Introduction to Cosmology

Teacher: ΗΛΙΑΣ ΚΥΡΙΤΣΗΣ

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Text: **E. Linder: First Principles of Cosmology**

Plan of the Course

- Introduction

- General: Units, General Theory of Relativity, Equivalence Principle Metric, Homogeneity and Isotropy, Energy-Momentum tensor, Riemannian Geeometry.

- The Dynamics of the Universe: Friedman equations, equation of state, temporal evolution, the end of the universe.

- Cosmological models and cosmological tests.

- The history of the universe: the physics. The Planck era, baryogenesis, neutrino background, inflation.

- The three main characteristics: Hubble expansion, Nucleosynthesis and Cosmic Microwave Background Radiation (CMBR)

- Structure Formation

Introduction

• Cosmology: The study of the universe at large scales.

• We will see that we need the physics of the very little (particle physics)

• The physics is mostly classical. We will still need some quantum physics (mostly for radiation and nucleosynthesis)

• The Physics: High Energy and Elementary particles, Nuclear physics, atomic physics, thermodynamics and statistical mechanics, electromagnetism and last but not least gravity.

Simple books and WWW sites:

▲ T. Ferris (1997) : "The whole Shebang"

▲ S. Weinberg (1993): "The first three minutes"

▲ T. Davis (1977): " Space and Time in the MOdern Universe"

◀ A simple introduction:
http://leandros.physics.uch.gr/cosmology/index.html

◀ New measurements:

http://leandros.physics.uch.gr/CosmoStatus99/index.html

Basic numbers and Units

Size of observable universe	$10^{26}\ m$
Total mass	$10^{22}\ M_O$
Baryon Number	10^{79}
Density	$10^{-29}\ gr\ cm^{-3}$
CMBR temperature	$2.7\ ^o$K
Age of the universe	$10^{10}\ years$
Entropy per baryon	$10^{10} \times k$
Pressure	$10^{-19}\ atm$

Units of length

- 1 Mpc (megaparsec) $=10^6$ pc $\simeq 3 \times 10^{22}\ m$.
- 1 pc=3.3 light years
- Hubble length: $H_0^{-1} \simeq 3000\ Mpc \simeq 10^{26}\ m$.

Units of mass

▷ Solar mass $M_O = 2 \times 10^{33}\ gr$

▷ Proton mass $m_p = 1.67 \times 10^{-24}\ gr$

Units of time

◂ 1 year $\simeq 10^{7.5}\ sec \qquad (e^x \simeq 10^{0.43x})$

◂ 1 Hubble time $H_0^{-1} \simeq 10^{10}\ years \simeq 3 \times 10^{17}\ sec$

Special Units

Geometrical Units: chosen so that $G = c = k = 1 \Rightarrow$ all quantities have dimensions of length only (choosing the units of time, mass and temperature).

$$c \sim \frac{[L]}{[T]} \quad , \quad G \sim \frac{[L]^3}{[M][T]^2}$$

In high energy physics $\hbar = c = 1$.

We can make combinations of the fundamental constants so that we make a length, a time and a mass (and temperature) all of them known as Planck ...

$$M_P^2 = \frac{\hbar c}{G} \quad , \quad l_p^2 = \frac{G\hbar}{c^3} \quad , \quad t_P^2 = \frac{G\hbar}{c^5} \quad , \quad T_P = \frac{1}{k}\sqrt{\frac{\hbar c^5}{G}}$$

$$M_P \simeq 2\times 10^{-5}\, gr \simeq 10^{19}\,\frac{GeV}{c^2} \quad , \quad l_P \simeq 4\times 10^{-33}\, cm \quad ,$$

$$t_P \simeq 10^{-43}\, sec \quad , \quad T_P \simeq 10^{32}\ {}^oK$$

- Mass: $[M] = [L]\frac{c^2}{G} \quad \longrightarrow \quad 1 M_O \sim\ (1.48\ \times\ )\ 1\ km$
- Time: $[T] = [L]/c \quad \longrightarrow \quad$ 1 year$\sim$ 1 light year$\simeq$ $10^{18}\ cm$
- Temperature: $[Temp] = \frac{[E]}{k} = [L]\frac{c^4}{kG} \quad \longrightarrow$ $300\ {}^oK \simeq 1/40\ eV$

$$10^{10}\ {}^oK \simeq 0.86\ MeV \quad ,$$

Nuclear length$\simeq 1\ fm \sim 10^{-13}\ cm \sim 200\ MeV$

Order of Magnitude Calculations

Important in Physics and especially in Cosmology.

▷ Ex1: How often we have a supernova in our Universe?

Most recent observable SN→ SN1987A in 1987.

Previous in 1604 (Kepler, Tycho Brahe)

We see 1 in 10 (due to dust, stars etc).

⇒ 1/40 per year in the galaxy. The observable Universe has $\sim$ 10^9 galaxies.

⇒ $1/40 \times 10^9$ per year$\sim 1/sec$.

▷ Ex1: How many CMBR photons pass through your palm per second?

Rate = (density)×(velocity)×(palm area)

density: $n = n_0\ T^3$ because the total energy is proportional to T^4 and the energy per degree of freedom is kT.

Normalize: $n = l_P^{-3}$ for $T = T_P \Rightarrow n_0 \sim 10 cm^{-3}K^{-3}$ and $n = n_0(3^oK)^3 = 270\ cm^{-3}$.

So: Rate $=270\ cm^{-3} \times 3\ 10^{10}\ \frac{m}{sec} \times 100\ cm^2 \sim 10^{15}\ sec^{-1}$

▷ Ex3: Luminosity: $\frac{\text{Energy}}{\text{Time}}$ We can make a combination of universal constants with the same dimensions: $c^5/G \sim 4 \times 10^{59} erg/sec$. (The luminosity of a body of Planckian mass transforming all the mass to radiation in a Planck time

General Relativity

The Equivalence Principle is in the heart of General Relativity (GR).

Simplest version: $\frac{M_{\rm inertial}}{M_{\rm gravitational}}$ = universal constant

Weak Equivalence Principle (WEP): The trajectory of a freely falling (small) body without gravitational slf-interaction is independent of its internal structure and composition.

Einstein Equivalence Principle (SEP): It is always possible to pick a coordinate system so that all the laws of physics are the same locally as those in a non-accelerating system in the absence of gravity. (So special relativity is enough locally)

Or better: ▲ (i) WEP is true.

▲ (ii) The outcome of any local non-gravitational experiment is independent of the velocity of the freely falling reference frame in which it is performed (LLI).

▲ (iii) The outcome of any local non-gravitational experiment is independent of where and when in the universe has been performed. (LPI)

Strong Equivalence Principle (WEP): Same as the Einstein principle, with WEP valid for small self-gravitating bodies (ex: stars).

Schiff's conjecture: WEP $\Rightarrow$ SEP

Tests of WEP

Parametrization :

$$M_G = M_I + \sum_I \eta^I \frac{E^I}{c^2}$$

We measure the Eötvös ratio:

$$\eta = \frac{2|a_1 - a_2|}{|a_1 + a_2|} = \sum_I \eta^I \left(\frac{E_1^I}{M_1 c^2} - \frac{E_2^I}{M_2 c^2} \right)$$

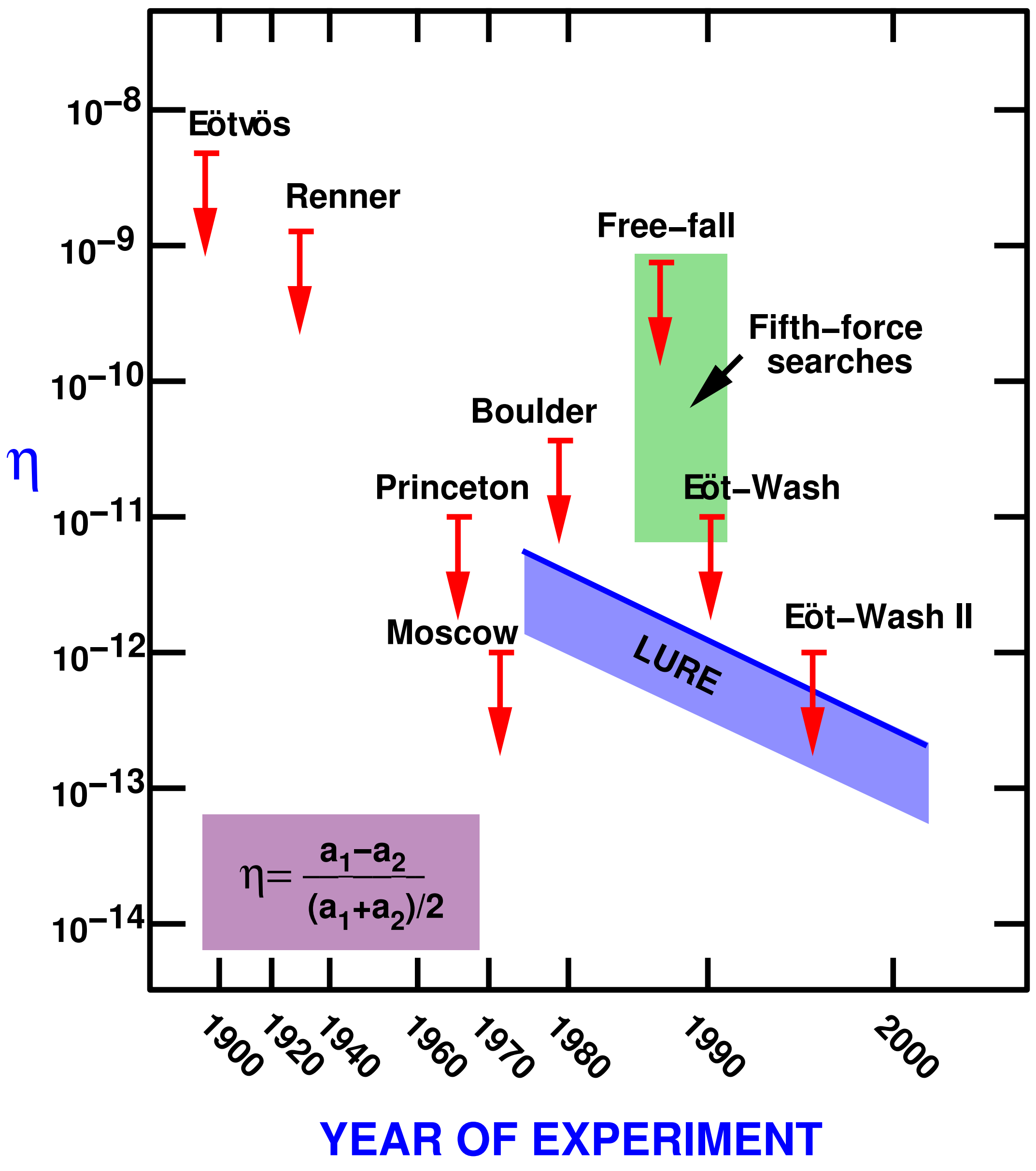
Tests of LLI

Measurement of $\delta = \frac{1}{c^2} - 1$.

Tests of LPI

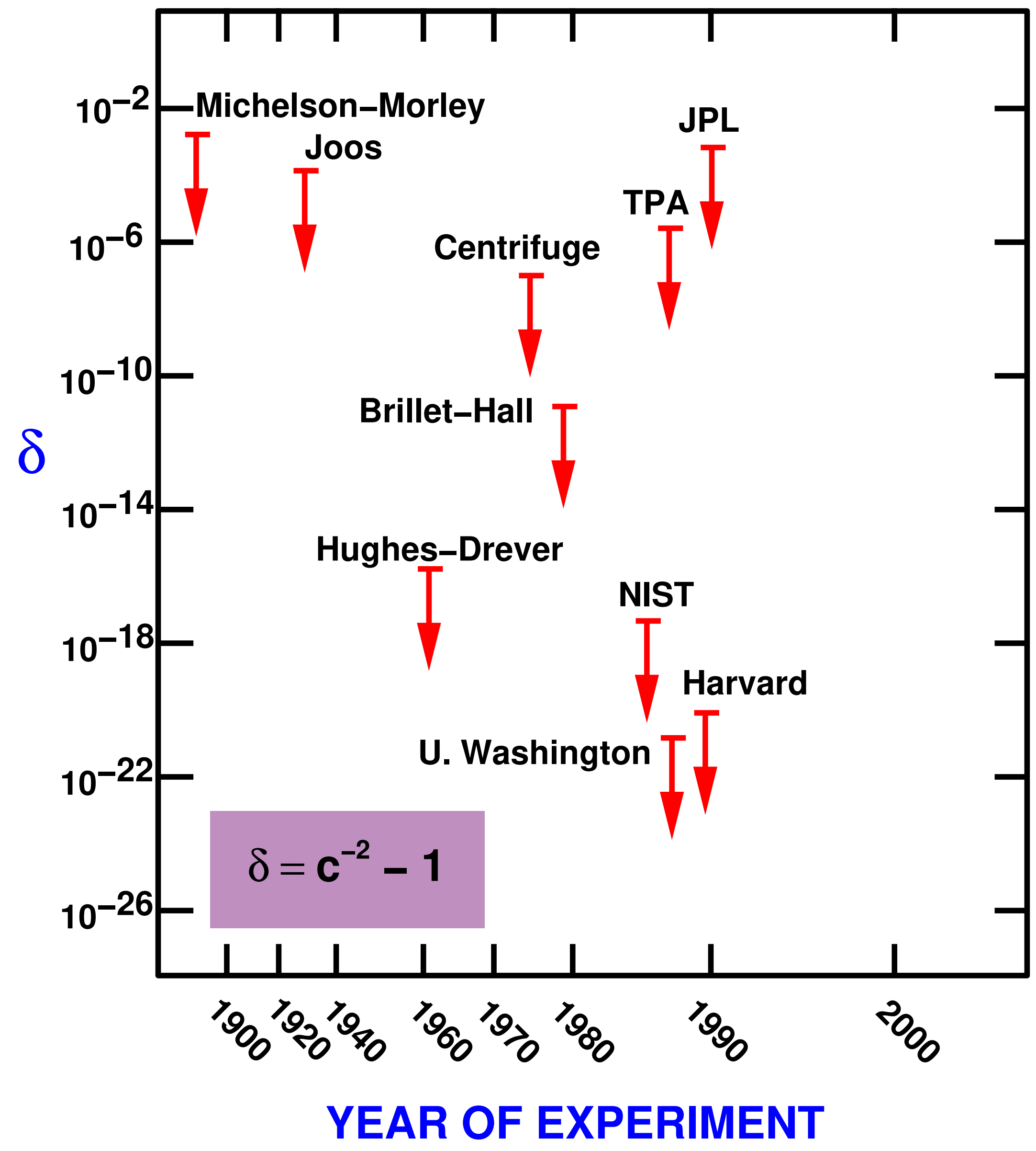
$$\frac{\Delta \nu}{\nu} = (1 + \alpha) \frac{\Delta U}{c^2}$$

