Structure Formation

An important issue is the generation of inhomogeneities (clusters, galaxies, stars) at short distance scales in the universe)

This is a difficult subject! Deviations from symmetry are much harder to analyse than the symmetric phase itself.

There are many possible forces that could amplify tiny inhomogeneities, but only one natural one: gravity. Gravity is always attractive. Thus, it can give rise to a runaway process of amplification (gravitational instability).

Naive Newtonian treatment

Consider deviations from homogeneity $\delta = (\rho - \bar{\rho})/\bar{\rho}$. The acceleration of a small mass adjacent to an overdense region (density $\bar{\rho} + \delta\rho$) is (we assume spherical symmetry)

$$\ddot{r} = -G\frac{\delta M}{r^2} = -\frac{4\pi}{3}\, G\ \delta\ \rho\ r$$

Consider a bigger region of radius R that, if the density was average, it would contain the same ammount of matter as in the overdense region. Let $r = R(1-\epsilon)$ with $\epsilon << 1$.

$$(\bar{\rho} + \delta\rho)r^3 = \bar{\rho}R^3 \quad \to \quad \delta \simeq 3\epsilon$$

Then Newton's equation becomes

$$\ddot{\delta} = 4\pi\bar{\rho}\ \delta \quad \to \quad \delta(t) \sim \delta_0\ e^{\sqrt{4\pi\bar{\rho}}\ t}$$

FRW treatement

This is the Jeans instability with characteristic time

$$t_{dyn} \sim \bar{\rho}^{-1/2}$$

Simplistic FRW treatment

Here again we assume that there is a "local" rate of expansion of the scale factor a and an average one $\bar{a}$. From

$$\ddot{a} = -\frac{4\pi}{3}(1+3w)\ \rho\ a \quad , \quad \ddot{\bar{a}} = -\frac{4\pi}{3}(1+3w)\ \bar{\rho}\ \bar{a}$$

$$a = \bar{a}(1-\epsilon) \quad , \quad \rho\ a^{3(1+w)} = \bar{\rho}\ \bar{a}^{3(1+w)}$$

We obtain

$$\ddot{\epsilon} + 2\frac{\dot{\bar{a}}}{\bar{a}}\dot{\epsilon} = \frac{16\pi}{3}\bar{\rho}(1+3w) \quad , \quad \delta = 3(1+w)\epsilon$$

so that

$$\ddot{\delta} + 2\frac{\dot{\bar{a}}}{\bar{a}}\dot{\delta} = 4\pi(1+w)(1+3w)\bar{\rho}\ \delta$$

There is a friction-like term the Hubble drag due to the expansion of the universe.

The analysis above is sketchy. A better treatement can be done using perturbation theory of the Einstein equations.

The Vlasov/Euler equations

One line of attack comes from the Liouville theorem: we expect that interactions will become important only when matter becomes dense enough (or the perturbation "goes non-linear"). Thus, Liouville's theorem is valid during the initial stages (small fluctuations, small velocities)

It says that the phase space density $f(t,\vec{x},\vec{v})$ is constant or

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt}\cdot\frac{\partial f}{\partial \vec{x}} + \frac{d\vec{v}}{dt}\cdot\frac{\partial f}{\partial \vec{v}}$$

and using the dynamical equations we obtain Vlasov's equation

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{x}} + \vec{\nabla}\phi\cdot\frac{\partial f}{\partial \vec{v}} = 0$$

We can obtain Euler-type equations from Vlasov's equation. We use the following definition

$$\rho(t,\vec{x}) = \int d^3v \; f(t,\vec{x},\vec{v})$$

and taking moments we obtain

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\phi - \frac{1}{\rho}\vec{\nabla}p$$

together with

$$\nabla^2\phi = 4\pi G\rho$$

Jeans' theory

Consider small perturbations $\rho_1, p_1, \vec{v}_1$ and ϕ_1 to the Euler equations around the stationary cosmic fluid $\rho, p =$ constant, and $\vec{v} = 0$

$$\frac{\partial \rho_1}{\partial t} + \rho \vec{\nabla} \cdot (\vec{v}_1) = 0$$

$$\frac{\partial \vec{v}_1}{\partial t} = -\frac{v_s^2}{\rho} \vec{\nabla} \rho_1 - \vec{\nabla} \phi_1 \quad , \quad \nabla^2 \phi_1 = 4\pi G \rho_1$$

and $v_s^2 = p_1/\rho_1 = \left(\frac{\partial p}{\partial \rho}\right)_{\text{adiabatic}}$ is the speed of sound.

We obtain the following differential equation

$$\frac{\partial^2 \rho_1}{\partial t^2} = v_s^2 \nabla^2 \rho_1 + 4\pi G \rho \rho_1$$

with a solution

$$\rho_1 \sim e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

and dispersion relation

$$\omega^2 = v_s^2 \vec{k}^2 - 4\pi G \rho$$

For $k < k_J = \frac{\sqrt{4\pi G \rho}}{v_s}$ there is an instability: ω is imaginary and ρ_1 grows exponentially with time.

This is strictly valid in a static universe.

Expanding universe modifications

In an FRW universe:

$$\frac{\dot{a}}{a} \simeq \sqrt{\frac{8\pi\rho G}{3}} = \sqrt{2/3}\ k_J\ v_s$$

and the expansion rate is of the same order as the time scale of the instability.

For large k the waves are ordinary sound waves and cannot lead to instability. The conditions for this are:

• Gravitational forces are negligible: the gravitational energy in a sphere of radius k^{-1} is smaller than its thermal energy

$$\frac{G(\rho k^{-3})^2}{k^{-1}} << \rho\ v_s^2\ k^{-3} \quad \to \quad G\rho << v_s^2 k^2$$

• The expansion rate is much smaller than the frequency $\omega \simeq v_s\ k$

$$\sqrt{G\rho} << \omega$$

Both conditions are valid for $k >> k_J$.

Expanding universe modifications

Even in the expanding universe, there is a critical wave number k_J above which perturbations do not grow but oscillate.

Since $k \sim 1/a$ we must define a better measure of the instability onset.

This is the Jeans' mass = the mass in a sphere of radius k_J^{-1} which is constant

$$M_J = \frac{4\pi n m_H}{3}\left(\frac{2\pi}{k_j}\right)^3 \simeq \frac{4\pi n m_H}{3}\left(\frac{\pi v_s^2}{G(\rho+p)}\right)^{3/2}$$

where n is the Hydrogen number density.

