Exact results in N=2 SCFTs

A review

Elli Pomoni

Heraklion Workshop on Theoretical Physics

10 September 2016

Supersymmetric Gauge theories in 4D

$\mathcal{N} = 4$ Super-Yang-Mills (SYM)

$A_\mu \, , \, \lambda^A \, , \, \phi^{AB}$ $\quad A, B = 1, \ldots, 4 \quad SU(4)_R$

No matter fields allowed! Only "gluons"!
Only the choice of the color group G.

- ✳ AdS/CFT
- ✳ Integrability
- ✳ Localization

$\mathcal{N} = 3$?

$SU(2)_R \times U(1)_r$

$\mathcal{N} = 2$
- Vector multiplet $\quad A_\mu \, , \, \lambda^{\mathcal{I}} \, , \, \phi \quad \mathcal{I} = 1, 2$
- Matter multiplet $\quad Q_{\mathcal{I}} \, , \, \psi \, , \, \tilde{\psi}$

Color group G = $G_1 \times G_2 \times \ldots \times G_n$
and reps of the matter fields

$\mathcal{N} = 1$
- Vector $\quad A_\mu \, , \, \lambda$
- Matter $\quad q \, , \, \psi$

$\mathcal{N} = 0$
- Vector $\quad A_\mu$
- Matter $\quad \psi$

Outline

- *Classification of theories with a **Lagrangian***
- *String/M-theory constructions (discover a 2D surface)*
- *Seiberg Witten theory*
- *S-duality: Gaiotto's class S: **non-Lagrangian** theories*
- *Supersymmetric Localization*
- *Relation between 4D partition functions and 2D CFT correlators*
- *Conclusions and Vision for the future*

Classification of Lagrangian N=2 SCFTs

[Bhardwaj,Tachikawa 2013]

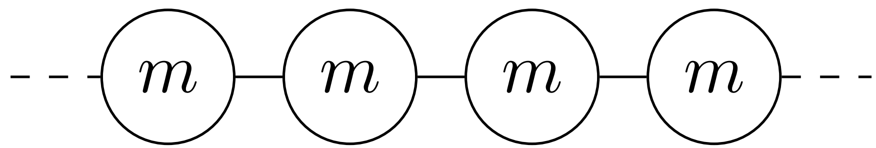
The UV Lagrangian is fixed by N=2 susy:

In the UV: only marginal (conformal theories)
and relevant operators (mass deformations)

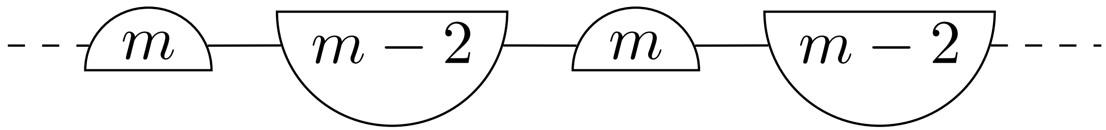
Bootstrap ideology: Begin with (solve) the conformal theory first
then add (study) relevant deformations

✱ *All data are stored in quiver diagrams s.t.* $\beta^{(1)} = 0$

- SU(m) chain



- SO(m)-USp(m-2) chain



- a mixture of the above
- some sporadic

N=2 theories are non-chiral (no arrows)

Moduli space of vacua

Supersymmetric vacua $V = |\,[\phi,\bar{\phi}]\,|^2 + |Q|^4 + |\phi Q|^2 + |mQ|^2 = 0$

- Coulomb Branch: $\langle Q \rangle = 0$ with $\langle \phi \rangle = \mathrm{diag}\,(a_1, \ldots, a_N)$
- Higgs Branch: $m_i = 0$ and $\langle \phi \rangle = a = 0$
- There exist also mixed Branches

Gauge invariant operators whose vevs parameterize the vacua:

- Coulomb operators $u_\ell = \langle \mathrm{tr}\phi^\ell \rangle$ chiral $\mathcal{E}_r$ with $\Delta = r$

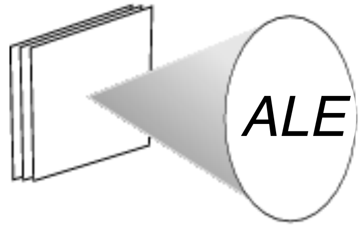
The Lagrangian = Q^4 descendant of the $\mathcal{E}_{r=2}$

- Higgs operators $\mu_{\mathcal{I}\mathcal{J}} = \langle \mathrm{tr}\left(Q_{\{\mathcal{I}}\bar{Q}_{\mathcal{J}\}}\right)\rangle$ real $\hat{\mathcal{B}}_R$ with $\Delta = 2R$
- There are operators that parameterize mixed branches

String/M-theory constructions

String/M-theory constructions

IIB	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$\mathbb{Z}_M$	.	.	.	.	.	.	—	—	—	—
N D3-branes	—	—	—	—	.	.	.	.	.	.