TESTS OF THE
WEAK EQUIVALENCE PRINCIPLE



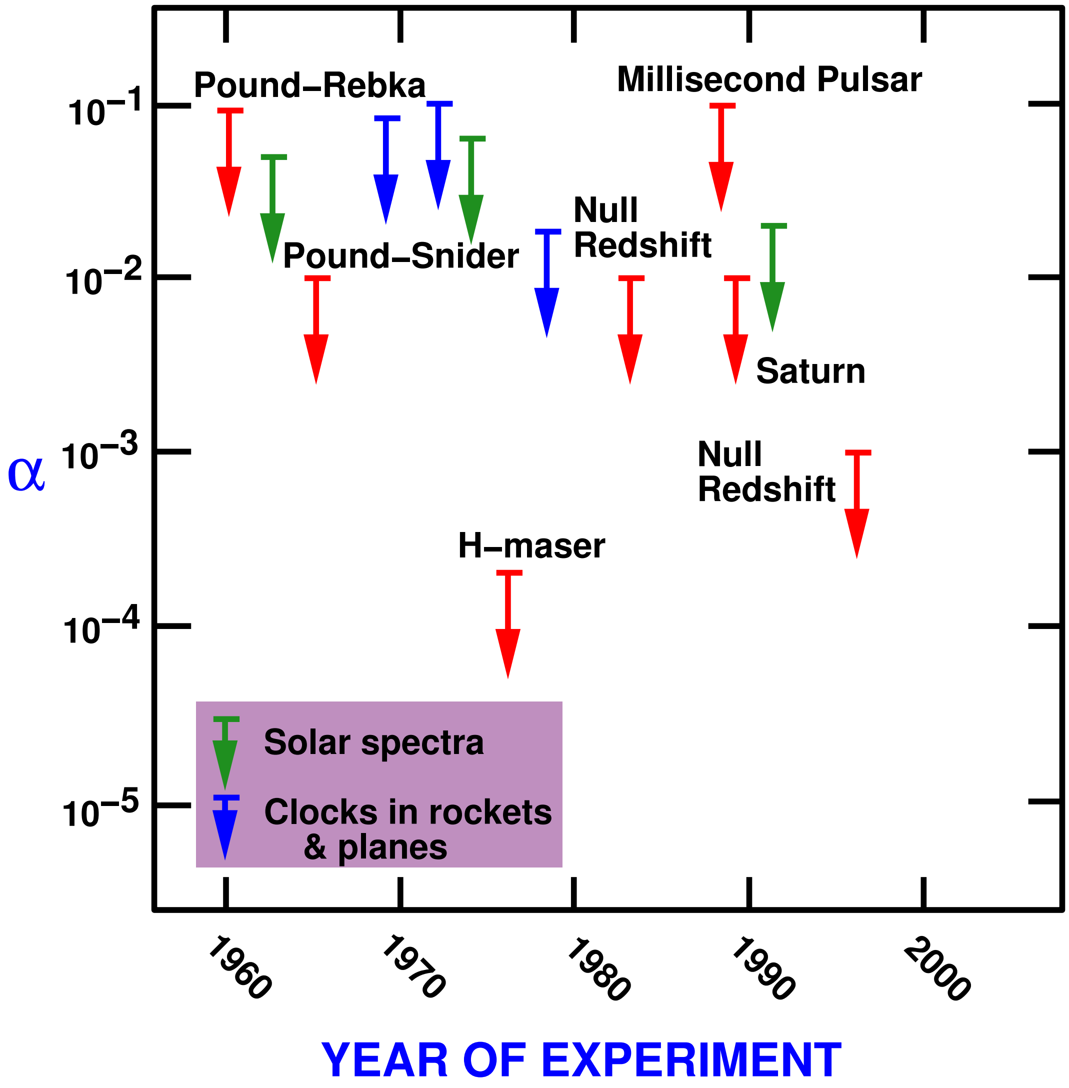
Selected tests of the Weak Equivalence Principle, Free-fall and Eöt-Wash experiments originally performed to search for fifth force. Hatched line shows current bounds on η for gravitating bodies from lunar laser ranging (LURE).

TESTS OF
LOCAL LORENTZ INVARIANCE



Michelson-Morley, Joos, and Brillet-Hall experiments test isotropy of round-trip speed of light. Centrifuge, two-photon absorption (TPA) and JPL experiments test isotropy of light speed using one-way propagation. Remaining four experiments test isotropy of nuclear energy levels.

TESTS OF LOCAL POSITION INVARIANCE



$$\boxed{\Delta\nu/\nu = (1+\alpha)\Delta U/c^2}$$

Selected tests of local position invariance via gravitational redshift experiments, showing bounds on α, which measures degree of deviation of redshift from the formula $\Delta\nu/\nu = \Delta U/c^2$.

Variation of fundamental constants

Constant k	Limit on $\dot{k}/k$ 1.2×10^{10}yr	Method
E+M $\alpha = e^2/\hbar c$	4×10^{-4} 6×10^{-7} 6×10^{-5}	H-maser vs Hg ion clock Oklo Natural Reactor 21-cm vs molecular absorption at $Z = 0.7$
Weak $\beta = G_f m_p^2 c/\hbar^3$	1 0.1 0.06	^{187}Re, ^{40}K decay rates Oklo Natural Reactor BB nucleosynthesis
M_e/M_p	1	Mass shift in quasar spectra at $Z \sim 2$
g_p	10^{-5}	21-cm vs molecular absorption at $Z = 0.7$

Bounds on cosmological variation of fundamental constants of non-gravitational physics.

References: C. Will, gr-qc/9811036 at xxx.lanl.gov
T. Damour, gr-qc/9711060

Gravity and Geometry

The SEP implies a deep description of gravity:

Acceleration — Gravity — Geometry (curvature)

- Gravity is always attractive (no negative masses)

- Contains many strange occurences: black holes with lots of puzzles, strange acausal solutions etc).

- Marrying quantum mechanics and general relativity is a formitable task!

Metric Geometry

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \equiv \sum_{\mu,\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu$$

$g_{\mu\nu}$ is the covariant metric tensor (symmetric)

The contravariant (inverse) metric tensor:

$$g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho \Leftrightarrow g^{-1} \cdot g = \mathbf{1}$$

Under a change of coordinates $x_\mu = x^\mu(y)$ $g_{\mu\nu}$ transforms as a covariant second rank tensor:

$$g'_{\mu\nu} = g_{\rho\sigma}\frac{\partial x^\rho}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\nu}$$

A covariant vector A_μ or a contravariant vector B^μ transform under the change of coordinates $x^\mu \to y^\mu$ as

$$A'_\mu = A_\nu \frac{\partial x^\nu}{\partial y^\mu} \quad , \quad B'^\mu = B^\nu \frac{\partial y^\mu}{\partial x^\nu}$$

In general a mixed tensor transforms as

$$A_{\mu_1 \cdots \mu_p}{}^{,\nu_1 \cdots \nu_q} = A'{}_{\mu'_1 \cdots \mu'_p}{}^{,\nu'_1 \cdots \nu'_q} \frac{\partial x^{\mu'_1}}{\partial y^{\mu_1}} \cdots \frac{\partial x^{\mu'_p}}{\partial y^{\mu_p}} \times$$

$$\times \frac{\partial y^{\nu'_1}}{\partial x^{\nu_1}} \cdots \frac{\partial y^{\nu'_q}}{\partial x^{\nu_q}}$$

Indices are raised and lowered with the metric:

$$\text{example} \quad A^\mu = g^{\mu\nu} A_\nu \quad , \quad A_\mu = g_{\mu\nu} A^\nu$$

If in some coordinate system $A_{\mu_1 \cdots \mu_p} = B_{\mu_1 \cdots \mu_p}$ then in any coordinate system $A'{}_{\mu_1 \cdots \mu_p} = B'{}_{\mu_1 \cdots \mu_p}$

Covariant derivatives

If A_μ is a tensor then

$$\frac{\partial A_\mu}{\partial x^\nu} \equiv \partial_\nu A_\mu$$

is not a tensor.

Christofel symbols:

$$[\mu\nu;\rho] = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

$$\Gamma^\rho_{\mu\nu} = g^{\rho\sigma}[\mu\nu;\sigma]$$

The covariant derivative is a (symmetric) two-tensor

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho$$

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho$$

In general

$$\nabla_\rho A_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q} = \partial_\rho A_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_q} -$$

$$-\Gamma^\sigma_{\rho\mu_1} A_{\sigma\cdots\mu_p}{}^{\nu_1\cdots\nu_q} - \cdots - \Gamma^\sigma_{\rho\mu_p} A_{m_1\cdots\sigma}{}^{\nu_1\cdots\nu_q} +$$

$$+\Gamma^{\nu_1}_{\rho\sigma} A_{m_1\cdots\mu_p}{}^{\sigma\cdots\nu_q} + \cdots + \Gamma^{\nu_q}_{\rho\sigma} A_{m_1\cdots\mu_p}{}^{\nu_1\cdots\sigma}$$

Tensor Densities

$$\det g' = \frac{\det g}{\left|\det \frac{\partial x'}{\partial x}\right|^2}$$

so that the volume element $\int d^n x \sqrt{|\det g|}$is invariant.

The Schwarschild Metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Solves $R_{\mu\nu} = 0$. In dimensionfull units

$$1-\frac{2M}{r} \to 1-\frac{2GM}{r\ c^2}$$

• Spherical Symmetry but no homogeneity. Describes the gravitational field outside a spherical body of mass M.

• Divergence at $r = 2M$ (horizon) is fake. At $r = 0$ the singularity is real.

When the surface of a body is inside the horizon, then we talk about a "black hole".

Rotating Spherical Body

The Kerr metric (at large distance)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{2M}{r^2}\left(\vec{r} + \frac{1}{r}(\vec{j}\times\vec{r})\right)\cdot d\vec{r}dt +$$

$$+\left(\delta_{ij} + \frac{2M}{r^3}\right)dx^i dx^j$$

wher $\vec{j}$ is the angular momentum.

Geodesic Motion

A particle of mass m moving in a gravitational field (metric) follows a path known as a geodesic: The equation defining the path is the geodesic equation:

$$D_\tau^2 x^\mu \equiv \frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\rho}{}^\mu \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

where $dt^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the proper time.

It can be obtained by minimizing the legth of the path of the particle:

$$L = m\int ds = m\int du\sqrt{\frac{ds^2}{du^2}} = m\int du\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

In flat space $\Gamma_{\nu\rho}{}^\mu = 0$ and the equation is $\ddot{x}^u = 0$.

Locally we can always choose a coordinate system (freely falling) so that $g_{\mu\nu}|_0 = \eta_{\mu\nu}$, $\Gamma_{\nu\rho}{}^\mu|_0 = 0$ (but the curvature cannot be made to vanish).

Newtonian Limit

Newtonian equations of motion (in a Newtonian potential Φ):

$$\frac{d^2x^i}{dt^2} + \frac{\partial\Phi}{\partial x^i} = 0$$

Field equation:

$$\nabla^2\Phi = 4\pi\rho$$

Rewriting ($t = a\tau + b$):

$$\frac{d^2x^i}{dt^2} + \left(\frac{dt}{d\tau}\right)^2 \frac{\partial\Phi}{\partial x^i} = 0 \quad , \quad \frac{d^2t}{d\tau^2} = 0$$

Agrees with geodesic equation if $\Gamma^i_{00} = \partial_i\Phi$ all others zero. This implies that

$$g_{00} \simeq -1 + 2\Phi \quad , \quad g_{0i} \simeq 0 \quad , \quad g_{ij} \simeq \delta_{ij}$$

The Riemann tensor is

$$R^i{}_{0;j0} = \partial_i\partial_j\Phi$$

so that the Einstein equation $G = 8\pi T$ becomes the Newton equation.

Surface Φ is $\sim 10^{-39}$ on the proton, 10^{-9} on earth, 10^{-6} on the sun and 10^{-4} on a white dwarf.

Redshift

A freely falling clock reads (proper) time

$$d\tau = \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

The equivqlence principle states that the clock works similarly in a gravitational field ($\eta_{\mu\nu} \to g_{\mu\nu}$).

In a general coordinate system

$$dt = \frac{d\tau}{\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}}$$

For a clock at rest $dx^\mu/d\tau = \delta_{\mu,0}$ so $dt = d\tau\ (-g_{00})^{-1/2}$

Consider a (freely falling) source of light (2) and an observer (1).

The period of light at the position of the source is $T_2 = T/(-g_{00}(x_2))^{1/2}$ whereas at the position of the observer $T_1 = T/\sqrt{-g_{00}(x_1)}$

For the frequencies

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}}$$

For weak fields $g_{00} \simeq -1 - 2\Phi$ so that

$$\frac{\Delta\nu}{\nu} \simeq \Delta\Phi$$

Can be used to resolve the asymetry of the twin story in special relativity

Curvature

The Riemann curvature tensor can be written in terms of the Christofell connections:

$$R^{\mu}{}_{\nu;\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} - \Gamma^{\tau}_{\nu\rho}\Gamma^{\mu}_{\tau\sigma} + \Gamma^{\tau}_{\nu\sigma}\Gamma^{\mu}_{\tau\rho}$$

The full covariant tensor $R_{\mu\nu;\rho\sigma} = g_{\mu\mu'} R^{\mu'}{}_{\nu;\rho\sigma}$ has symmetries

$$R_{\mu\nu;\rho\sigma} = R_{\nu\mu;\rho\sigma} \quad , \quad R_{\mu\nu;\rho\sigma} = R_{\rho\sigma;\mu\nu}$$

By tracing in one set of indices we obtain the Ricci tensor,

$$R_{\mu\nu} = R^{\lambda}{}_{\mu;\lambda\nu} \quad , \quad R_{\mu\nu} = R_{\nu\mu}$$

Tracing once more we obtain the scalar curvature

$$R = R^{\mu}{}_{\nu} = g^{\mu\nu} R_{\mu\nu}$$

Finally the Einstein tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Geodesic Deviation

$$D_\tau^2 \xi^\mu + R^{\mu}{}_{\nu;\rho\sigma} \dot{x}^\nu \xi^\rho \dot{x}^\sigma = 0$$

The Stress Energy Tensor

We need a tensor to describe energy and momentum (the sources of gravitational fields).

In special relativity the energy and momentum are components of a 4-vector. In field theory we must talk about densities.

Consider the energy density T_{00}. In relativity this is completed by the energy flux in i-direction T_{0i} to a four vector (like the EM case, $(\rho, \vec{j})$)

Similarly the momentum density (it is the same as T_{0i}) is completed by the momentum fluxes T_{ij}. The total is the the symmetric stress-energy tensor $T_{\mu\nu}$

Conservation:

$$\nabla^\mu T_{\mu\nu} = 0 \to -\partial_0 T_{0\nu} + \partial_i T_{i\nu} = 0$$

The Einstein Equation

We must relate the sources (charges)$\to$ stress-energy tensor to the effect (curving of spacetime).

$$G_{\mu\nu} = 8\pi \; T_{\mu\nu}$$

Reinstating units $8\pi \to 8\pi/G$.