Fluctuations with masses larger than the Jeans' mass are unstable and grow

Full Newtonian theory

There is a solution to the Euler equations

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0 \quad , \quad \frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\vec{\nabla}\phi-\frac{1}{\rho}\vec{\nabla}p$$

$$\rho=\rho_0\left(\frac{a_0}{a}\right)^3 \quad , \quad \vec{v}=\vec{r}\,\frac{\dot{a}}{a} \quad , \quad \vec{\nabla}\phi=\frac{4\pi G\rho}{3}\,\vec{r}$$

where the scale factor satisfies

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}\rho-\frac{k}{a^2}$$

We now search for small perturbations $\rho_1, \vec{v}_1, \phi_1$ of the solution above. They must satisfy:

$$\dot{\rho}_1+3\frac{\dot{a}}{a}\rho_1+\frac{\dot{a}}{a}(\vec{r}\cdot\vec{\nabla})\rho_1+\rho\vec{\nabla}\cdot\vec{v}_1=0$$

$$\dot{\vec{v}}_1+\frac{\dot{a}}{a}\vec{v}_1+\frac{\dot{a}}{a}(\vec{r}\cdot\vec{\nabla})\vec{v}_1=-\frac{1}{\rho}\vec{\nabla}p_1-\vec{\nabla}\phi_1$$

$$\nabla^2\phi_1=4\pi G\rho_1 \quad , \quad p_1=v_s^2\;\rho_1$$

Due to spatial homogeneity we expect solutions of the form $\rho_1(\vec{r},t)=\rho_1(t)e^{i\vec{r}\cdot\vec{k}/a(t)}$ and similarly for $\vec{v}_1, \phi_1$. We obtain

$$\dot{\rho}_1+3\frac{\dot{a}}{a}\rho_1+\frac{i}{a}\rho\vec{k}\cdot\vec{v}_1=0$$

$$\dot{\vec{v}}_1+\frac{\dot{a}}{a}\vec{v}_1=-\frac{iv_s^2}{\rho a}\vec{k}\rho_1-\nabla\phi_1 \quad , \quad \phi_1=\frac{4\pi G a\rho_1}{\vec{k}^2}$$

Full Newtonian theory

Split the velocity into parallel and perpendicular components

$$\vec{v}_1 = \vec{v}_{1\perp} + i\vec{k}\epsilon(t) \quad , \quad \vec{k}\cdot\vec{v}_{1\perp} = 0 \quad , \quad \epsilon = -i\frac{\vec{k}\cdot\vec{v}_1}{\vec{k}^2}$$

and define the fractional density $\delta(t) = \rho_1(t)/\rho(t)$. The equations become

$$\dot{\vec{v}}_{1\perp} + \frac{\dot{a}}{a}\vec{v}_{1\perp} = 0$$

$$\dot{\epsilon} + \frac{\dot{a}}{a}\epsilon = \left(-\frac{v_s^2}{a} + \frac{4\pi G\rho a}{\vec{k}^2}\right)\delta \quad , \quad \dot{\delta} = \frac{\vec{k}^2}{a}\epsilon$$

The rotational modes corresponding to $\vec{v}_{1\perp}$ die off a $1/a$ and will not be considered further.

Eliminating ϵ we obtain

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\delta + \left(\frac{v_s^2\vec{k}^2}{a^2} - 4\pi G\rho\right)\delta = 0$$

This becomes the Jeans' equation if $a \to$constant and we redefine $\vec{k} \to a\vec{k}$.

It is applicable at the onset of the matter dominated era. No general solution exists but several special solutions.

(see Weinberg: Gravitation and Cosmology)

Solution of zero-pressure case

A galaxy is suposed to have grown out of small fluctuations present at recombination (left over from previous growth).

How much such fluctuations have grown till today? Did they become galaxies?

For $k << k_J$ ($M >> M_J$) we can neglect the pressure term $v_s^2\vec{k}^2/a^2$.

We have seen that after recombination, the Jeans' mass is very low ($\sim 10^6 M_\odot$) and drops further as $a^{-3/2}$, so this is a good approximation. We need (we assume k=0)

$$\frac{a(t)}{a_0} = \left(\frac{3H_0 t}{2}\right)^{2/3} \quad , \quad \rho = (6\pi G t^2)^{-1}$$

for this period. Then the equation becomes:

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

with solutions

$$\delta = \delta_+ + \delta_- = C_+ \; t^{2/3} + \frac{C_-}{t}$$

In that case the amplification factor $\delta_+(\text{today})/\delta_+ = 1 + z \sim 10^3$. This implies that there is enough time for perturbations to grow although the equations are invalid for $\delta \geq 1$.

We find that growth is powerlike unlike the exponential growth in a static universe.

Problem 1: Derive the general fluctuation equations for arbitrary matter density and arbitrary expansion rate.

Problem 2: Show that during radiation domination $\delta \sim t$.

Relativistic treatement

This is needed to handle lepton dominated eras, or any time that p is comparable to ρ. Gravitational radiation should always be treated that way.

The theory is very complicated and we will just present the final equations.

If δ_w is the fluctuation in the matter component with coefficient w then

$$\ddot{\delta}_w + 2\frac{\dot{a}}{a}\dot{\delta}_w - \frac{v_s^2}{a^2}\nabla^2\delta_w = 4\pi G(1+w)\sum_\nu \rho_\nu\delta_\nu(1+3\nu) - \frac{3}{2}w(1+w)\frac{\dot{a}}{a}\dot{h}$$

where h is the trace of the perturbation to the metric.

The first is the growth term, the second is the friction term (Hubble drag) and the third is the source term. We can use $a(t)$ instead of t as "time" variable:

Problem 3: Derive

$$\delta'' + \frac{\sum_\nu(1-\nu)\rho_\nu}{\sum_\nu \rho_\nu}\frac{3}{2a}\delta' = \frac{3(1+w)}{2a^2}\frac{\sum_\nu(1+3\nu)\rho_\nu\delta_\nu}{\sum_\nu \rho_\nu} - \frac{3w(1+w)}{2a}h'$$

where prime stands for $\frac{\partial}{\partial a}$.

The General Case

It is difficult to have an analytic solution in the general case.

The fluctuations can be computed though by the method of "varying constant" This implies that

$$\frac{d\log a}{dk} \quad \text{and} \quad \frac{d\log a}{dt_0}$$

are good fluctuation solutions. Using the Friedman equation

$$\dot{a}^2 = \Omega \ H_0^2 a^2 - k$$

we obtain that

$$\delta \sim \frac{\dot{a}}{a}\int_0^a \frac{da}{\dot{a}^3} \qquad \text{growing mode}$$

and

$$\delta \sim \frac{\dot{a}}{a} \qquad \text{decaying mode}$$

which solve

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi[(\rho + p) - \frac{a}{\dot{a}}\dot{p}]\delta$$

Explicit formulae can be obtained only by numerical integration. A very good approximation is provided by

$$\frac{\delta(z=0,\Omega)}{\delta(z=0,\Omega=1)} \simeq \frac{5}{2}\Omega_m \left[\Omega_m^{4/7} - \Omega_V + \left(1+\frac{1}{2}\Omega_m\right)\left(1+\frac{\Omega_V}{70}\right)\right]^{-1}$$

where Ω_V is the vaccum energy.

The General Case

When w-matter dominates the expansion, the equation for fluctuations

$$\ddot{\delta}_w + 2\frac{\dot{a}}{a}\dot{\delta}_w = 4\pi(1+w)(1+3w)\rho\delta_w$$

becomes

$$\ddot{\delta}_w + \frac{4}{3(1+w)t}\dot{\delta}_w - \frac{2(1+3w)}{3(1+w)t^2}\delta_w = 0$$

with general solution

$$\delta = C_1 \; t^{\frac{2(1+3w)}{3(1+w)}} + \frac{C_2}{t}$$

Thus, the growing mode has the following behavior when w-matter dominates the expansion

$$a(t) \sim t^{\frac{2}{3(1+w)}} \quad , \quad \delta_w \sim t^{\frac{2(1+3w)}{3(1+w)}}$$

or

$$\delta_w \sim a^{1+3w} + \text{constant}$$

In the presence of $w < 0$ components during w=0 domination

$$\frac{\delta_f}{\delta_i} = \frac{a_f}{a_i}\left[1 + \frac{1-w}{w(5-6w)}\frac{\Omega_w}{\Omega_0}a^{-3w}\right]$$

At late times this cuts off growth.

