report No. EFI-86-19 (unpublished).

[5]D. Friedan, Z. Qiu, and S. Shenker, Phys. Lett. **151B**, 37 (1985).

[6]M. J. Tejwarni, O. Ferreira, and O. E. Vilches, Phys. Rev. Lett. **44**, 152 (1980); W. Kinzel, M. Schick, and A. N. Berker, *Ordering in Two Dimensions,* edited by R. Sinha (North-Holland, Amsterdam, 1980).

[7]V. Knizhnik, Phys. Lett. **160B**, 403 (1985).

[8]J. Cohn, D. Friedan, Z. Qiu, and S. Shenker, Nucl. Phys. **B278**, 577 (1986).

[9]A. B. Zamolodchikov and V. A. Fateev, Zh. Eksp. Teor. Fiz. **90**, 1553 (1986) [Sov. Phys. JETP **63**, 913 (1986)].

[10]W. Boucher, D. Friedan, and A. Kent, Phys. Lett. B **172**, 316 (1986).

[11]P. DiVecchia, J. L. Petersen, and H. B. Zheng, Phys. Lett. **162B**, 327 (1986); P. DiVecchia, J. L. Petersen, and M. Yu, Phys. Lett. B **172**, 211 (1986); P. DiVecchia, J. L. Petersen, M. Yu, and H. B. Zheng, *ibid.* **174**, 280 (1986).

[12]S. Nam, Phys. Lett. B **172**, 323 (1986).

[13]E. Kiritsis, Caltech Report No. CALT-68-1347 (unpublished); V. K. Dobrev, Phys. Lett. B **186**, 43 (1987); Y. Matsuo, Prog. Theor. Phys. **77**, 793 (1987).

[14]M. Ademolo *et al.*, Phys. Lett. **62B**, 105 (1976); Nucl. Phys. **B111**, 77 (1976).

[15]P. Candelas, G. Horowitz, A. Strominger, and E. Witten, Nucl. Phys. **B258**, 46 (1985); C. Hull and E. Witten, Phys. Lett. **160B**, 398 (1985).

[16]G. Waterson, Phys. Lett. B **171**, 77 (1986).

[17]S. K. Yang, Nucl. Phys. **B285**, 183 (1987).

[18]J. L. Petersen, in *Proceedings of the 19th International Symposium: Special Topics in Gauge Field Theories,* Ahrenshoop, Germany, 1985 (Akademie der Wissenschaften der DDR, Zeuthen, East Germany, 1985), p. 1.

[19]L. Crane and J. Rabin, EFI Report No. EFI-86-25 (unpublished).

[20]P. DiVecchia, J. L. Petersen, M. Yu, and G. B. Zheng, Phys. Lett. B **174**, 280 (1986); M. Yu and H. B. Zheng, Nucl. Phys. **B288**, 275 (1987).

[21]V. S. Dotsenko, Nucl. Phys. **B235** [FS11], 54 (1984).

[22]M. A. Bershadsky, V. G. Knizhnik, and M. G. Teitelman, Phys. Lett. **151B**, 31 (1985); H. Eichenherr, *ibid.* **151B**, 26 (1985).

[23]Z. Qiu, Nucl. Phys. **B270** [FS16], 205 (1986).

[24]G. M. Sotkov, I. T. Todorov, and V. Yu. Trifonov, Lett. Math. Phys. **12**, 127 (1986); G. M. Sotkov and M. S. Stanishkov, Sofia Report No. INRNE-4/86 (unpublished).

[25]A. Schwimmer and N. Seiberg, Phys. Lett. B **184**, 191 (1987).

[26]M. Kato and S. Matsuda, Phys. Lett. B **184**, 184 (1987); E. Kiritsis (unpublished).

[27]J. Cohn, Nucl. Phys. **B284**, 349 (1987).

[28]Z. Qiu, Phys. Lett. B **188**, 207 (1987).

[29]D. Gepner and E. Witten, Nucl. Phys. **B278**, 493 (1986).