This equation follows from general arguments of covariance and the strong equivalence principle.
$(\nabla^\mu G_{\mu\nu} = 0)$

If there is a cosmological constant Λ then

$$T_{\mu\nu} \to T_{\mu\nu} - \Lambda \; g_{\mu\nu}$$

Experimentally,

$$\frac{\Lambda}{M_P^2} \leq 10^{-120}$$

Symmetries of metrics

- A <u>stationary</u> metric$\to$ $\partial_0 g_{\mu\nu}=0$.
- A <u>static</u> metric: stationary and $g_{0i}=0$
- Isotropy=Invariance under rotations in space.

$$ds^2=g_{00}(r,t)dt^2+f(r,t)dr^2+g(r,t)d\Omega^2$$
$$d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$$

- Homogeneity: invariance under translations (constant curvature).

In Cosmology

$\to$ Isotropy is verified at the 10% level on galaxy distributions and 10^{-5} for CMBR.

$\to$ Homogeneity is a principle (plaussible yes, but an assumption)

The universe looks homogeneous at $\geq 10^8 lys$.

stars	galaxies	galactic clusters
$\sim$ 1ly	10^6 lys	3×10^7 lys

$$< v_{galactic} >\sim 10^{-3} \to < \frac{E_{kin}}{\rho} >=\frac{1}{2}< v^2 >\sim 10^{-6}$$

$$\frac{p}{\rho}=\frac{1}{3}< v^2 >\sim 10^{-6}$$

$$\rho_{CMBR}=3\ p_{CMBR}\simeq 4\times 10^{-34} gr/cm^3\sim T^4$$

Robertson-Walker metric

The most general metric satisfying homogeneity and isotropy (Robertson-Walker):

$$ds^2 = -dt^2 + a(t)^2 \frac{dr^2 + r^2 d\Omega^2}{(1 + kr^2/4)^2}$$

k is a constant that can be rescaled to $(0, \pm 1)$ It determines the curvature.

- Comoving Coordinates

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + \chi^2(r) d\Omega^2\right)$$

$$\chi_{k=0} = r \quad , \quad \chi_{k=1} = \sin r \quad , \quad \chi_{k=-1} = \sinh r$$

- Standard Coordinates

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right)$$

- Conformal coordinates

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \frac{dr^2 + r^2 d\Omega^2}{(1 + kr^2/4)^2}\right)$$

What are the dimensions of $a(t)$ and k?

Stress-Energy tensor of an ideal fluid

Consider an <u>ideal fluid</u> filling the spacetime which is homogeneous and isotropic, with metric $g_{\mu\nu}$.

The four-velocity tensor is given by

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} \quad , \quad u^{\mu}u_{\mu} = -1$$

In the frame of reference that is at rest with fluid, $u^i = 0$ and $u^0 = 1$ when $g_{00} = -1$.

The stress tensor can be the most general tensor constructed out of u^{μ} and $g_{\mu\nu}$

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + p\ g_{\mu\nu}$$

Go to rest frame, ($u^i = 0$, $u^0 = 1$): Then

$$T_{00} = \rho \quad , \quad T_{ij} = p\ g_{ij}$$

This identifies $\rho \equiv density$ and $p \equiv pressure$.

Microscopic Considerations

r an ideal fluid made out of $non-interacting$ particles with co-ordinates $\vec{x}_a(t)$. The four-momentum density is given by

$$T^{\mu 0}=\sum_a\ p_a^\mu \delta^{(3)}(\vec{x}-\vec{x}_a(t))$$

The current is

$$T^{\mu i}=\sum_a\ p_a^\mu\ \frac{dx_a^i}{dt}\ \delta^{(3)}(\vec{x}-\vec{x}_a(t))$$

for all $x_a^0=t$ so that the covariant expression is

$$T^{\mu\nu}=\sum_a\ p_a^\mu\ U_a^\nu\ \delta^{(3)}(\vec{x}-\vec{x}_a(t))$$

with $U^0=1$, $U^i=\dot{x}^i$. Use the relativistic expression $\vec{p}=E\ \vec{v}\to U^\mu=p^\mu/E$ we can write the covariant expression

$$T^{\mu\nu}=\sum_a\ \frac{p_a^\mu p_a^\nu}{E_a}\ \delta^{(3)}(\vec{x}-\vec{x}_a(t))$$

$$\rho=T^{00}=\sum_a E_a\ \delta^{(3)}(\vec{x}-\vec{x}_a(t))\quad ,\quad n=\sum_a\ \delta^{(3)}(\vec{x}-\vec{x}_a(t))$$

$$p=\frac{1}{3}\sum_i T^{ii}=\frac{1}{3}\sum_a\frac{|\vec{p}_a|^2}{E_a}\ \delta^{(3)}(\vec{x}-\vec{x}_a(t))$$

$$Radiation\to\rho\simeq 3p$$

$$non-relativistic\quad matter\to\rho=m_0\ n+\frac{3}{2}p$$

Friedmann Equations

Putting the most general homogeneous and isotropic metric and stress tensor in the Einstein equation we obtain the Friedman equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$$

Together they imply energy conservation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

Using basic Newtonian theory the Friedman equations can be derived (except for the pressure and curvature dependence).

History of Cosmology

1915	General Relativity
1917	De Sitter Solution (static universe)
1922	Friedmann Equations
1927	Lemaitre universe (with pressure)
1929	Confirmation of Hubble expansion
1931	Einstein accepts expanding universe
1935	Robertson-Walker metric
1965	Discovery of CMBR
1980	Inflation from Guth-Kazanas
1992	Observation of anisotropies in CMBR
1998	Observation of acceleration of Hubble expansion

Equation of State

Matter has equation of state in the form $p = f(\rho)$. This, plus the Friedmann equations determine the evolution of the universe.

In the case of dilute matter the relation is linear

$$p = w\ \rho \quad , \quad \rho \sim a^{-3(1+w)}$$

w	Type of Energy
0	Non-relativistic matter
1/3	Ultra-relativistic matter (photons, ν's etc)
1	Massless Scalars
-1/3	Curvature energy (cosmic string networks)
-2/3	Domain wall energy
-1	Cosmological constant

Causality: $c_s \equiv \sqrt{\left|\frac{dp}{d\rho}\right|} \leq 1$

A given type of energy dominates at any given time the evolution of the universe. In between there are transition regions.

When the universe evolves adiabatically (compared to decay rates) $N \sim$ constant.

Consider a non-relativistic particle X decaying to relativistic products C. Let the decay width be Γ

$$\dot\rho_X = -3\frac{\dot a}{a}\rho_X - \Gamma\rho_X$$

$$\dot\rho_C = -4\frac{\dot a}{a}\rho_C + \Gamma\rho_X$$

Set $\dot a = 0 \Rightarrow \rho_X \sim e^{-\Gamma t}$.

The solution is

$$\rho_X(t) = \frac{\rho_X(t_0)}{a(t)^3}e^{-\Gamma t}$$

$$\rho_C(t) = \frac{1}{a(t)^4}\left[\rho_C(t_0)a_0^4 + \Gamma\rho_X(t_0)\int_{t_0}^{t} du\frac{e^{-\Gamma u}}{a^3(u)}\right]$$

$$\rho_i = \frac{e_i N_i}{V}$$

$N_X = N_0 e^{-\Gamma t}$, $N_C = N_R(0) + N_0(1 - e^{-\Gamma t})$.

$$w_{eff} = -\frac{1}{3}\sum_i\left[\frac{\partial\log e_i}{\partial\log a} + \frac{\partial\log N_i}{\partial\log a}\right]$$

Evolution of the Universe

Define a new time variable: the redshift (dimensionless)

$$z = \frac{a_0}{a(t)} - 1$$

where $a_0 = a(now)$. $z < 0$ in the future.

Dust dominates over radiation when $z \simeq 2 \times 10^4$ (estimate it!).

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$$

Define the critical density

$$\rho_c = \frac{3}{8\pi}H^2 = \frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)^2$$

Define also the normalized densities

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_c(t)} \quad , \quad \Omega_i(0) \equiv \Omega_i$$

so

$$k = [\Omega(z) - 1]\, a^2 H^2 = [\Omega - 1]\, a_0^2 H_0^2$$

$$\frac{a}{a_0} \sim (t - t_0)^{\frac{2}{3(1+w)}}$$

$$t_0 \sim \frac{2}{3(1+w)} H_0^{-1}$$

The fate of the Universe

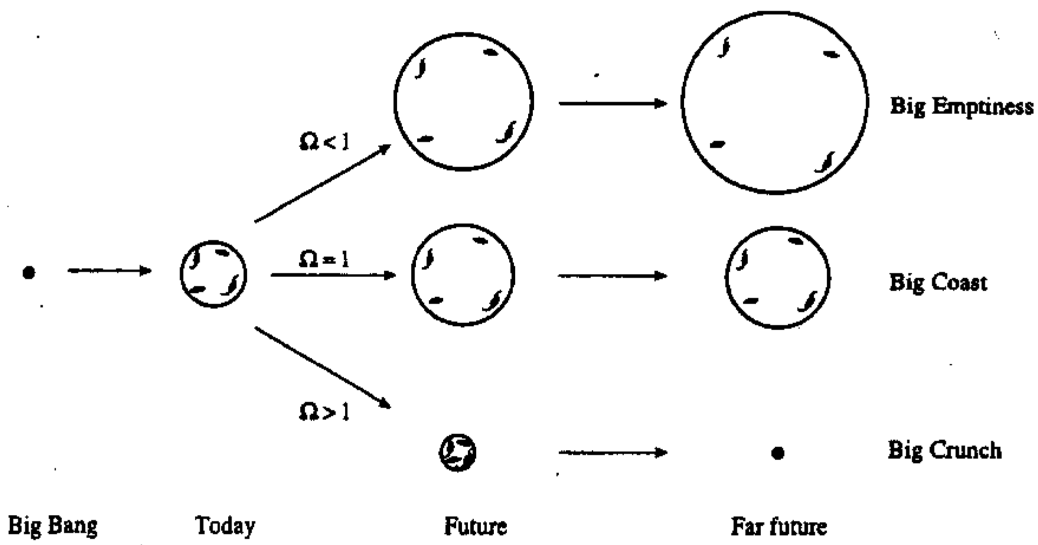


Figure Evolution of open, flat, and closed universes from origin in a Big Bang to ultimate fate.

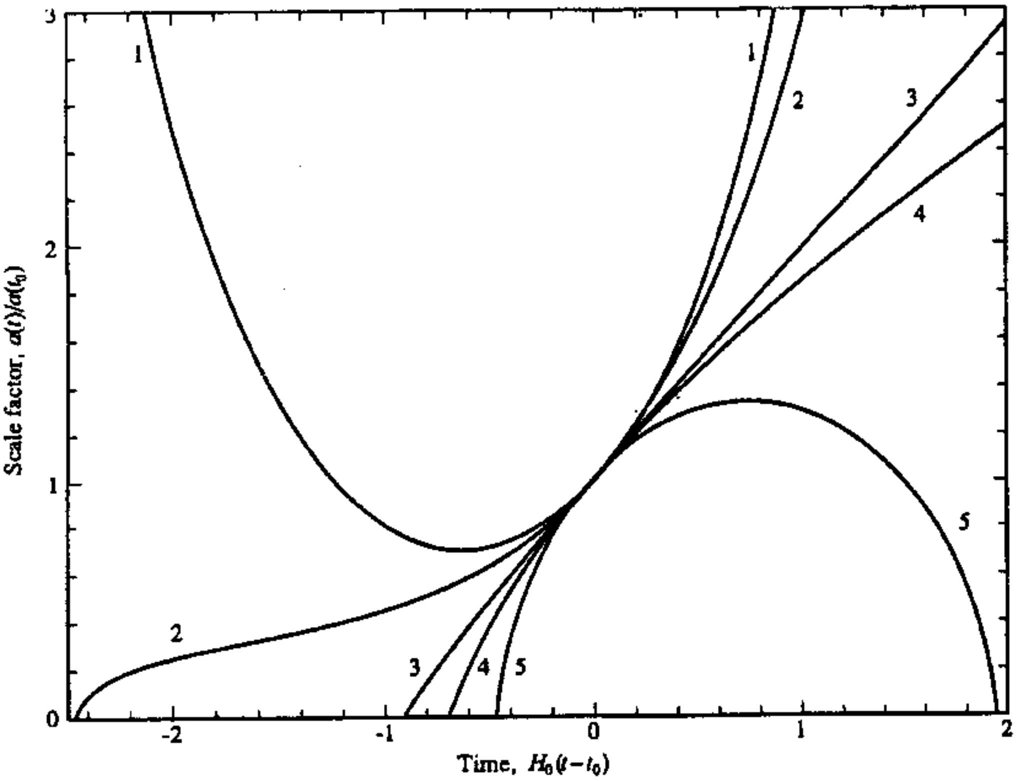


Figure The five types of currently expanding universe, showing their history and fate plotted in scale factor vs. time. A Big Bang model has $a = 0$ in the past. The cases are (1) bounce universe: must have some $\sigma < -1/3$; (2) Eddington–Lemaître universe: has *coasting* phase, must have some $\sigma < -1/3$; (3) open Universe: $\Omega < 1$ in $\sigma > -1/3$; (4) critical (Einstein–de Sitter) universe: $\Omega = 1$ in $\sigma > -1/3$; (5) closed universe: $\Omega > 1$ in $\sigma > -1/3$. Note that model 4 has a asymptotically constant (horizontal).

Cosmological Parameters

From the variables entering the Friedman equations we can make the following combinations that are useful in the analysis of observational data.

$$H_0 = \left(\frac{\dot{a}}{a}\right)_0 \qquad \text{Hubble constant}$$

$$\Omega_0 = \left(\frac{\rho}{\rho_c}\right)_0 \qquad \text{Density parameter}$$

$$q_0 = -\left(\frac{1}{H^2}\frac{\ddot{a}}{a}\right)_0 \qquad \text{Deceleration parameter}$$

H_0 has dimensions $[T^{-1}]$ while Ω_0, q_0 are dimensionless.

Taylor expanding the Friedman equations around $t_0 = today$ we obtain

$$q_0 = \frac{1}{2}\Omega_0 + \frac{3}{2}\sum_w w\ \Omega_w(0)$$

The second Friedman equation can also be written as

$$H(z) \equiv \frac{\dot{a}}{a} = H_0\ y\ \sqrt{\sum_w \Omega_w(0) y^{1+3w} - \Omega_0 + 1}$$

where $y = 1 + z = \frac{a_0}{a}$.