Thus when $w < -1/3$ perturbations do not grow.

Collisionless Matter and Radiation

We will assume that radiation dominates the expansion of the universe. Also matter and radiation do not interact ("wimpy" matter). We are looking also the region where radiation wavelengths are inside the horizon (below the Jeans' limit so they cannot grow). Thus, matter fluctuates alone while radiation drives the expansion: The equations are

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}=4\pi\rho_m\delta \quad , \quad H^2=\frac{8\pi}{3}(\rho_m+\rho_r)$$

which translate into ($\delta_r=0$)

$$\delta''+\frac{3\rho_m+2\rho_r}{3(\rho_m+\rho_r)}\frac{3}{2a}\delta'=\frac{3}{2a^2}\frac{\rho_m\delta}{\rho_m+\rho_r}$$

Using the variable $y=\rho_m/\rho_r=a/a_{eq}$ we obtain

$$\delta''+\frac{2+3y}{2y(1+y)}\delta'-\frac{3}{2y(1+y)}\delta=0$$

The hypergeometric equation has a simple growing mode $\delta\sim y+\frac{2}{3}$ and a logarithmic second solution. In a radiation-dominated universe $y<<1$ and perturbations cannot grow.

In general for fluctuations in $w>0$ we can obtain (see Problem)

$$\frac{\delta_f}{\delta_i}=1+\frac{1}{w(1+3w)}\frac{\Omega_0}{\Omega_w}a^{3w}$$

This gives a ratio of order one if w-matter dominates. No structure can be formed during radiation domination.

Coupled Perturbations

We will consider perturbations in non-relativistic matter coupled to w-matter. The resulting system of equations is

$$\delta_m''+\frac{(6w+1)z+3w+1}{6w\ z(z+1)}\delta_m'=\frac{1}{6w^2}\frac{z\delta_m+(1+3w)\delta_w}{z^2(z+1)}$$

$$\delta_w''+\frac{(6w+1)z+3w+1}{6w\ z(z+1)}\delta_w'=\frac{(1+w)}{6w^2}\frac{z\delta_m+(1+3w)\delta_w}{z^2(z+1)}$$

where as usual $y=\rho_m/\rho_w$.

The system can be diagonalized by noting that the "adiabatic" combination $\zeta=\delta_m-\frac{\delta_w}{(1+w)}$ satisfies

$$\zeta''+\frac{(6w+1)z+3w+1}{6w\ z(z+1)}\zeta'=0$$

with general solution

$$\zeta=C_1+C_2\int_{z_0}^{z}du\ u^{\frac{3w+1}{6w}}\sqrt{1+u}$$

• $C_1=C_2=0$. This corresponds to initially locked perturbations (due to potentially tight coupling between matter and w-matter) Thomson scattering in radiation case is an example. Then,

$$\delta_m''+\frac{(6w+1)z+3w+1}{6w\ z(z+1)}\delta_m'-\frac{1}{6w^2}\frac{z+(1+w)(1+3w)}{z^2(z+1)}\delta_m=0$$

Coupled Perturbations

Define $\delta_m = z^{(3w+1)/3w}\vartheta$ and $z \to -z$: we obtain the hypergeometric equation,

$$\vartheta'' + \frac{5(3w+1) - (18w+5)z}{6w\ z(1-z)}\vartheta' - \frac{6w+5}{6w}\frac{\vartheta}{z(1-z)} = 0$$

For w-matter driving the expansion, $z << 1$ and the two independent solutions are

$$\delta_1 \sim z^{(3w+1)/3w} \quad , \quad \delta_2 \sim z^{-(w+1)/2}$$

- $w > 0$: In this case z grows with time. The growing mode is δ_1.

- $w < 0$: Here z decreases with time. For $w > -0.24$ δ_1 is still the growing mode while for $-.24 > w. - 1$, δ_2 is the growing mode. At $w = -1$, growth stops.

When non-relativistic matter drives the expansion, $z >> 1$.

The two independent solutions behave as

$$\delta_1 \sim z^{-\frac{1}{2w}} \quad , \quad \delta_2 \sim z^{\frac{1}{3w}}$$

- $w > 0$: δ_2 is the growing mode.

- $w < 0$: δ_2 is still the growing mode.

Isocurvature Perturbations

• $C_1 \neq 0,\ C_2 = 0$. Whatever initial conditions we choose, further evolution is adiabatic. We can set

$$\delta_w = (1+w)(\delta_m - \delta_i)$$

The relevant equation now picks an inhomogeneous term

$$\delta''_m + \frac{(6w+1)z+3w+1}{6w\ z(z+1)}\delta'_m - \frac{1}{6w^2}\frac{z+(1+w)(1+3w)}{z^2(z+1)}\delta_m = J$$

$$J = -\frac{(1+w)(1+3w)}{6w^2\ z^2(z+1)}\delta_1$$

There is an exact particular solution for w=1/3 (radiation):

$$\frac{\delta_m}{\delta_i} = 1 - \frac{z}{2} \quad , \quad \frac{\delta_r}{\delta_i} = -\frac{2}{3}z$$

This mode is decreasing for δ_m while increasing for radiation.

Existing perturbations are washed out from the matter sector and carried into the radiation sector.

Baryons and Dark Matter

In this case we have two decoupled equations

$$\left(\partial_t^2 + 2\frac{\dot{a}}{a}\partial_t\right)(\delta_b - \delta_d) = 0$$

$$\left(\partial_t^2 + 2\frac{\dot{a}}{a}\partial_t - 4\pi(\rho_d + \rho_b)\right)\left(\delta_b + \frac{\rho_d}{\rho_b}\delta_d\right) = 0$$

During matter domination we obtain

$$\delta_b - \delta_d = C_1 + C_2\ t^{-1/3} \quad , \quad \delta_b + \frac{\rho_d}{\rho_b}\delta_d = D_1 t^{2/3} + \frac{D_2}{t}$$

If we start with $\delta_b = 0$ since $\frac{\delta_b - \delta_d}{\delta_b + \delta_d} \to 0$ we will obtain $\delta_b \simeq \delta_d$.

Baryonic matter falls into the potential wells of dark matter

This is a mechanism in order that seeds of structure formation produce smaller CMBR anisotropies.

Baryons and Dark Matter:Full treatment

Perturbation theory of Einstein equations

• Vector modes die out

• Gravitational waves dilute away (but may be important during inflationary reheating)

• Scalar perturbations relevant for structure formation.