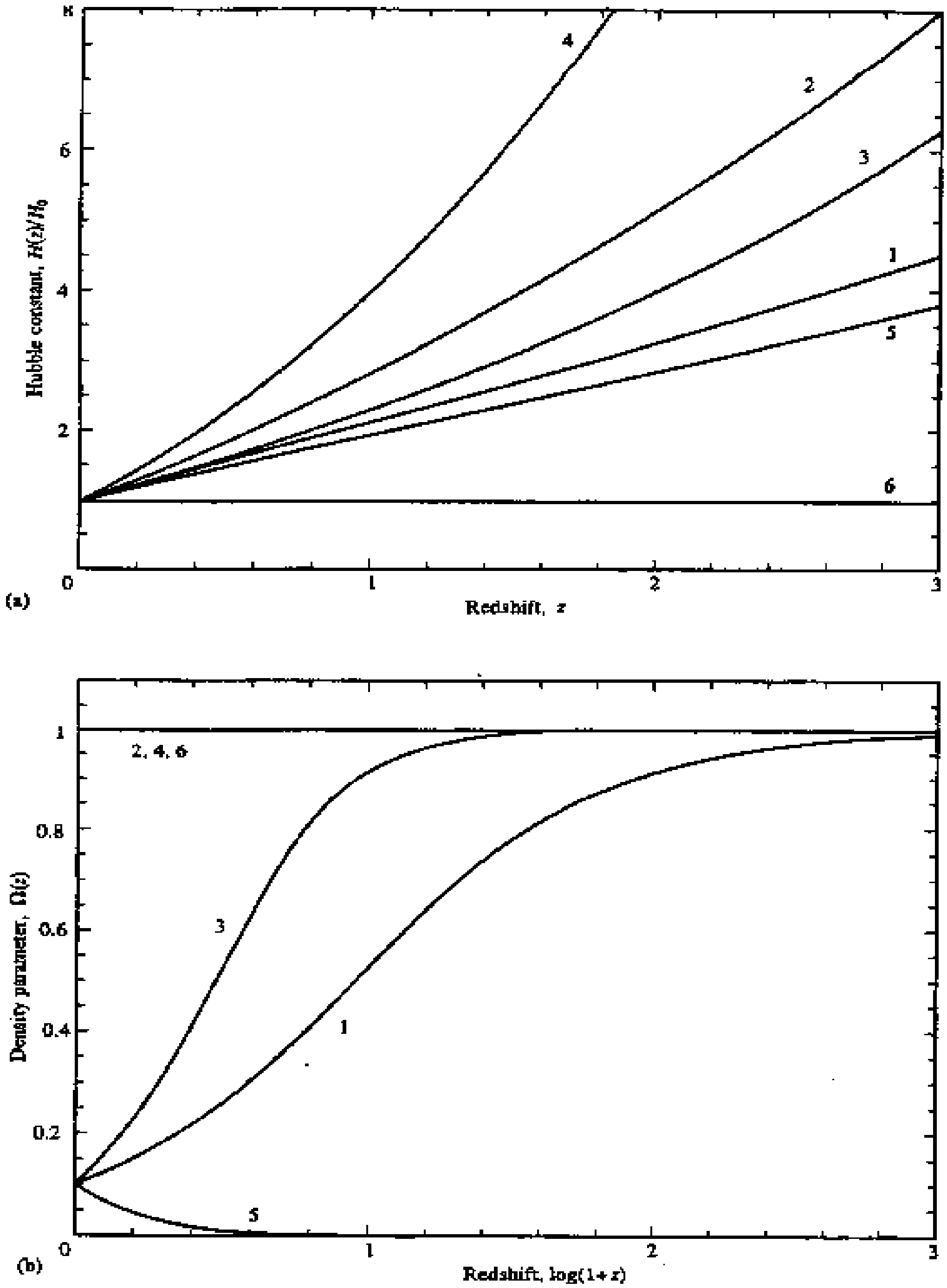


Figure Cosmological parameters plotted vs. redshift: (a) $H(z)/H_0$ vs. z and (b) $\Omega(z)$ vs. $\log(1+z)$. The parameter $q(z) = (1/2, 1, -1)\,\Omega(z)$ for the (first, second, third) pair of models. The models are (1) $\sigma = 0$, $\Omega = 0.1$; (2) $\sigma = 0$, $\Omega = 1$; (3) $\sigma = 1/3$, $\Omega = 0.1$; (4) $\sigma = 1/3$, $\Omega = 1$; (5) $\sigma = -1$, $\Omega = 0.1$; (6) $\sigma = -1$, $\Omega = 1$.

Lookback time

$$\Delta t(z_1, z_2) = \int_{1+z_1}^{1+z_2} \frac{dy}{y} \frac{1}{H(y)}$$

For a single component we obtain the age of the universe as

$$t_0 = \frac{2}{3(1+w)} \frac{1}{H_0} f_w(\Omega) = \int_0^{a_0} \frac{da}{a} \left[\frac{8\pi\rho_0}{3} \left(\frac{a_0}{a}\right)^{3(1+w)} - \frac{k}{a^2} \right]^{-1/2}$$

with $f_w(\Omega)$ being unity for $\Omega = 1$ otherwise being a slowly decreasing function.

Distance Tests

Comoving distance r_c traveled by a photon:

$$\int_0^{r_c} \frac{dr}{\sqrt{1-kr^2}} = \int_{t(r_c)}^{t_0} \frac{dt}{a(t)}$$

Proper distance: it is given by the photon proper time times the speed of light (=1). The photon proper time is the same as comoving time (dr=0 etc). Thus

$$t = \int dt = \int \frac{da}{\dot{a}} = \int_1^{1+z} \frac{dy}{yH(y)}$$

Angular distance

$$r_a = d/\theta$$

where d is the size of the source and θ is the observation angle.

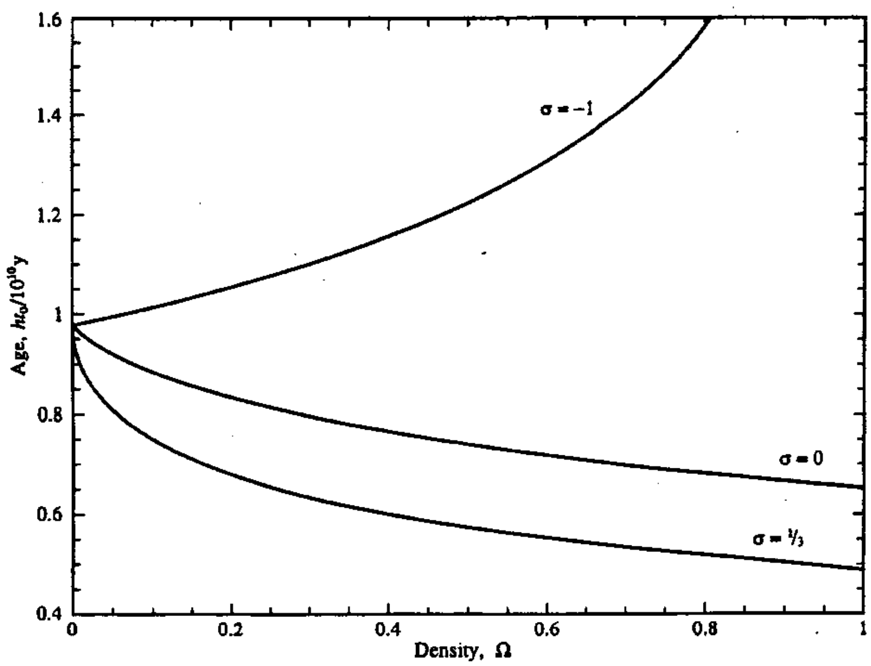


Figure Age of the universe for various components and densities. The models are the same as in Figure 3.1. For $\sigma = -1$ the age becomes infinite as Ω approaches one.

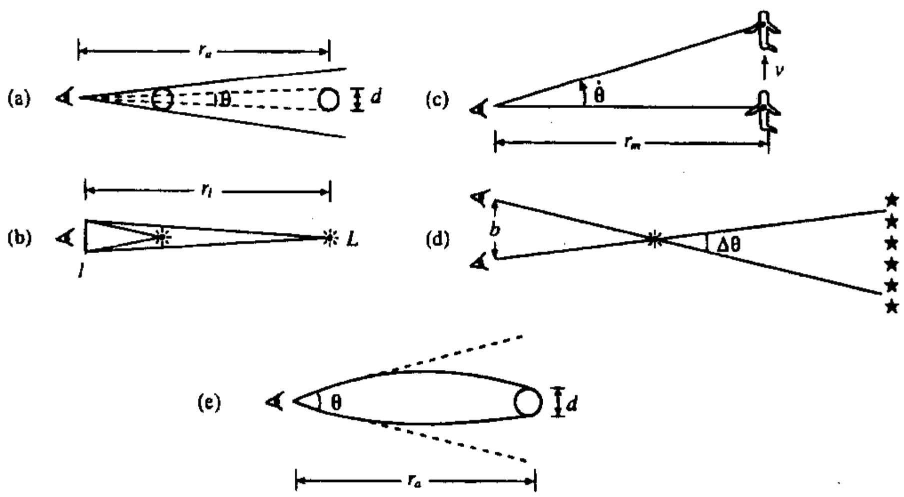


Figure Illustration of different distance measures defined in terms of their Euclidean equivalents: (a) angular diameter distance, (b) luminosity distance, (c) proper motion distance, (d) parallax distance. Part (e) shows how one of them, the angular diameter distance, really looks in curved space.

Proper Motion distance

$$r_m = \frac{v}{\dot{\theta}}$$

where v is the velocity of the source transverse to the line of sight and $\dot{\theta}$ is the angular velocity.

Luminosity Distance

$$r_l = \sqrt{\frac{L}{4\pi l}}$$

where L is the intrinsic luminosity while l is the apparent luminosity.

In general

$$r_a = \frac{r_m}{1+z} = \frac{r_l}{(1+z)^2}$$

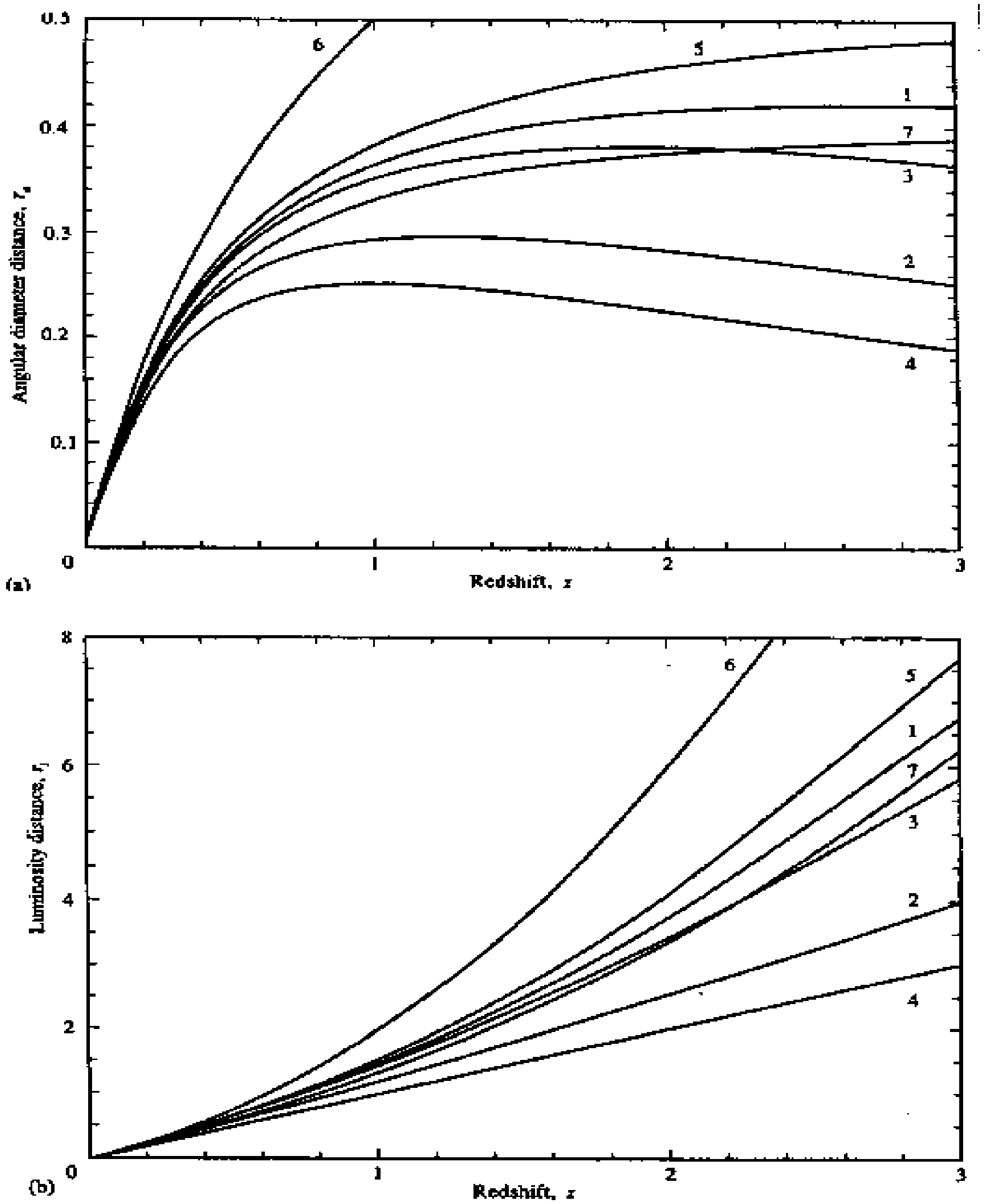


Figure Tests for the models of Figure 3.1: (a) angular diameter distance (b) luminosity distance, and (c) magnitude. Model 7 is for a completely clumpy, $\alpha = 0$, fla dust universe (see Section 3.6). The magnitude used is $m = 22.95 + 5 \log r_l$ corresponding to $m(z = 0.1) = 18$ for model 2. Note that the ordering of the models is the same for the three tests.

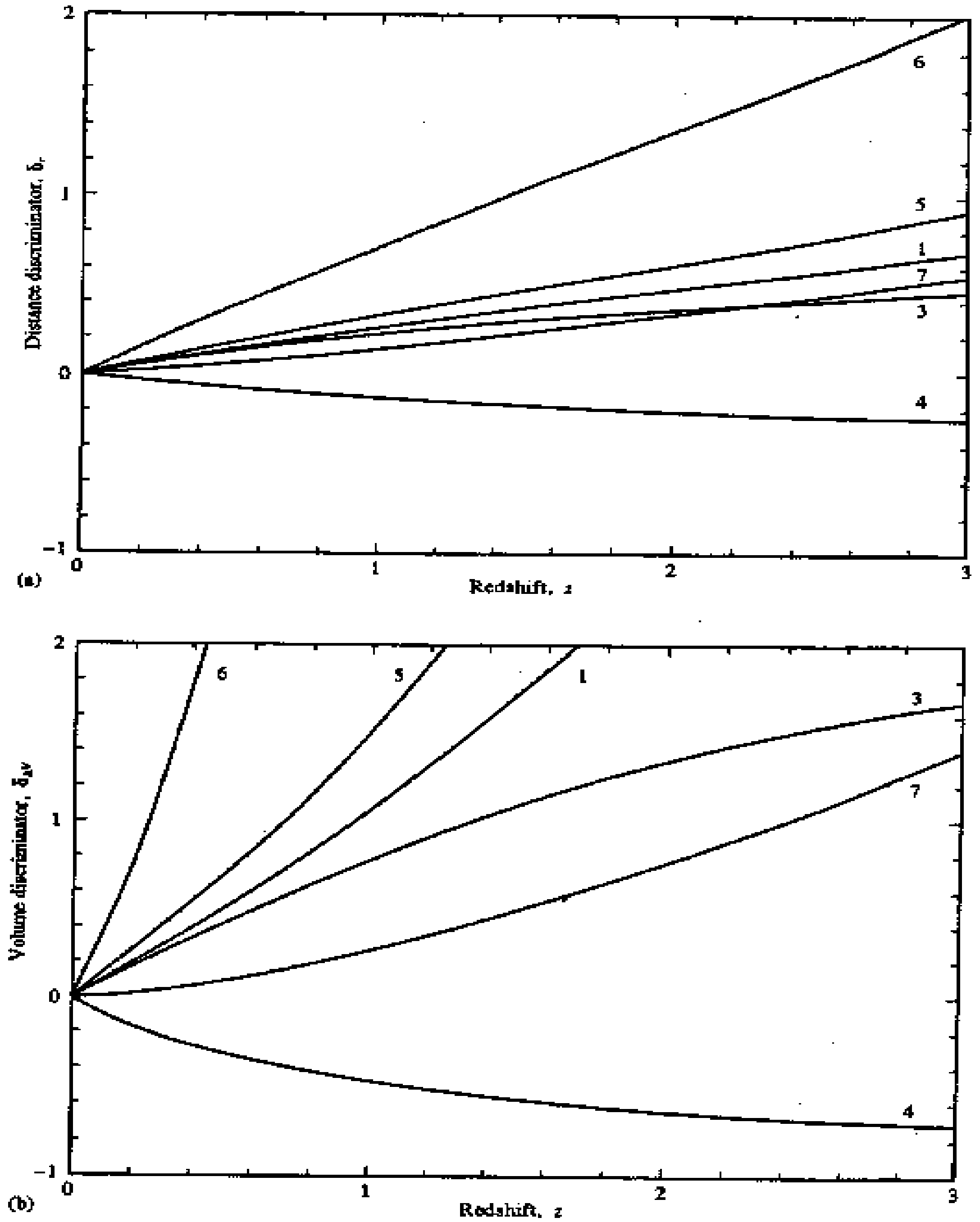


Figure Model discriminators for (a) the distance, (b) the volume element, and (c) the magnitude tests for the models of Figure 3.1, all on the same scale; the fiducial model is taken to be model 2, i.e. $\sigma = 0$, $\Omega = 1$. The discriminators for the angular diameter, luminosity, and proper motion distances are all identical. For δ_m models 1, 3, 5 are almost indistinguishable.

VOLUME TEST

Extension of the concept of distance: volume

$$dV = dA\ dp_{tr} = r_a^2\ d\Omega\ dp_{tr}$$

In physical units

$$V = \Omega \int_1^{1+z} dy\ \frac{y^2}{H(y)}\ r_a^2$$

NUMBER TEST

This amounts to measuring the number of objects above a certain luminosity (given by the distribution $\Phi(L)$)

$$\mathcal{N} = n(z) \int_{L_0}^{\infty} dL\ \Phi(L)$$

where $n(z)$ is the comoving density.

$$N(z) = \Omega \int_1^{1+z} dy\ \frac{y^2}{H(y)}\ r_a^2\ n(z) \int_{L_0}^{\infty} dL\ \Phi(L)$$

gives the number of a "survey complete out to z".

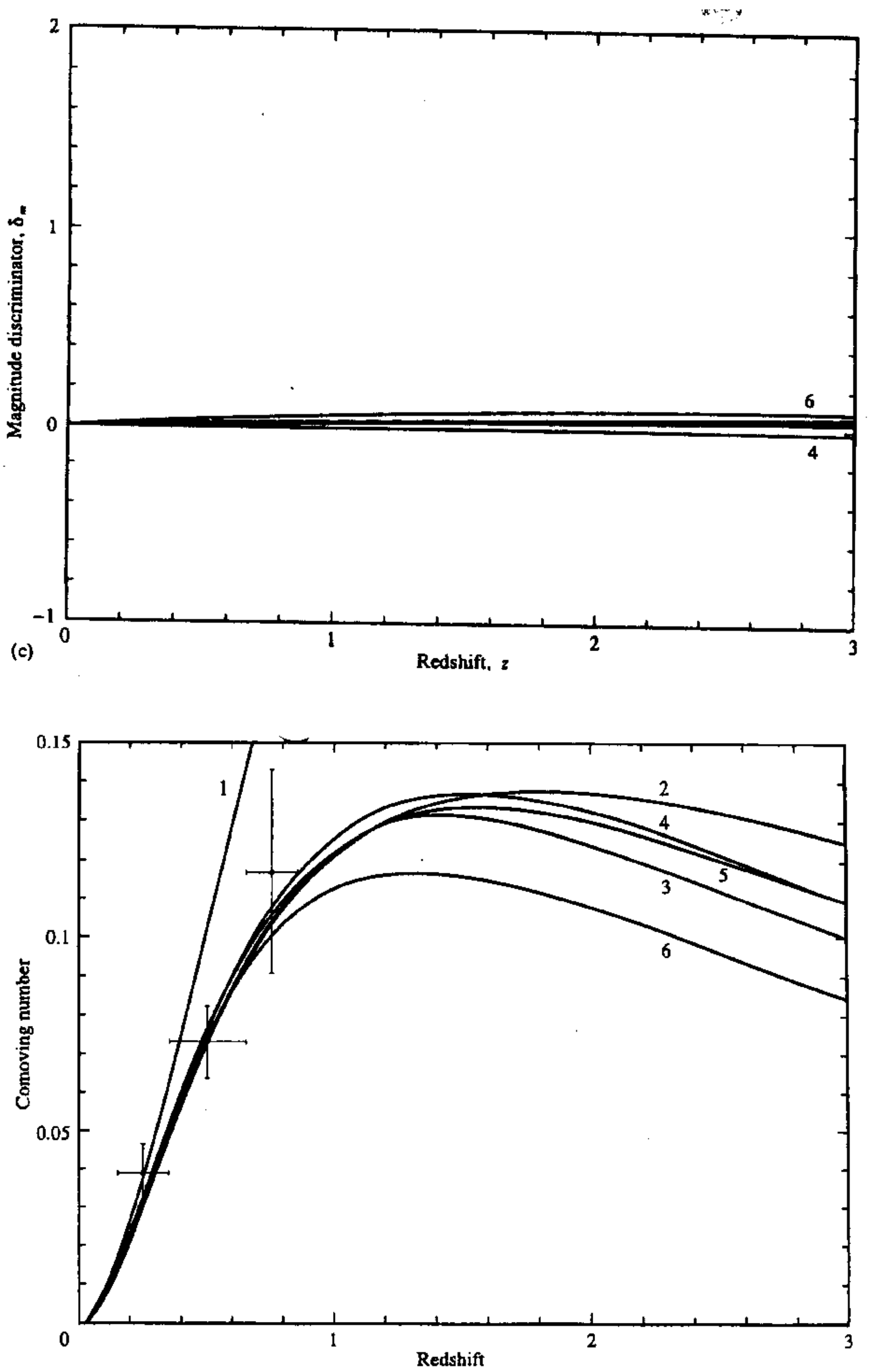


Figure 3.6 The number–redshift test, illustrating the equivalence of components under the exchange rule (3.27). What is actually plotted is the comoving volume element per solid angle, $dV/dz\,d\omega$. Models 1 and 2 are $\Omega_0 = 0.1$, 1 as in Figure 3.1; models 3 to 6 are the four listed in (3.27), respectively. The data points are adapted from Loh and Spillar (1986) *Astrophysical Journal* **307**, L1.

Scaling

L$\sim$ a	$v_{pec} \sim 1/a$
$\rho_{matter} \sim 1/a^3$	$\rho_{rad} \sim 1/a^4$
$\lambda \sim a$	$\nu \sim 1/a$
$T_{rad} \sim 1/a$	$n \sim 1/a^3$

$$s \sim T^3/n \sim constant$$

Hubble's law:

$$v = \frac{dr}{dt} = \frac{d(ar_0)}{dt} = r_0 \ \dot{a} = \frac{\dot{a}}{a} \ r = H \ r$$

modulo peculiar velocities.

Since $T \sim 1/a$ we obtain

$$T(z) = T_0(1+z) = 2.73(1+z)\ {}^oK = 2.4\times 10^{-4}(1+z)\ eV$$

The temperature, especially measured in units of energy is very useful since it alludes to the type of physics that is relevant for a given period in the evolution of the universe.

For $z < z_{eq} \simeq 2.3 \times 10^4\ \Omega_0 h^2$ non-relativistic matter dominates. This corresponds to $t_{eq} \simeq 1500(\Omega_0 h^2)^{-3/2} yr$. The timescale is given by

$$t = t_{eq}(z_{eq}/z)^2 \simeq 10^{12}\ z^{-2}(\Omega_0 h^2)^{1/2} yr \quad , \quad z > z_{eq}$$

$$t = t_0(1+z)^{-3/2} \simeq 10^{10}(1+z)^{-3/2} yr \quad , \quad z < z_{eq}$$

Combining the temperature and time relations we can write

$$T \simeq \left(\frac{t}{1\ sec}\right)^{-1/2}\ MeV\ ,\ T > T_{eq} \simeq 10\ eV\ ,\ t < t_{eq} \simeq 4\times 10^{10}\ sec$$

$$T = 2.73(t/t_0)^{-2/3}\ {}^oK\ ,\ T < T_{eq} \simeq 10^5\ {}^oK\ ,\ t > t_{eq} \simeq 1500\ yr$$

Table 15.4 Thermal History of the Universe, from the Annihilation of $\mu^+\mu^-$ Pairs Until the Decoupling of Matter and Radiation[a]

T (°K)	R/R_0	T/T_ν	t (sec)
10^{12}	1.9×10^{-12}	1.000	0
6×10^{11}	3.2×10^{-12}	1.000	1.94×10^{-4}
3×10^{11}	6.4×10^{-12}	1.000	1.129×10^{-3}
2×10^{11}	9.6×10^{-12}	1.000	2.61×10^{-3}
10^{11}	1.9×10^{-11}	1.000	1.078×10^{-2}
6×10^{10}	3.2×10^{-11}	1.000	3.01×10^{-2}
3×10^{10}	6.4×10^{-11}	1.001	0.1209
2×10^{10}	9.6×10^{-11}	1.002	0.273
10^{10}	1.9×10^{-10}	1.008	1.103
6×10^{9}	3.1×10^{-10}	1.022	3.14
3×10^{9}	5.9×10^{-10}	1.081	13.83
2×10^{9}	8.3×10^{-10}	1.159	35.2
10^{9}	2.6×10^{-9}	1.346	1.82×10^{2}
3×10^{8}	9.0×10^{-9}	1.401	2.08×10^{3}
10^{8}	2.7×10^{-8}	1.401	1.92×10^{4}
10^{7}	2.7×10^{-7}	1.401	1.92×10^{6}
10^{6}	2.7×10^{-6}	1.401	1.92×10^{8}
10^{5}	2.7×10^{-5}	1.401	1.92×10^{10}
10^{4}	2.7×10^{-4}	1.401	1.92×10^{12}
4×10^{3}	6.3×10^{-4}	1.401	1.20×10^{13}

[a] The values for R/R_0 are derived assuming a present radiation temperature $T_{\gamma 0} = 2.7$°K. The last few values for t are derived under the assumption that the energy density of matter was still negligible compared with that of photons and neutrinos. Values of T/T_ν and t for $T > 10^8$°K are taken from P. J. E. Peebles, Ap. J., **146**, 542 (1966).

From Weinberg's book " Gravitation and Cosmology"

the history

- At $T \sim 10^{19}$ GeV quantum gravity effects cease to be important (around $t \sim 10^{-43}$ sec). The graviton gas decouples here. Today it has $T \simeq 1\ ^oK$ and relative density $\Omega_{gr} \simeq 10^{-6.5}$.

- GUT phase transition (?) Around 10^{16} GeV.

- Inflation. Around 10^{15} GeV

- Baryogenesis. WE need one baryon per 10^8 photons. The 3 Sakharov postulates must concur for it to happen:(1) CP violation (2) Baryon number violation (3) Out of thermal equlibrium (phase transition).

- Quark confinement phase transition. Temperature uncertain.

- Electroweak phase transition. Decoupling of neutrinos.

- Nucleosynthesis. $T \simeq 1$ MeV.

- Decoupling of neutrinos. Today $T_\nu \simeq 2\ ^oK$.

- Positrons fall out of equilibrium and disappear.

- Nuclei capture electrons to form atoms.

- Electromagnetic radiation decouples from atoms. Today $T_\gamma \simeq 2.73\ ^oK$.

- Large scale structure starts forming.

The particle horizon (physical distance) is

$$r_H(t) = a(t)\int_0^t \frac{dt}{a(t)}$$

It is finite provided $w > -1/3$. The angular extend of the particle horizon at recombination ($z \simeq 10^3$) is

$$\theta_{hor} = \frac{r_H}{r_a} = 2^o\ \sqrt{\Omega\frac{10^3}{z}}$$

and is much smaller than the homogeneity we observe. This is the **horizon problem**.

Also

$$1-\Omega(z) \simeq (1-\Omega_0)(1+z)^{-(1+3w)}\Omega_w(0)$$

since $\Omega(0)$ today is close to one, it has been

$$1-\Omega \sim 10^{-61} \quad ,$$

at the Planck time ($z_P \sim 10^{32}$) We need **fine tuning**. This is the **flatness problem**.

Monopole/defect problem: During GUT phase transitions too many topological defects are produced. These have not been observed.

All of the above can be solved provided there was a fast expansion of the universe somewhere in the past.

We assume that a scalar field drives a phase transition (Figure) When in the false vaccum the evolution of the scale factor is dominated by the vaccum energy (cosmological constant). During this period the universe expands exponentially. Then the field relaxes to the true vaccum. During relaxation its oscillations reheat the universe.