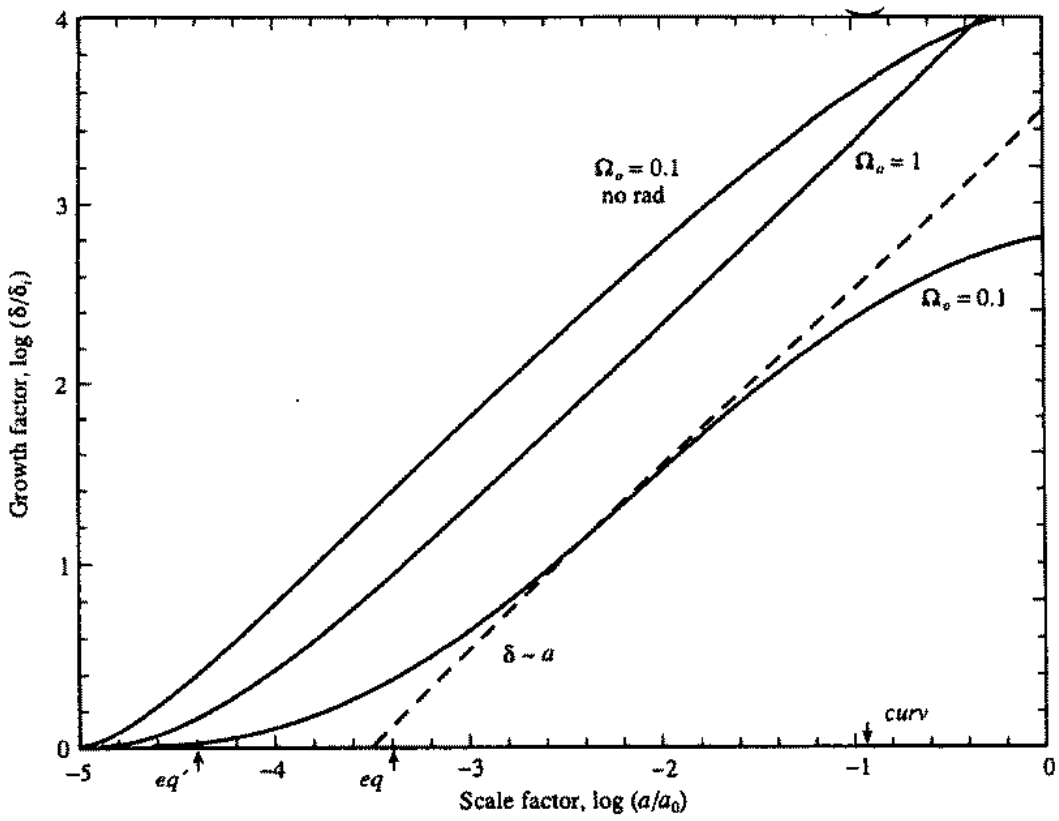
Use:

• Gauge invariant variables

• Boltzman equation for various components (including interactions).

• In realistic cases numerical integration is needed.

Growth Curves



The growth of linear density perturbations with scale factor, showing the effects of radiation at early times and curvature at late times. In the matter-dominated regime the density contrast grows as a. The $\Omega_0 = 0.1$ curve shows the effects of curvature. The perturbations do not immediately cease. The curve labeled "no-rad" takes $\Omega_{1/3} = 0$. No radiation-matter coupling effecta are taken into account, only effects of the radiation energy density on the expansion rate, so these curves are essentially for WIMPS.

Summary

We have seen that:

• Perturbations of total mass bigger than the Jeans' mass can efficiently grow (gravity drives them).

• Perturbations of total mass smaller than the Jeans' mass behave like sound and do not lead to structure (pressure drives them). It is interesting to note that the invariant wavelength of such perturbations is larger than the horizon during the radiation era

$$\lambda a > v_s \frac{\pi}{\rho} \simeq \sqrt{\frac{8}{3}} \frac{\pi v_s}{H} \sim \mathcal{O}(H^{-1})$$

• When matter is tightly coupled with radiation, the perturbations are adiabatic. They grow as $\delta \sim a^2$ during the radiation era, but must be bigger than the horizon. They are irrelevant for todays' structure.

• When matter is not coupled with radiation, during the radiation era, then perturbations cannot grow (the universe expands too fast). The above implies that structure cannot be generated during the radiation era.

• Density perturbations in dark matter can drive similar perturbations in baryonic matter just because of their gravity. This is a concrete popular mechanism for transfering earlier perturbations to the visible universe

• When dark matter is hot, free streaming will erase structure up to the horizon. This will cease when particles become non-relativistic (Cold DM). For light massive neutrinos this happens at z_{eq}. Thus all structure up to the horizon break is erased.

A more careful treatment

From $e^+ - e^-$ era ($T \sim 10^{10o}K$) to recombination $\sim 4000^o K$ it is a good approximation to treat the contents of the universe as nonrelativistic ionized Hydrogen and blackbody radiation.

Since the photon entropy is very large we can neglect the pressure, thermal energy and entropy of matter.

$$\rho = n m_H + \alpha\ T^4 \quad , \quad p = \frac{1}{3}\alpha T^4 \quad , \quad \sigma = \frac{4\alpha T^3}{3 n k_B}$$

During adiabatic perturbations σ is constant (and $n \sim T^3$) so

$$\delta\rho = (3 n m_H + 4\alpha T^4)\frac{\delta T}{T} \quad , \quad \delta p = \left(\frac{4}{3}\alpha T^4\right)\frac{\delta T}{T}$$

Jean's mass calculation

Thus

$$v_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\text{adiabatic}} = \frac{1}{3}\left(\frac{k_B T \sigma}{m_H + k_B T \sigma}\right)$$

$$M_J = \frac{2\pi^{5/2} k_B^2 \sigma^2}{9\sqrt{\alpha} m_H^2 G^{3/2}(1 + \sigma k_B T/m_H)^3} = \frac{9.06 \ M_\odot \sigma^2}{(1 + \sigma k_B T/m_H)^3}$$

After recombination, radiation decouples and we have a mono-atomic ideal gas with

$$\rho = n m_H + \frac{3}{2} n k_B T \quad , \quad p = n k_B T \quad , \quad \sigma = n k_B \log(V T^{3/2})$$

$$v_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\text{adiabatic}} = \frac{5}{3}\frac{k_B T}{m_H}$$

and the Jeans' mass has the value

$$M_J = 4\left(\frac{\pi}{3}\right)^{5/2}\left(\frac{5 k_B T}{G}\right)^{3/2} \ n^{-1/2} m_H^{-2}$$

At recombination T is the photon temperature and this gives $M_J = 102 M_\odot \sigma^{1/2}$.

From that point on, since $T \sim a^{-2}$, $n \sim a^{-3}$ and $M_J \sim a^{-3/2}$