Inflation solves the horizon problem by expanding distances superluminally (exponential expansion).

Inflation solves the flattness problem since it brings Ω arbitrarily close to one with eponentiall acuracy

Inflation sdoves the topological defect problem by diluting their density to the point that they are not observable

It turns out we need Inflation to produced around 60 e-foldings.

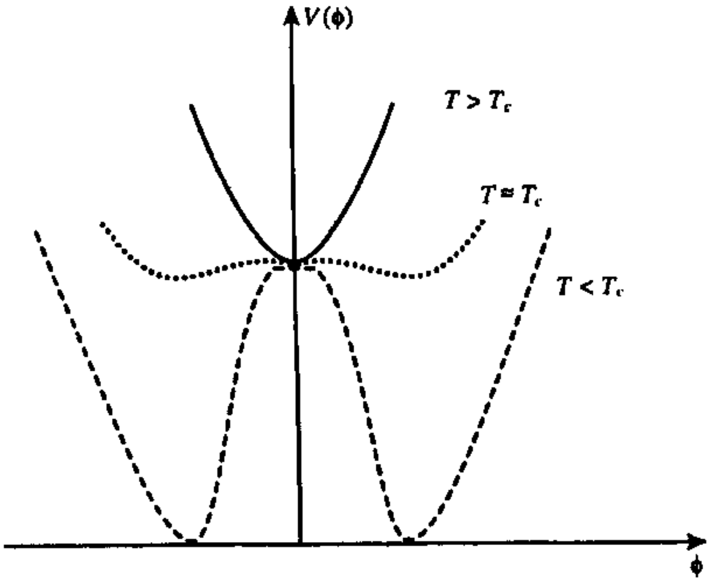


Figure Schematic scalar potential for inflation, showing a phase transition as the universe cools below some critical energy T_c. The important points are the displacement from the minimum energy value and the relative flatness of the potential, acting as a slowly changing energy density (slow roll). Reheating occurs as the field finds the minimum and converts the excess energy into particle production (and perturbations).

The Hubble Expansion

Based on simple physics: atomic transitions and general relativity.

Predicted by General Relativity (assuming symmetry)

1912	Slipher observed a source with $\frac{\Delta\lambda}{\lambda} = -0.001$
1922	30 redshifts measured
1982	$\sim 3 \times 10^3$ redshifts measured
1992	$\sim 10^5$ redshifts measured (~ 10 negative)
1999	$\sim 10^6$ redshifts (Sloan Digital Sky Survey)

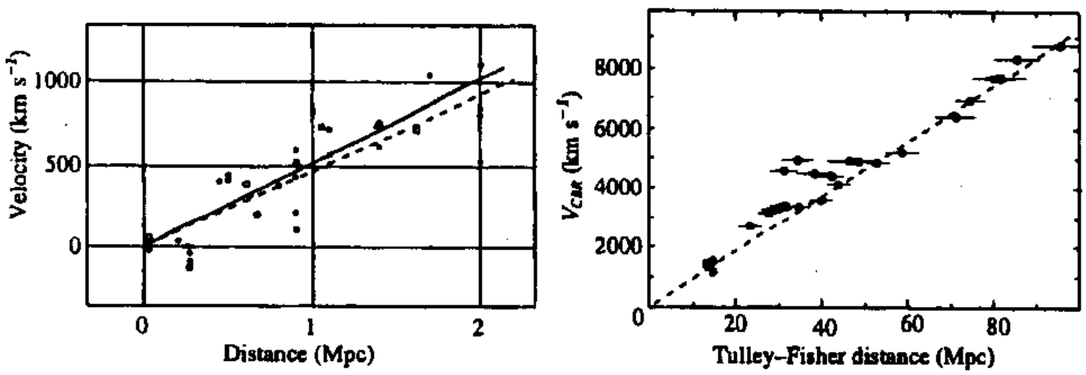


Figure Hubble diagrams showing the detected spectral shift, equivalent to a Doppler velocity of recession, vs. distance to the objects obtained by other astrophysical methods. The left diagram is Hubble's original plot from 1929 (*Proc. Nat. Acad.* **15**, 168); the right is a modern version (from Mould, J. *et al. Ap. J.* **383**, 467). Note the much greater distance out to which the modern data shows linearity.

Examples of spectra

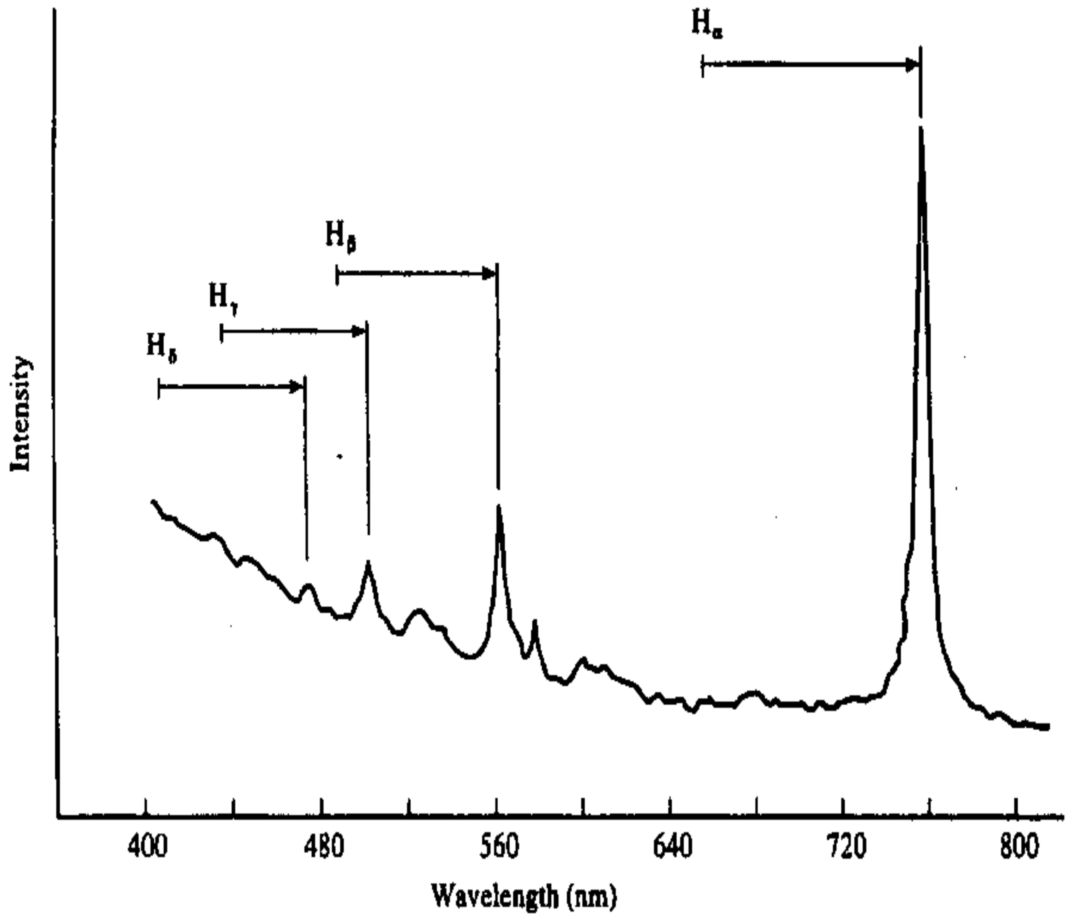


Figure Astronomical spectrum of quasar 3C 273 showing the shift of the emitted electromagnetic radiation. The features indicated are hydrogen Balmer lines. Comparison with the laboratory wavelengths of these atomic transitions gives a redshift of $z = \Delta\lambda/\lambda = 0.158$. (From Kaufmann, W.J., 1994, *Universe*, 4th edn.)

Properties

• **Positivity**: Almost all $z > 0$. A few negative come from local Virgo cluster.

• **Frequency independence**: The same redshift is measured from radio frequencies to γ-rays, over a range 10^{12} in frequency. This is impressive.

• **Linearity**: In some case the distance can be measured by independent means. Measurements then indicate that $z \sim r$ with a 10% accuracy.

• **Isotropy**: Once the distance dependence is removed, then redshifts are isotropic.

Potential explanations:

• **Movement of the source relative to the observer**

This is also known as the Doppler effect.

• **Time dependent gravitational phenomena**

This com from the geometry. The expansion of the universe is the prime example.

• **Static gravitational phenomena** This comes from static gravitational fields. The redshift due to static fields is the prime example.

• **Interactions (scattering, absorption etc)**

This is an envirionmental explanation and relies on particle physics processes.

Galilean Doppler Shift

A source is moving with velocity $\vec{v}$ with respect to an observer $\vec{x} = \vec{x}_0 + \vec{v}t$.

A pulse is emitted at time t_e and moves (isotropically) with velocity v_s. It is observed at time

$$t_o = t_e + \frac{d}{v_s} \quad , \quad d = \widehat{n} \cdot \vec{x} = \widehat{n} \cdot \vec{v}\; t_e$$

so that

$$\frac{\Delta t_o}{\Delta t_e} = 1 + \frac{\widehat{n} \cdot \vec{v}}{v_s} \quad , \quad \frac{\Delta \nu}{\nu} = \frac{\widehat{n} \cdot \vec{v}}{v_s} = \frac{v_{\text{radial}}}{v_s}$$

For electromagnetic waves $v_s = 1$. If the sources is receding $v_{\text{radial}} > 0$ and $z > 0$

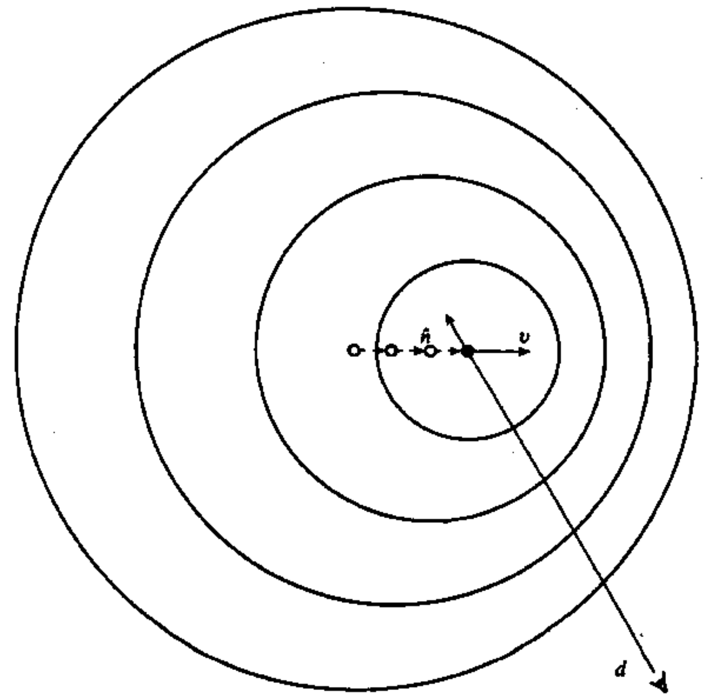


Figure Wave fronts from a moving source. In the direction of motion the fronts are closer together (shorter wavelengths) so the frequency is higher (blue shift). In the story of the battling armies, the equivalent interpretation is that the general stays stationary in the center and the circular front lines move.

Relativistic Doppler Shift

There is no transverse Doppler shift in galilean physics.

There is however in special relativity. The previous relations are modified

$$t_o = \gamma \left(t_e + \frac{d}{v_s} \right)$$

$$\frac{\Delta t_o}{\Delta t_e} = \gamma \left(1 + \frac{v_{\text{radial}}}{v_s} \right)$$

$$\Rightarrow z = -1 + \frac{1 + \frac{v_{\text{radial}}}{v_s}}{\sqrt{1 - v^2}}$$

For $v_r = 0$

$$z = -1 + \frac{1}{\sqrt{1 - v_t^2}} \simeq \frac{1}{2} v_t^2 + \dots$$

which is quadratic in velocity.

Kinematic redshift

The Hubble relation is due to kinematics only.

Weyl: Homogeneity+Isotropy→ FRW metric → expansion and redshift

$$ds^2 = -dt^2 + a(t)^2 F^2(r)(dr^2 + r^2 d\Omega_2^2)$$

a photon path is given by $ds = 0$

$$\frac{dt}{a(t)} = dr F(r)$$

Consider light emitted at t_e with period T_e and received at t_o with period T_o:

$$\int_{t_e+NT_e}^{t_o+NT_o} \frac{dt}{a(t)} = \int_0^R dr F(r)$$

This should be independent of N so that:

$$\int_{t_e+T_e}^{t_o+T_o} \frac{dt}{a(t)} = \int_{t_e}^{t_o} \frac{dt}{a(t)}$$

take T_e, T_o small

$$\frac{T_o}{a_o} = \frac{T_e}{a_e}$$

with $T_i\ a_i << 1$. This is good for all observations.

$$z = \frac{a_o}{a_e} - 1$$

as advertised.

Expansion $\to z > 0$

At small distances, $a(t_e) \simeq a(t_o) - (t_o - t_e)\dot{a}(t_o) + ...$ Use $r = t_o - t_e$ and $H_o = \dot{a}(t_o)/a(t_o)$

$$z = H_0 \ r$$

If we think in terms of Doppler shift, $z = v_r$ and then $v_r \simeq H_0 \ r$. At larger distances, ther are different types of distances with different relations to redshift.

If the source is local then it can also have a "peculiar" velocity and

$$z = H_0 \ r + v_r^{\text{peculiar}}$$

Typically $|v_r^{\text{peculiar}}| \sim 10^{-3}$ so they do not contribute to the redshift for objects further than a few Mpc.

General Redshift Expression

$$z = -1 + \frac{(k \cdot u)_e}{(k \cdot u)o}$$

where k is the photon 4-momentum and u is the emitter 4-velocity, and $k \cdot u \equiv g_{\mu\nu} k^\mu u^\nu$.

Special Cases

Special Relativity: $g_{\mu\nu} \sim$ diag(-1,1,1,1)

$k \cdot u = -k^0 u^0 + \vec{k} \cdot \vec{u}$, $k^2 = 0 \to \vec{k} = -k^0 \hat{n}$

In general $u = (\gamma, \gamma\vec{v})$ so that $u_o = (1, \vec{0})$.

$$z = -1 + \frac{k \cdot u_e}{k \cdot u_o} = -1 + \frac{k^0 \gamma(-1 - \hat{n} \cdot \vec{v})}{-k^0} = -1 + \gamma(1 + \hat{n} \cdot \vec{v})$$

FRW: From mass-shell condition $k^0 = a(t)|\vec{k}|$ and both emitter and observer are comoving $u = (1, 0)$. Thus,

$$z = -1 + \frac{(k^0)_e}{(k^0)_o} = \frac{a_o}{a_e}$$

In general redshifts appear because one component in $u \cdot k = g_{\mu\nu} k^\mu u^\nu$ changes:

- u changes $\to$ Doppler.
- g changes $\to$ gravitational redshift.
- k changes $\to$ interactions.