History of perturbations

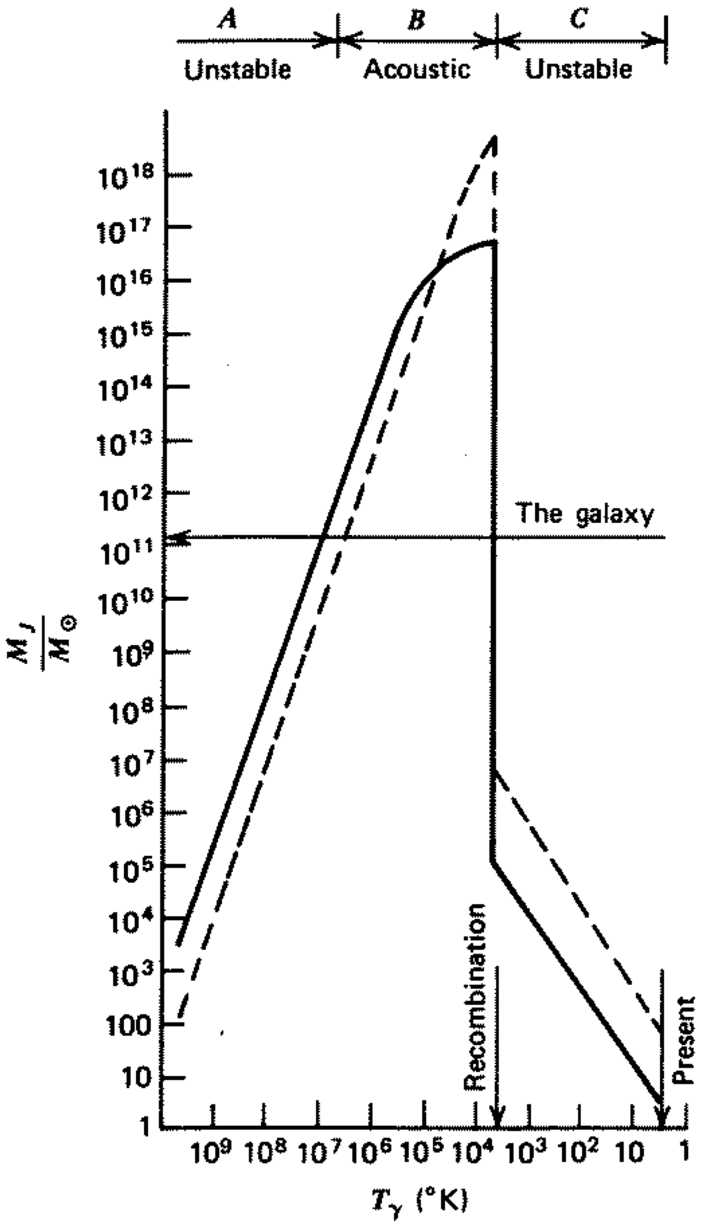
• The Jeans' mass is less than the galactic mass ($\sim 10^{11} M_\odot$) untill the temperature falls to $10^7\ {}^oK$. In this region, energy is dominated by radiation and the problem is relativistic. It can be shown that for perturbations $\delta\rho/\rho \sim t$.

• From that time till recombination, the Jeans' mass grows much larger than the galactic mass. No growth of perturbations can happen in that period.

• After recombination, Jeans' mass becomes again low so perturbations can again grow as $\delta\rho/\rho \sim t^{2/3}$

This makes us expect that we must have a smooth distribution of galaxies with masses ranging up to $10^{17}\ M_\odot$. The observational upper bound is $10^{12}\ M_\odot$. There is no explanation so far (but non-linearities can transfer energy to shorter wavelegths were dissipations is less).

History of perturbations



Jeans' mass as a function of radiation temperature. The solid line is for $\sigma = 0.8 \times 10^8$ corresponding to $T_\gamma = 2.7^o K$, $\rho_0 = 3\times 10^{-29} gr/cm^3$. Dashed line is for $\sigma = 2.4\times 10^9$ corresponding to $T_\gamma = 2.7^o K$, $\rho_0 = 10^{-30} gr/cm^3$. The drop in the Jeans' mass at recombination is somewhat more gradual than shown here.

Transfer Functions

Definition:

$$T_k \equiv \frac{\delta_k(z=0)}{\delta_k(z)D(z)}$$

where $D(z)$ is the linear growth factor between redshift z and today.

There are two ways in which the power spectrum today may differ from that before.

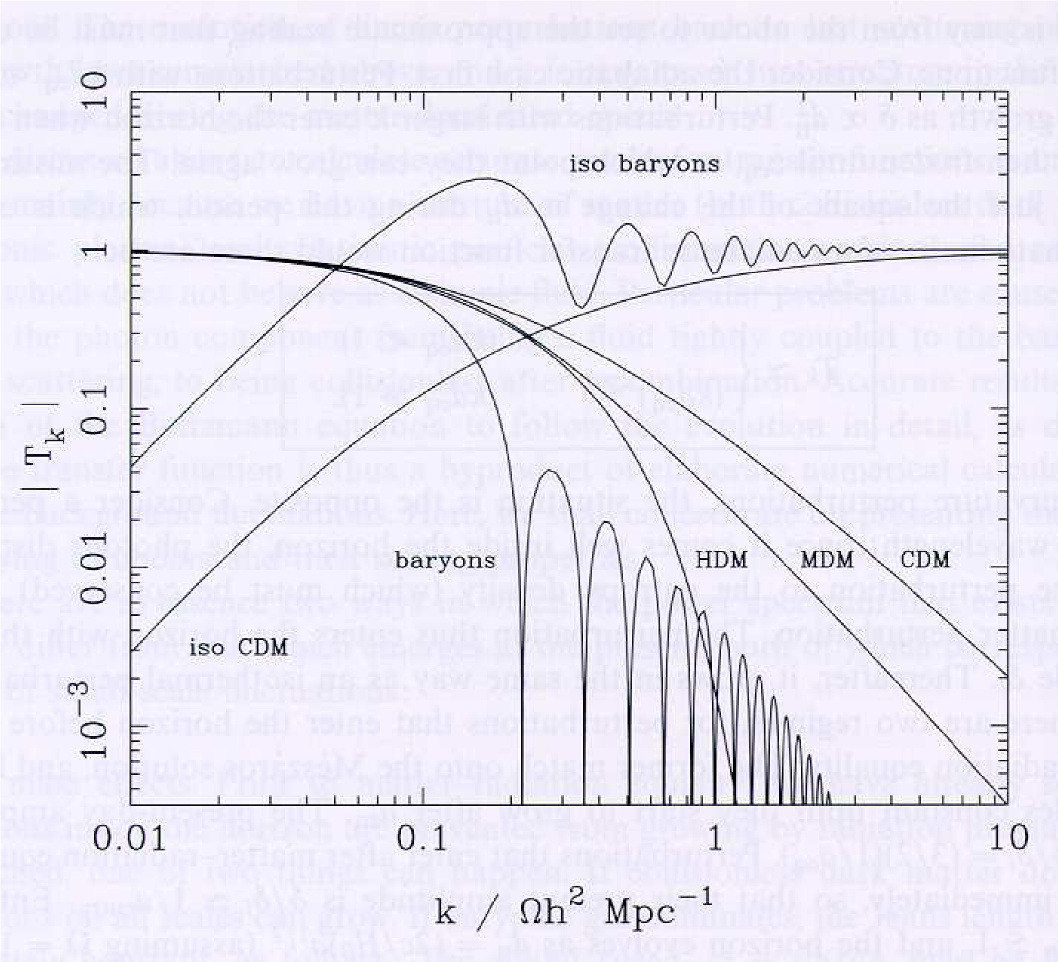
• Jeans' mass effects. After z_{eq} if collisionless dark matter dominates then perturbations on all scales can grow. If baryonic matter dominates the Jeans mass is approximately constant: with a comoving Jeans legth $\lambda_J = 50(\Omega_b h^2)^{-1}$ Mpc. Another critical length scale is the horizon at matter-radiation equality $d_{eq} = 16(\Omega_h^2)^{-1}$ Mpc.