Gravitational lensing

Imagine there is a gravitating body in-between source and observer. Then,

$$z = -1 + \frac{(u \cdot k)_e}{(u \cdot k)_d} \frac{(u \cdot k)_d}{(u \cdot k)_o}$$

If the system is static there is no effect!

If not then $\Delta p = F \ \Delta t$, $F = \frac{GM}{b^2} \frac{p}{c}$, $\Delta t = b/c$. The (small) angular deflevtion δ is then given by

$$\delta \simeq \frac{\Delta p}{p} = \frac{GM}{bc^2} = \frac{\Phi}{c^2}$$

$$1 + z = \frac{(u \cdot k)_e}{(u \cdot k)_{d_+}} \frac{(u \cdot k)_{d_+}}{(u \cdot k)_{d_-}} \frac{(u \cdot k)_{d_-}}{(u \cdot k)_o} = \frac{(u \cdot k)_{d_+}}{(u \cdot k)_{d_-}}$$

In our case, $u = \gamma(1, \vec{v})$, $k_- = (|\vec{k}|, \vec{k})$, $k_+ = k_- + \Delta k$, $\Delta k = (0, \Delta\vec{k})$ $(\Delta E = \mathcal{O}(\Delta k^2))$ so that

$$1+z = 1 - \frac{\Delta k \cdot u}{k_- \cdot u} = 1 - \frac{\Delta\vec{k} \cdot \vec{v}}{-\vec{k} + \vec{k} \cdot \vec{v}} \simeq 1 + \frac{\Delta\vec{k} \cdot \vec{v}}{\vec{k}} \simeq 1 + |\vec{v}|\delta$$

$$z \simeq v \ \Phi$$

Fitting the Hubble expansion

• $z > 0$ Naturally explained by expansion.

• Frequency independence. Always true if geometric optics approximation is valid → "small" wavelegths.

• Linearity. Arises naturaly from taylor expansion without the use of equations of motion. (only homogeneity+isotropy)

$$\vec{v}_{rel} = H_0 \ \Delta\vec{r}$$

A shift of coordinates does not affect the relation.

WE can imagine the most general linear relation

$$v^i = H^i j_{x_j}$$

H can be decomposed into the trace $H_{ij} = H\delta_{ij}$, symmetric traceless and antisymmetric parts.

• The trace is uniform expansion.

• The antisymmetric piece is rotation (experimentally $\omega < 10^{-3}$)

• The tensor piece is shear (experimentally $\omega < 10^{-6}$).

Non-trivial O(3) reps break rotation invariance.

The Hubble Constant

This is one of the most important cosmological paran-

meters. Its value

$$H_0 = (70 \pm 10) \quad Km/sec/Mpc$$

is known to a 15 % accuracy. Measuring it is no trivial feat:

• At klarge distances we cannot measure distance accurately

• At small distances data are tainted from peculiar velocities.

This is the only dimensionful cosmological parameter $\rightarrow$ sets the scale.

Agrees well with other "dating" methods (using radioisotopes and nuclear energy production). (see exercize).

Redshift Tests

• Microwave background temperatures: Test $T \sim 1/a$ or

$$s \sim \frac{T^3}{n} \sim \text{constant}$$

or

$$T_o = T_e(1+z)^{-1}$$

Look at rotational energy transition in CN or other molecules. The lines are in the microwave part of the spectrum and can be excited by the CMBR.

▷ From CI we obtain $T = 7.4 \pm 0.8^o K$ at $z = 1.78$ agreeing well with theory $T = 7.6^o K$.

▷ From CII we obtain $T < 13.5^o K$ at $z = 2.9$ agreeing with $T = 10.7^o K$.

• **Surface Brightness**

I_ν=energy/(area · time · frequency · solid angle)

$$I_o = I_e(1+z)^{-4}$$

$$I_\nu \sim \nu^3 \Rightarrow I = \int I_\nu d\nu \sim \nu^4$$

Olbers Paradox

If the universe is filled with stars sp that every lines meets a star the sky would be as bright as the surface of a star.

The energy flux of an emitter dies off as $L \sim 1/r^2$ but the number of sources grows as $\sim r^2$ so that the enrgy flux scales like constant. Integrating over r from zero to infinity gives an infinte result.

Flaw: there is a cosmological horizon.

alternative theories for redshift

- Doppler shifts (incompatible with z almopst always positive).

- Tired light. Interactions must be postulated abnd from $dE/dr = -kE$ we obtain $z = e^{kr} - 1$ linear only for small z.

- Photon mass: $m_\gamma < 10^{-27}$ eV.

- Wolf shift. Correlation of light from sources that can cause shifting of frequencies. Redistributes the number of photons in the frequency range.

- Static gravitational redshift. Fails isotropy and size.

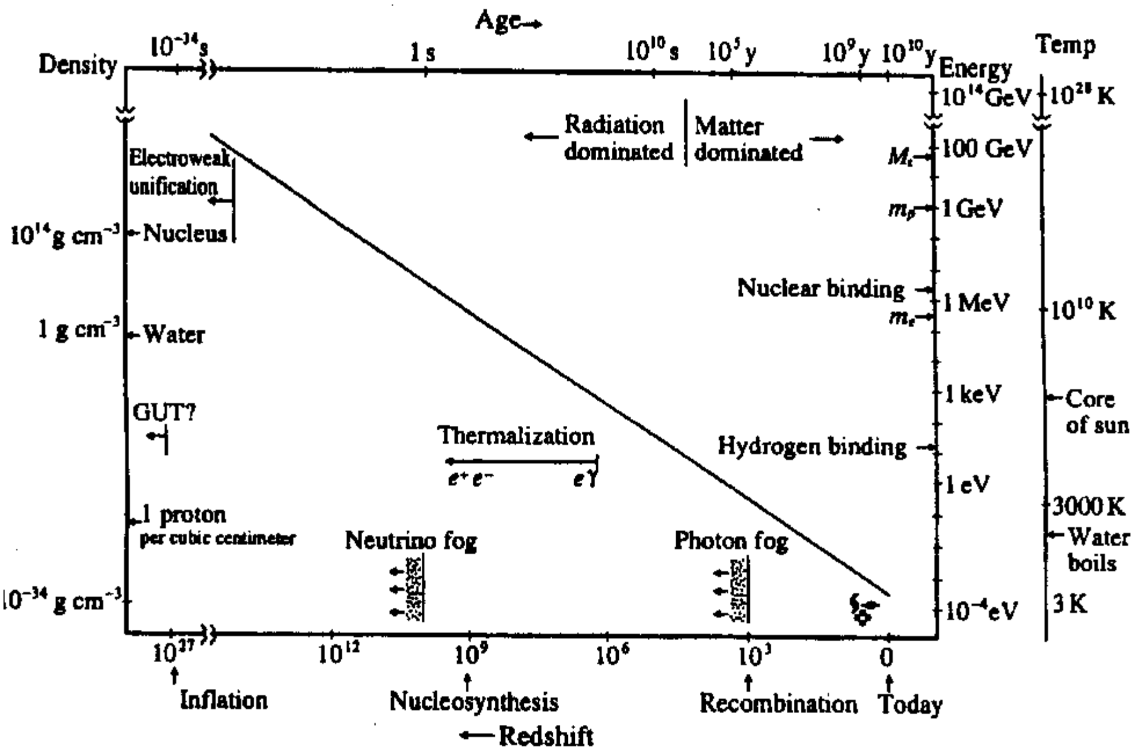


Figure History of the universe showing significant events, particle interactions, and barriers in terms of the radiation temperature, energy, or energy density vs. time or redshift. The radiation behavior is exquisitely simple, yet packed with information as shown along the axes.

Nucleosynthesis

Nucleosynthesis probes times scales a few minutes after the Big Bang far earlierr than CMBE ($t > 10^5$yrs) or Hubble expansion data ($t > 10^9$ yrs). It rests on well founded high energy nuclear and statistical physics.

He^4 must be synthesized during the hot early stage of the universe. The luminoscity of a typical galaxy is $L \simeq 10^{11} L_\odot = 2.5 \times 10^{56}$ eV/sec. Over a Hubble time the total energy output is $E = L \times 10^{10} yrs \simeq 7.5 \times 10^{73}$ eV. There are $M/m_p \sim 10^{12} M_\odot/m_p \sim 10^{69}$ nucleons in a galaxy. So the total energy produced per nucleon is 0.06 MeV. This well below the one (7 MeV produced for $H \to He$ conversion.

Nuclei starting with ^{12}C or heavier are synthesized in stars via the triple α process. The nuclear density during nucleosynthesis is $10^{20}/cm^3$ while in stellar cores it is $10^{29}/cm^3$.

Important concepts in nucleosythesis are statistical equilibrium and its breakdown: freezout

- At $T > 1$ GeV there are baryon-antibaryon pairs in equilibrium (hadron era). Below that temperature antibaryons dissappear and we are left with baryons only (baryogenesis).

- The lepton era: pairs of leptons and antileptons $(e^+ - e^-)$ dominate the soup. Below $T = 1$ MeV positrons dissappear.

- We enter the photon era. Nucleosynthesis takes place here.

Nucleosynthesis

The simplest nucleus H^2 has binding energy 2.2 MeV. Synthesis starts when the nuclear reactions fall out of equlibrium.

A reaction timescale is given by

$$\Gamma = 1/t_{reac} = n < \sigma v >$$

where n is the density, σ is the cross section and v is the velocity.

If there is no expansion the temperature remains constant and the reactions $X \to Y$ and $Y \to X$ remain at equilibrium (detailed balance ot time reversal).

Expansion provides another time scale H. If $\Gamma >> H$ the interaction remains in equilibrium. If $\Gamma << H$ the expansion is fast and interactions are not relevant (freezeout).

Consider the interactions after time t:

$$N_{int}(> t) = \int_t^\infty dt' \Gamma(t')$$

. $\Gamma \sim T^n$ and for radiation domination $T \sim t^{-1/2}$ and $H \sim 1/(2t)$. So that $N_{int} = (n-2)^{-1}\Gamma/H$. For $n > 2$ $N_{int} < 1$ if $\Gamma < H$ (freezeout).

From the definition $\Gamma \sim T^3\ \sigma$ for relativistic particles. For weak interactions $\sigma \sim T^2$ while for electromagnetic $\sigma \sim 1/T^2$.

Interactions and Timescales

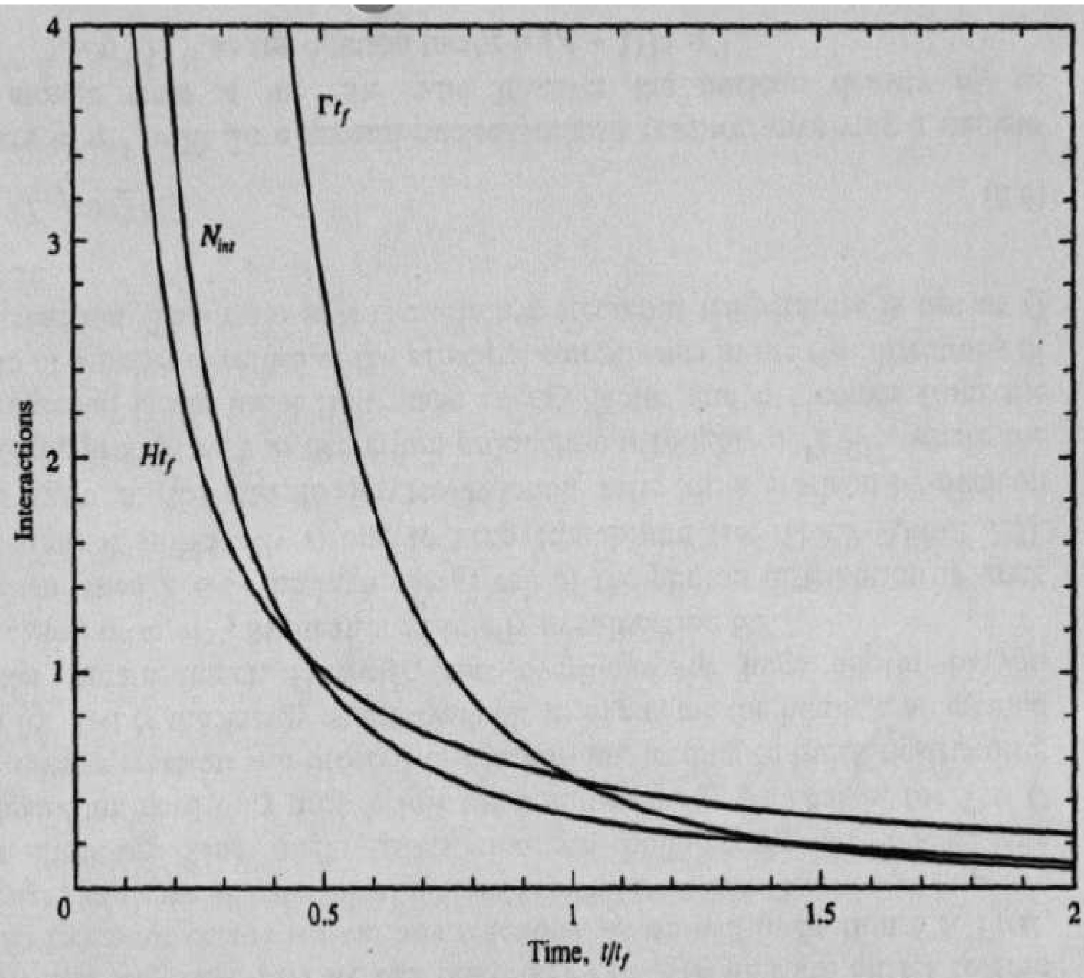


Figure 6.1 Interaction and timescales in the expanding universe. The interaction rate Γ, expansion rate H, and the number of interactions after time t are plotted vs. time. The two rates become equal at the freezeout time t_f; the number of interactions N_{int} drops below one for $\Gamma < H$.

Statistical expressions and kinetics

Using

$$\rho_i(T_i) = \frac{1}{(2\pi)^3} g_i \int d^3\vec{p}\ f(\vec{p})\ E(\vec{p})$$

with the standard distribution function

$$f = 1/(exp(E-\mu)/T_i \pm 1)$$

we obtain the energy density of radiation as

$$\rho = \frac{\pi^2}{30} g_*\ T^4$$

$$g_* = \sum_{bosons} g_i (T_i/T)^4 + \frac{7}{8} \sum_{fermions} g_i (T_i/T)^4$$

where T is the photon temperature.

So, $t(sec) \simeq [T(MeV)]^{-2}$ to a very good accuracy. For $1MeV < T < 100MeV$ the species that contribute are photons, electrons and positrons (each with g=2) and as many neutrinos are relativistic. This gives $g_* \simeq$ 10.75 during that period.

If Q is the nuclear binding energy, because of the high entropy,

$$T_{freezeout} \simeq Q/20$$

The ration of forward to reverse reactions is $\eta^{-1} e^{-Q/T}$ where η is the entropy per baryon.

Entropy

From thermodynamics we have

$$S \sim \rho V/T \quad \text{and} \quad \rho \sim T^4 \Leftarrow S \sim VT^3 \to s \sim n_\gamma \sim T^3$$

The exact expression is

$$s = \frac{2\pi^2}{45} g_* T^3$$

Today $s = 7.04\ n_\gamma$ and

$$\eta = n_b/n_\gamma = 7.04(n_b/s) = 2.7 \times 10^{-8}\ \Omega_b h^2$$

so the entropy per baryon is huge $s/n_b \simeq 10^{10}\ k$ for $\Omega_b \simeq 0.01$ (During stellar nucleosynthesis $s/n_b \simeq k$.)

From adiabatic expansion we get

$$g_* T^3 a^3 = \text{constant}$$

so $T \sim 1/a$ when no decoupling occurs. During decoupling g_* chnages.

For example after the neutrinos have decoupled

$$T_\gamma/T_\nu = (g_*^{\text{before}}/g_*^{after})^{1/3} = (5.5/2)^{1/3} = 1.4$$

.

Calculation of abundances

The basic building blocks are protons and neutrons and the relevant reactions

$n \leftrightarrow p + e^- + \bar{\nu}$, $n + \nu \leftrightarrow p + e^-$, $n + e^+ \leftrightarrow p + \bar{\nu}$

Use the notation: $c/d = n_c/n_d$, $X_A = A n_A/n_N$ where A is the atomic mass and $n_N = n_n + n_p + \sum_A A\ n_A \simeq n_n + n_p$.

Heuristic derivation of He abundance

Before freezeout the reactions are in equilibrium and the abundance of n/p is determined by Boltzman factors:

$$\frac{n}{p} = \frac{X_n}{X_p} = e^{-Q/T}$$

with $Q = m_n - m_p = 1.29$ MeV. After freezeout the only interaction going is free neutron decay.

The freezeout condition $\Gamma = H$ gives $T_f = 0.8$ MeV, $T_f \sim \tau_n^{-1/3}$ and $(n/p)_f = 0.2$ All neutrons make He so

$$X_4 = 2X_n(t) = 2\ \frac{e^{(t-t_f)/\tau_n}}{1 + e^{Q/T}}$$

($X_4 = 4He/(n+p) = 2n/(n+p) = 2/(1+e^{Q/T})$ modulo neutron decay). Freezeout of He interactions at $t =$ 180 sec so that $X_4 = 27\%$ (exact number is 24%)

There are three methods of calculation:

- Reaction networks.
- Approximate integration.
- Quasistatic analysis.

Nuclear Reaction Networks

If in a reaction $i + j \to k + l$ then $\dot{n}_i = n_i n_j < \sigma v >_{ij\to kl}$ and so on. As an example

$$\dot{X}_d = X_n X_p[npd\gamma] - 2X_d^2([ddpT]+[ddn3]) - X_d X_T[dTn\alpha] -$$

$$- X_d X_3[d3p\alpha] - X_d X_\gamma[d\gamma np] - X_d X_p[dp3\gamma]$$

(T is tritium, 3 is He^3, and α is He^4)

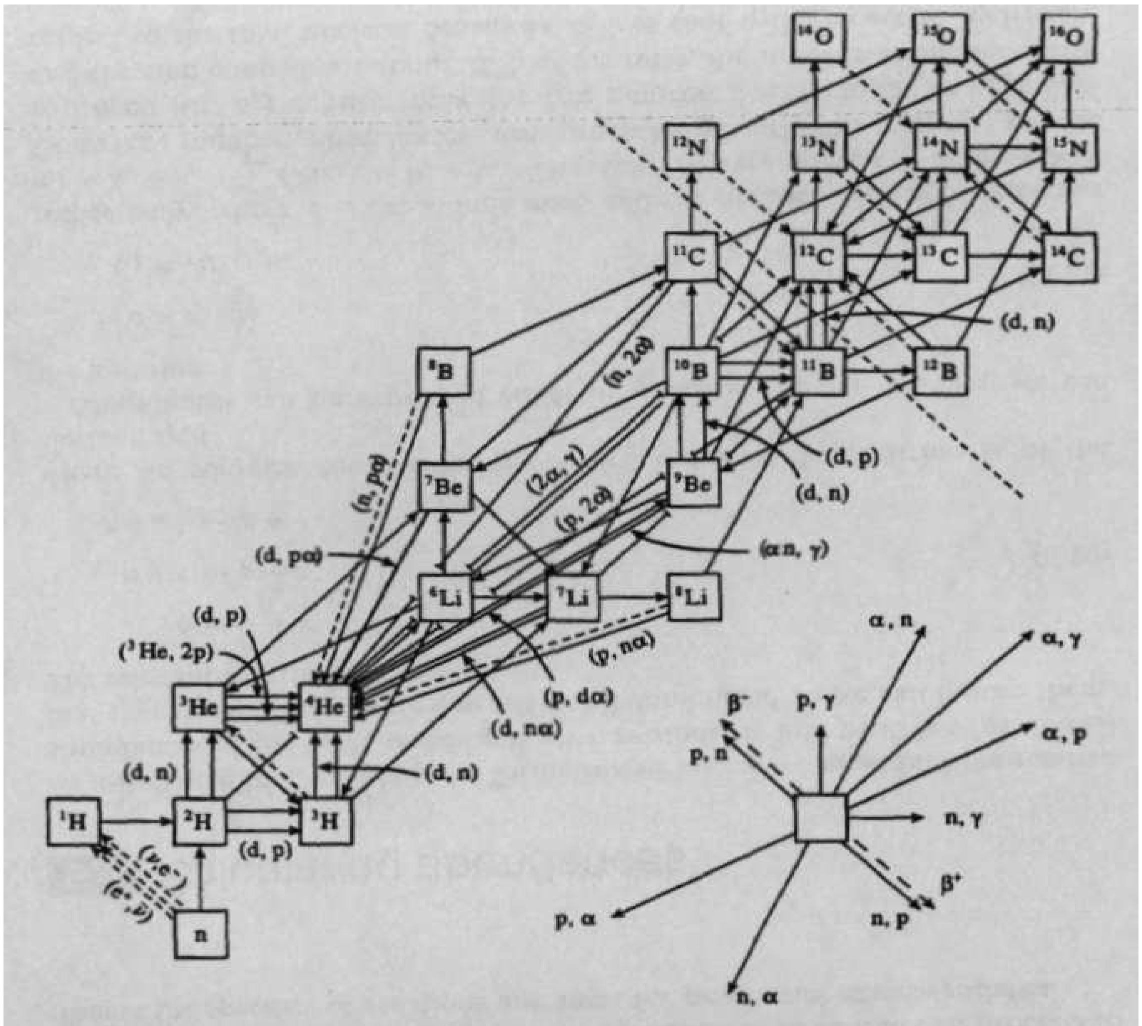


Figure 6.2 A typical nuclear reaction network employed in a primordial nucleosynthesis calculation. The lower right part of the diagram indicates the particles absorbed and emitted to move in the direction of the arrows. (From Wagoner, R.V., 1990, in *Modern Cosmology in Retrospect*, eds B. Bertotti *et al.*, Cambridge University Press.)

• Quasistatic analysis.

The equations are of the form $\dot{X}_i = J - \Gamma X_i$ where J is the source term and ΓX_i the sink term. The decay becomes quasistatic is $X_i = J/\Gamma$ if $\Gamma >> H$. The fixed point solutions is the final abundance.

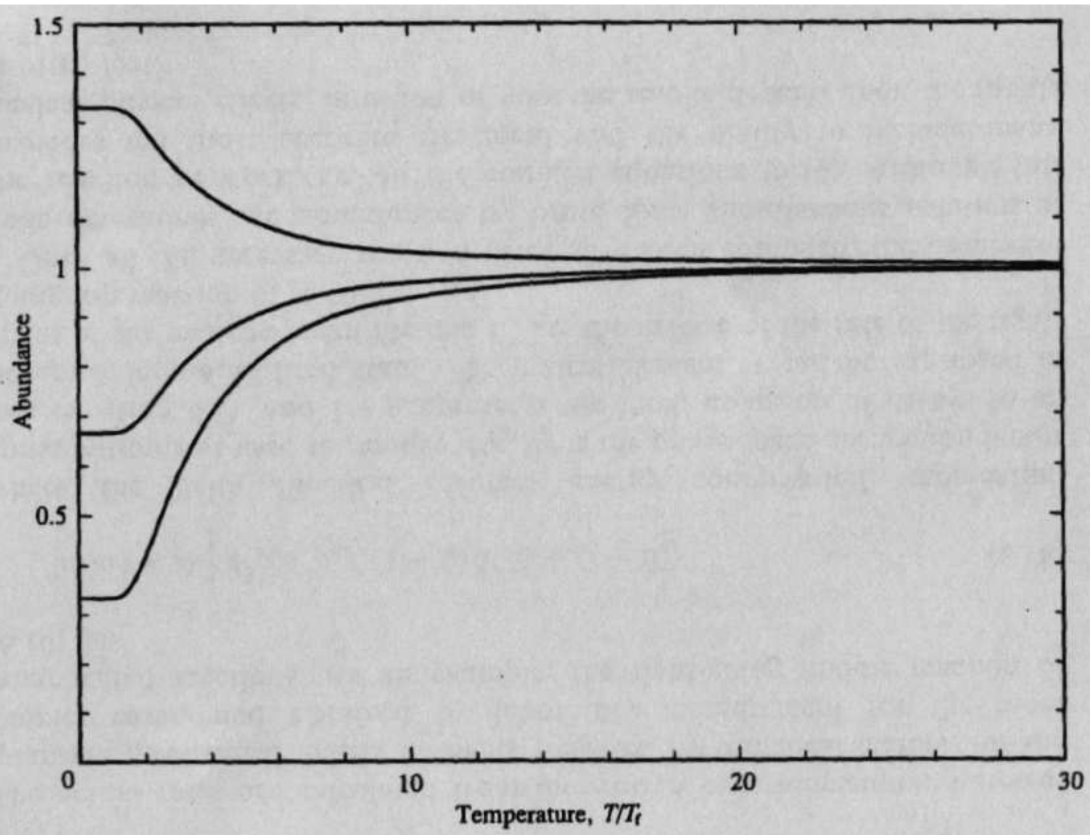


Figure 6.3 Quasistatic and freezeout abundances for interaction products in units of the equilibrium abundance. The curves represent the solutions f to $\dot{f} = J - \Gamma f$, with the source and sink terms having the form as in the deuterium abundance equation. The quasistatic solution $f = J(T)/\Gamma(T)$ is achieved for $\Gamma \gg H$ and frozen in at T_f (note time increases to the left). It is displayed for $J(T_f)/f_0\Gamma(T_f) = 4/3, 2/3, 1/3$ respectively.

Observations of abundances

Nuclosynthesis is a accurate cornerstone of the standard theory. There is agreement with observations over a range of 10 orders of magnitude with inputs: $t_n = 889$ sec, $N_\nu = 3$, $\eta = 4 \times 10^{-10}$.

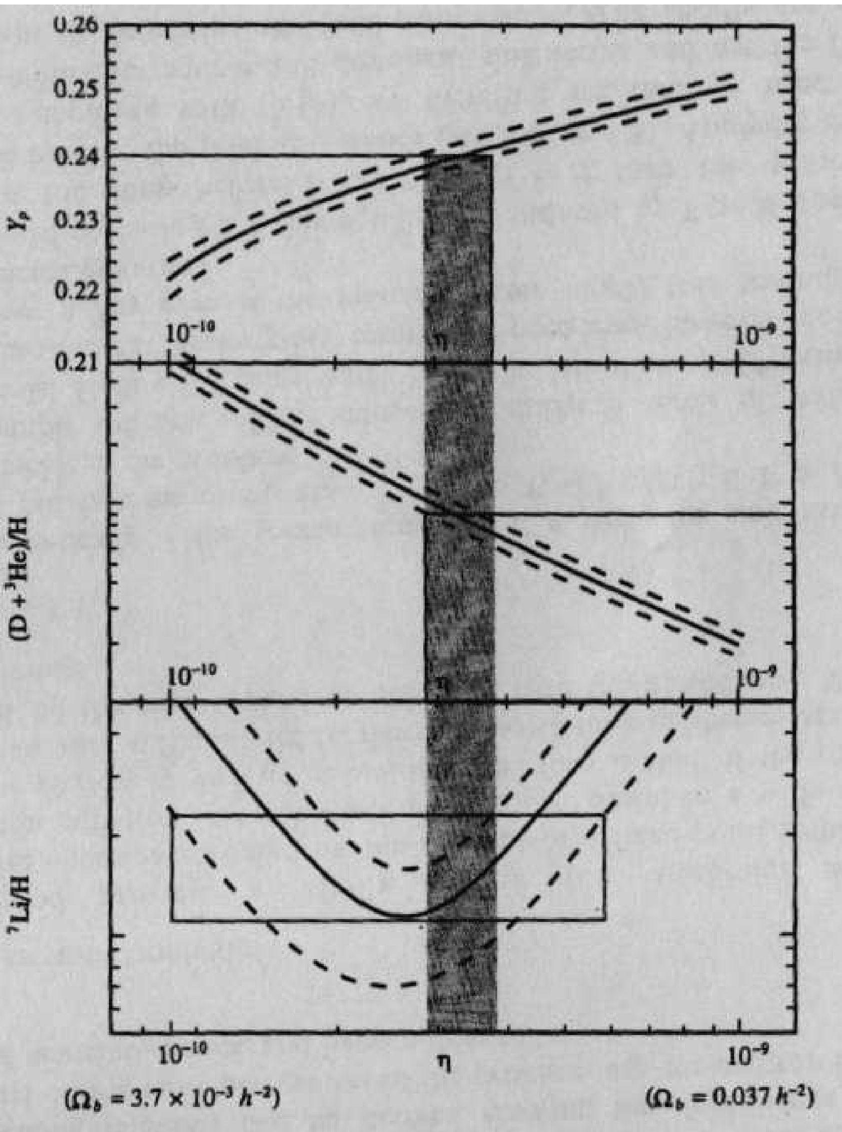


Figure 6.4 The primordial light element abundances are plotted vs. the baryon–photon ratio (entropy). The corresponding baryonic density parameter Ω_b is also shown. Observational constraints are the horizontal cuts, yielding the allowed values for the entropy. The vertical shaded bar shows the range simultaneously satisfying all the elements measured. (After Smith, M.S., Kawano, L.H., and Malaney, R.A., 1993, *Ap. J. Suppl.* 85, 219.)