▴ Adiabatic case. Perturbations with $\lambda >> d_{eq}$ grow as a^2. Small wavelegths enter the horizon at $\lambda \sim a$ and then freeze until z_{eq} when they can start growing again. They miss a growth factor which is the square of the change in a, $\sim k^2$ Thus, $T_k = 1$ when $kd_{eq} << 1$ and $T_k = (kd_{eq})^{-2}$ for $kd_{eq} >> 1$.

▴ Isocurvature perturbations. The situation here is the opposite. $T_k \simeq (2/15)(kc/H_0)^2$ when $kd_{eq} << 1$ and $T_k = 3/2a_{eq}^{-1}$ for $kd_{eq} >> 1$.

• Damping. Small scale fluctuations appart from growth inhibition can be erased completely. This can happen in two ways. For collisionless dark matter the mechanism is free streaming: random particle velocities cause the perturbation to disperse. when $kT \geq mc^2$ this certainly happens. It will stop when particles become non-relativistic. For massive particle this happens long before z_{eq} (CDM) For massive neutrinos around z_{eq} (HDM)

Transfer Plots



A plot of transfer functions for various models. For adiabatic models $T_k \rightarrow 1$ at small k, whereas the oppposite is true for isocurvature models. A number of possible matter contents are illustrated: pure baryons;, pure CDM; pure HDM; MDM (30% HDM, 70% CDM). For dark matter models the characteristic wave number scales as Ωh^2. The scaling for baryonic models does not obey this exactly. The plotted cases correspond to $\Omega = 1$, $h = 0.5$.

Non-linear evolution

N-body models The matter in the universe is approximated with N particles that move under the influence of their gravitational field. The system is solved numerically by calculating the gravitational field, then the change in velocities and then back again.

The Zeldovich approximation. Use the following ansatz for the motion of particles

$$\vec{x}(t) = a(t)\vec{q} + b(t)\vec{f}(\vec{q})$$

The first term is the Hubble flow. The perturbation is assumed factorized. $\vec{q}$ are the comoving coordinates. In terms of them

$$\frac{\rho}{\rho_0} = det\left[1 + \frac{b}{a}\{\partial_j f^i\}\right]$$

Collapse takes place first around the largest negative eigenvalue of $\{\partial_j f^i\}$.

Non-linear evolution (cont.)

Gravity accentuates asphericity $\mapsto$ pancakes. If the original mode is growing then $\vec{f}$ is irrotational so $\vec{f} = \vec{\nabla}\psi$. Linearizing the density and the comoving velocity:

$$\delta = -\frac{b}{a} Tr[\partial_i \partial_j \psi] \equiv -\frac{b}{a} \vec{\nabla} \cdot \vec{f}$$

$$\vec{u} = \frac{1}{a}\left(\dot{\vec{x}} - \frac{\dot{a}}{a}\vec{x}\right) = \left(\frac{\dot{b}}{a} - \frac{b\dot{a}}{a^2}\right)\vec{f}$$

which satisfies the conservation equation $\vec{\nabla} \cdot \vec{u} = -\dot{\delta}$. Using the Friedman equations and the equation for δ we obtain

$$\frac{\ddot{b}}{b} = -2\frac{\ddot{a}}{a} = -\frac{8\pi}{3}\rho_0$$

which implies $b \sim t^{4/3}$ when $\Omega = 1$.

The spherical Model: Approximates perturbations as mini-spherical universes

Statistical measures

We think that the generation of perturbations comes from a statistical process and thus makes sence to consider the correlations of $\delta(x)$ at various positions.

- Do we have an ensemble?
- How can we make correlators $\mapsto$ Ergodic hypothesis

ensemble average $\simeq$ volume average

Thus, the data are <u>connected</u> correlators. Since $<\delta>=0$ the first non-trivial case is the 2-point correlator $\langle\delta(\vec{x})\delta(\vec{y})\rangle$ which because of rotational and translational invariant depends only on $|\vec{x}-\vec{y}|$.

One can consider correlations in the distibution of number densities. They depend on the probability to find one clump next to another. The probability to find a galaxy in a volume dV_1 is $dP_1 = n\ dV_1$ where is the number density. If the galaxies are uncorrelated the probability to also find another one in volume dV_2 is $dP_{12} = n^2 dV_1 dV_2$. Deviations from this are due to correlations and we define the correlation function as

$$dP_12 = n^2 dV_1 dV_2[1+\xi_{gg}(x_{12})]$$

Thus, the probability that there is another galaxy at distance r froma given galaxy is $dP = 4\pi n r^2 dr[1+\xi_{gg}(r)]$.

For $\xi > 0$ we have clustering. The overdensity of galaxies in a sphere of radius r is

$$\delta N = 4\pi n \int_0^r r^2 dr \xi_{gg}(r)$$

Sometimes mass and number distributions are equivalent (mass traces light). Sometimes not (biasing).

Power spectrum

Observationally

$$\xi_{gg}(r) \simeq \left(\frac{r}{r_0}\right)^{-\gamma}$$

with $r_0 \simeq 5\ h^{-1}$ Mpc. The cluster-cluster distribution is similar with $r_0 \simeq 20\ h^{-1}$ Mpc

Many times we cannot have volume data. In that case we consider angular correlations. Typically they scale as $(\theta/\theta_0)^{-(\gamma-1)}$ with $\theta_0 \simeq r_0/D$ for a survey with depth D.

The spectrum distribution of fluctuations is of primary interest. It can be measurent and various physical processed can create or destroy fluctuations with a specific range in wavelegths (ex: Jeans' limit).

We need to Fourier transform

$$\delta(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x\ \delta(\vec{x})\ e^{i\vec{k}\cdot\vec{x}}$$

We are interested in correlators in momentum space $\langle\delta(\vec{k})\delta(\vec{k}')\rangle$

In linear perturbation theory, each mode evolves independently.

$$\langle\delta(\vec{k})\delta(\vec{k}')\rangle = P(|k|)\delta^{(3)}(\vec{k}-\vec{k}')$$

$P(|k|)$ gives the power spectrum

$$\xi(r) = \frac{1}{(2\pi)^3} \int d^3k\ P(k)\ e^{i\vec{k}\cdot\vec{x}} = \frac{1}{2\pi^2\ r} \int_0^\infty kdk \sin kr P(k)$$

Problem: Do the angular integrals above. Show also that $P_{gg}(k) \sim (kr_0)^{\gamma}$

Gaussian fluctuations

In the case of gaussian fluctuations (generic and easy, expected from quantum fluctuations) all higher correlation functions are determined by the two-point function. The probability distribution is

$$f(\delta(\vec{k})) = \frac{1}{(2\pi P(k))^{3/2}} \exp\left[-\frac{|\delta(\vec{k})|^2}{2P(k)}\right]$$

Various physical processes give rather simple power spectra. Gravity provides a scale-free process so we expect $P(k) = A\ k^n$ with cutoffs where other physics intervenes. The important data is the amplitude A, the exponent (spectral index) n and the cutoffs.

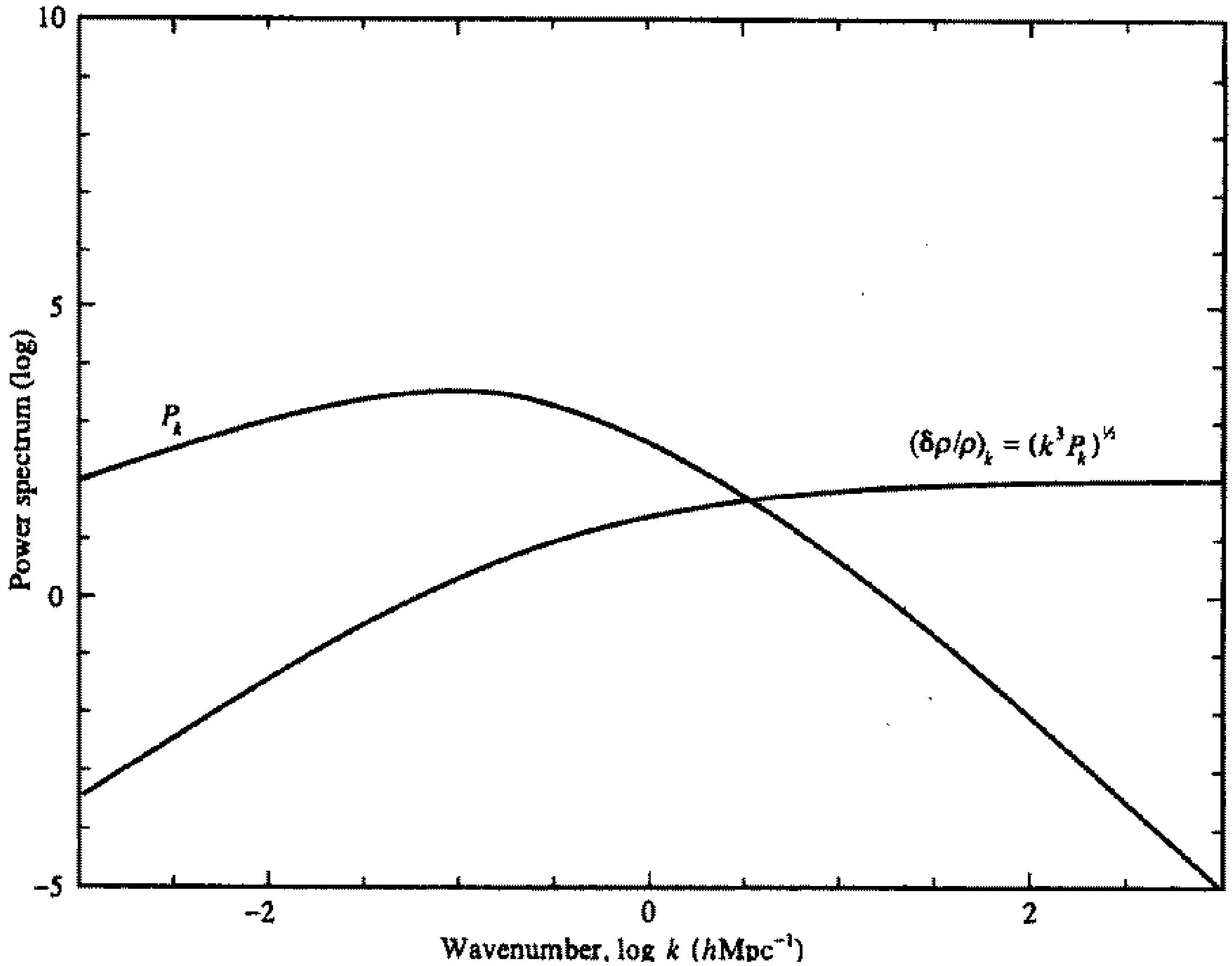
Consider the density contrast at a particular comoving scale L. It will get it main contribution from $k \sim L^{-1}$. using some scaling we obtain

$$\langle\delta^2\rangle \sim \int d^3k\ P(k) \sim k^3|\delta(k)|^2 \sim L^{-(3+n)}$$

$$\langle\Delta\phi\rangle \sim L^2\delta\rho \sim L^2\langle|\delta|\rangle \sim L^{(1-n)/2}$$

Larger scales are expected to go non-linear later than smaller scales since they start out smaller and the perturbations beyond the horizon grow the same way. The case $n = 1$ is special. Here the potential is constant and the spectrum is scale invariant. This is the Harrison-Zeldovich spectrum and is a prediction of inflation. We must have $n > -3$ so that the fluctuations do not diverge at large scales. There is no problem at small scales due to the Jeans' cutoff. One can also put an upper bound on n from gravitational motions at small k: $-3 < n < 4$.

Power spectrum



Power spectrum versus wavenumber. This example is a scale invariant CDM model so that at long wavelengths $P(k) \sim k$ and for $k >> 1$ Mpc^{-1}, $P(k) \sim k^{-3}$. Also plotted is the density fluctution in a sphere of radius k^{-1}

Connections to data

The power spectrum contains information about the density distribution of the fluctuation: for large objects like clusters good predictions of their density distribution can be made.

For large clusters we can observe the mass-radius relation. We expect $\delta\rho/\rho \sim L^{-(n+3)/2}$ and evolves in time as $t^{2/3}$ until it goes nonlinear at $\delta\rho/\rho \sim L^{(n+3)/2} t_{nl}^{2/3} \sim 1$

We obtain a relation of the scale of the cluster and the time at which it decouples from the expansion of the universe.

Its proper scale at this time is $a(t_{nl})L \simeq L^{(5+n)/2}$ The overall density evolves following gravitational dynamics, namely $\rho \sim t^{-2}$ so that $\rho \simeq t_{nl}^{-2} \simeq L^{-3(3+n)/2}$ Combining the two we obtain

$$\rho \sim r^{-\beta} \quad , \quad \beta = 3\frac{3+n}{5+n}$$

$n = 1$ gives an isothermal distribution $\rho \sim r^{-2}$ in agreement with observations on clusters.

Similarly we can obtain

$$\xi_{nl}(r) \sim r^{-\beta} \quad , \quad \xi_l(r) \sim r^{-(3+n)}$$

for non-linear (small) and linear (large) scales respectively.

From observations we have seen that $\beta \simeq 1.8$ so that both $n = 0, 1$ fit the data.

Dark Matter

There are several sources of evidence for the existence of matter that does not shine:

• From mass to light ratios: typical stars have $M/L = 1 - 10$. At the extreme, comets have $M/L = 10^{12}$.

Surveys indicate that $(M/L)_{\text{critical}} = 1390\ h$ with 35% error. Thus there must be dark matter.

• In the milky way, mass can be computed from M/L surveys knowing the relation between mass and luminosity. Mass can also be determined by rotation curves. The results indicate some dark matter inside the galaxy.

• One expects that the rotation velocity $1/r$ in the galaxy halo. However measurements show a flat rotation curve indicating the existence of dark matter in galaxy halos. This was predicted by Ostriker, Peebles and Yahil in 1974 in order to argue for the absence of bar instabilities in gravitating disks.

• Similar atrguments apply to groups and clusters of galaxies. The earliest observations of dark matter in 1933 by Zwicky were in this domain. The arguments and measurements are similar to those of individual galaxies.

• Masses can be estimated from hot gas in galaxy clusters emiting X-rays. This gives for the Coma cluster $M_B/M_{\text{total}} = 0.01 \pm 0.05 h^{-3/2}$.

• Cluster masses can be estimated from gravitational lensing indicating the presence of dark matter.

Galaxy Rotation Curves

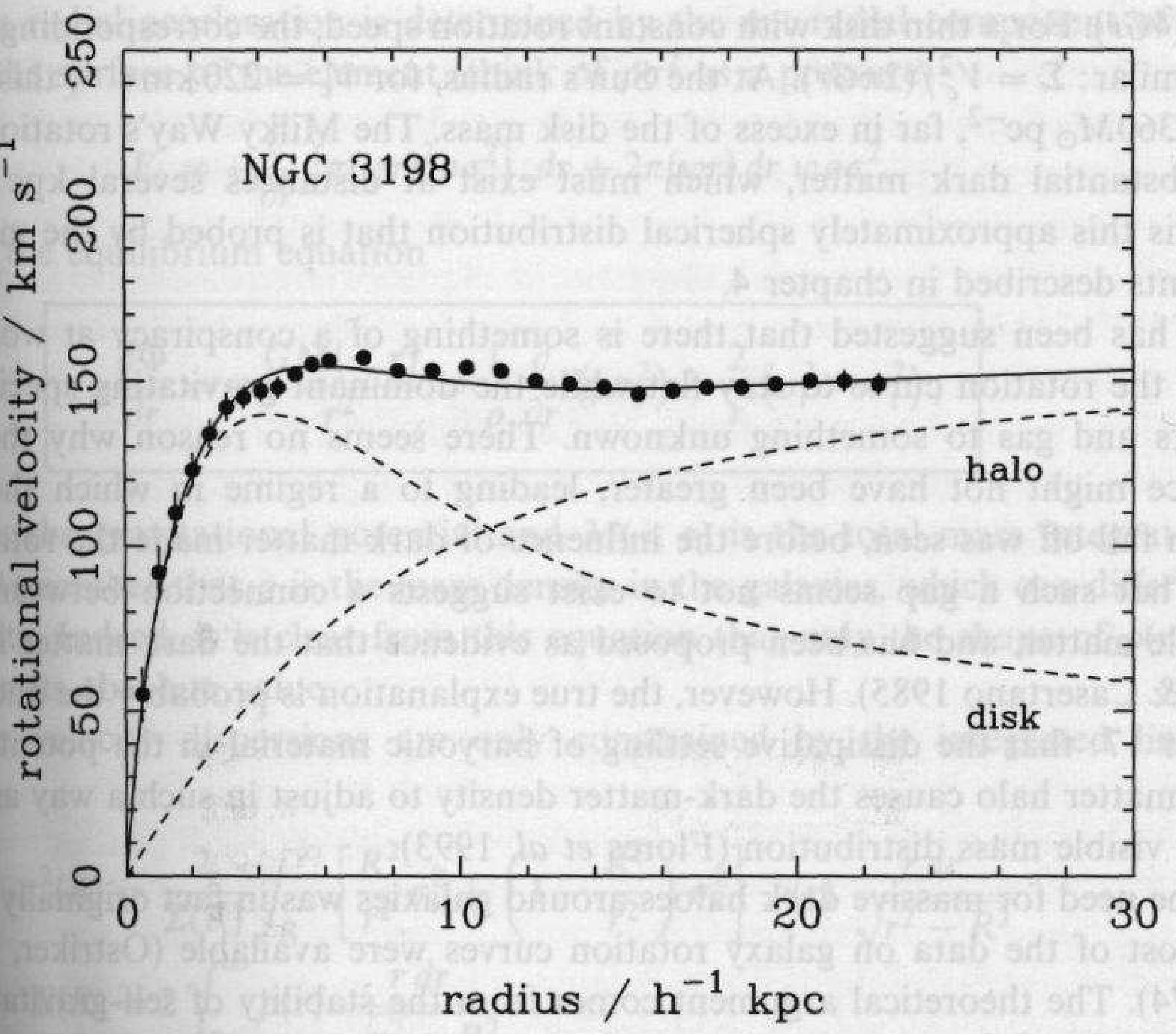


Figure 12.4. The brightness profile and rotation curve of the spiral galaxy NGC3198 (adapted from van Albada *et al.* 1985). Note the flat form of the curve at large radii by comparison with the contribution expected from the luminous disk.

Baryonic Dark Matter

An important ingredient is the Initial Mass Function

$$\frac{dn}{d\log m} \sim m^{-1.35}$$

The total density is $\rho = \int_{m_c}^{\infty} m \ dn$ To obtain a factor of two in ρ we need a factor of 7.2 in m_c. So: reasonable amounts of dark matter exist if there are enough low mass bodies that do not count as stars.

NUCLEAR BURNING LIMIT: If a mass shines its gravitational binding energy then the Kelvin timescale is given by

$$L \sim \frac{\partial}{\partial t}\left(\frac{GM^2}{R}\right) \Rightarrow t_K \equiv \frac{R}{|\dot{R}|} \sim \frac{GM^2}{RL}$$

For the Sun $t_K \sim 10^7$ years, so stars need nuclear reactions to shine.

Gravitational collapse in a star will make the temperature rise untill nuclear burning starts. From the virial theorem the temperature of the star is given by

$$T = \alpha \frac{GM^2}{3NkR} \sim \frac{M}{R}$$

where $\alpha \sim \mathcal{O}(1)$. The only thing that can prevent this is fermionic degeneracy when $p < \hbar \ n^{1/3}$.

For uniform bodies degeneracy becomes important at

$$T_{max} \simeq 6 \times 10^8 \ \left(\frac{M}{M_\odot}\right)^{4/3} \ {}^oK$$

Baryonic Dark Matter (contd.)

Fusion rates drop exponentially when $T < 10^6\ ^oK$. One obtains that the minimum mass for normal stars is $0.08 \pm 0.01 M_\odot$. Thus, less massive objects do not shine after t_K. These are known as brown dwarfs.

Can one form a brown dwarf? Its mass must be below the Jeans mass

$$M_J = \rho\lambda_J^3 = \left(\frac{\gamma\pi kT}{Gm_H}\right)^{3/2} \rho^{-1/2}$$

with $\lambda_J = c_s\sqrt{\pi/(G\rho)}$.

In practical units the Jeans mass is

$$M_J = 10^{5.3}(T/K)^{3/2}(n/m^{-3})^{-1/2}\ M_\odot$$

To achieve $M_J = 0.01\ M_\odot$ we need $n \geq 10^{16} m^{-3}$. This is no good since the densest molecular clouds have $n \sim 10^{11}$.

The only way for this to happen is via fragmentation. Analysis of this gives $M > 0.01\ M_\odot$

Recent microlensing surveys indicate that brown dwarfs provide more mass than visible matter but they are still an order of magnitude smaller than non-baryonic dark matter.

Non-baryonic dark matter

Constraints from nucleosynthesis indicate that $\Omega_B \simeq 0.0125\ h^{-2}$.

Dark matter must interact very weakly (WIMPS) in order not to disturb nucleosynthesis.

• HOT DARK MATTER. These are particles that decouple when relativistic and have a number density roughly that of photons. eV-mass neutrinos is a popular example.

• WARM DARK MATTER. These decouple when relativistic but their density is much smaller than that of photons and today they are much slower than HDM. Examples are difficult to come by.

• COLD DARK MATTER. Decoupling happens when particles are non-relativistic, the thermal velocities are zero and the mass as high as desired.

For small masses we get the neutrino result:

$$\Omega_\nu\ h^2 = \frac{\sum m_i}{93.5\ eV}$$