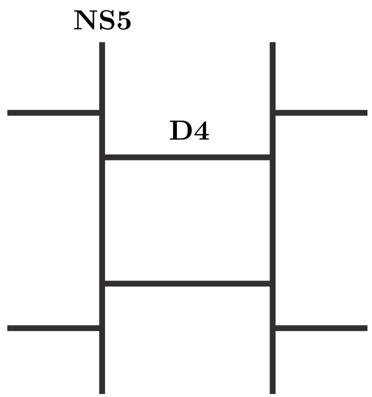


T-duality along x^6

4D N=2 on M^4:

SU(N)^M: Hanany Witten

IIA	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	(x^{10})
M NS5-branes	—	—	—	—	—	—	.	.	.	.	.
N D4-branes	—	—	—	—	.	.	—	.	.	.	—

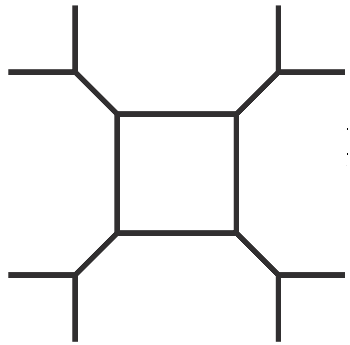


T-duality along x^5

5D N=1 on M^4 x S^1:

(p,q) 5-brane web

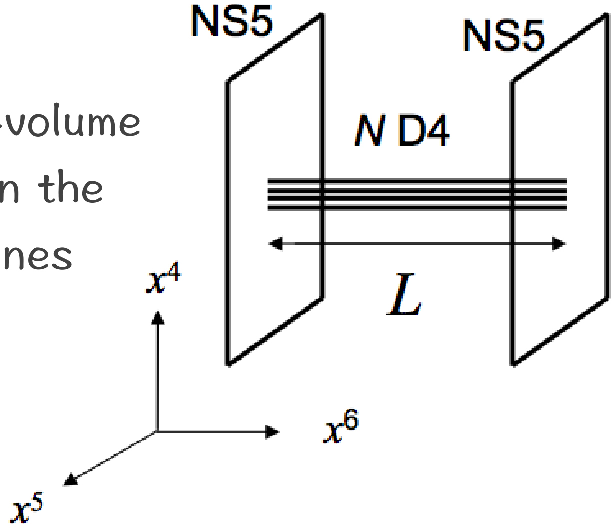
IIB	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
M NS5-branes	—	—	—	—	—	—	.	.	.	.
N D5-branes	—	—	—	—	.	—	$\underline{7}$	.	.	.



The Hanany Witten set up

[Hanany,Witten 1996]
[Witten 1997]

The world-volume theory on the D4-branes

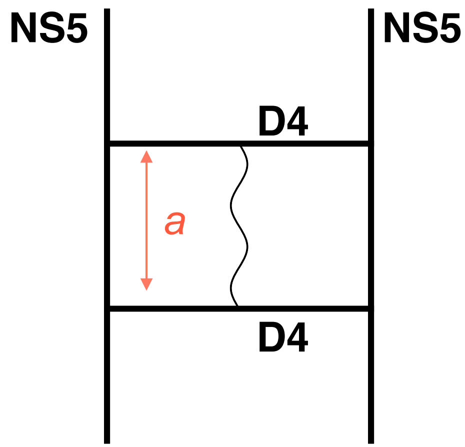


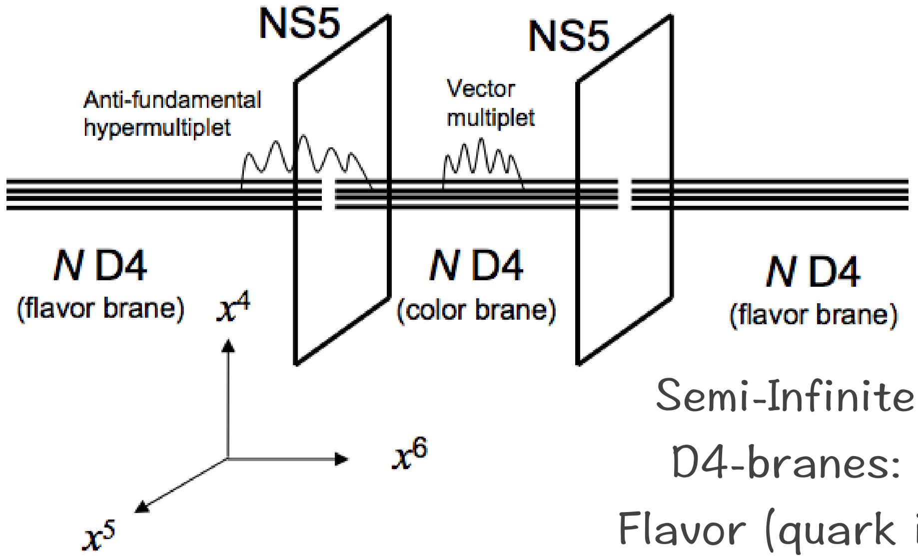
$$S = \int d^4x\, dx^6\, (\cdots) = L \int d^4x\, \mathrm{Tr} F^{\mu\nu} F_{\mu\nu} + \cdots$$

$$L \simeq \frac{1}{g^2}$$

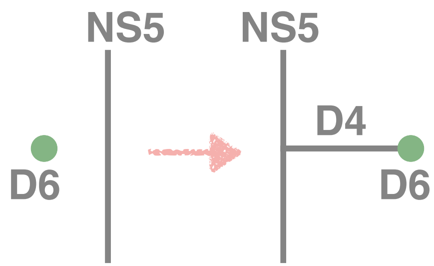
$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$$

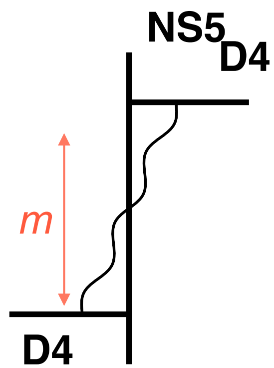
Separating the D4-branes:
Go to the Coulomb Branch



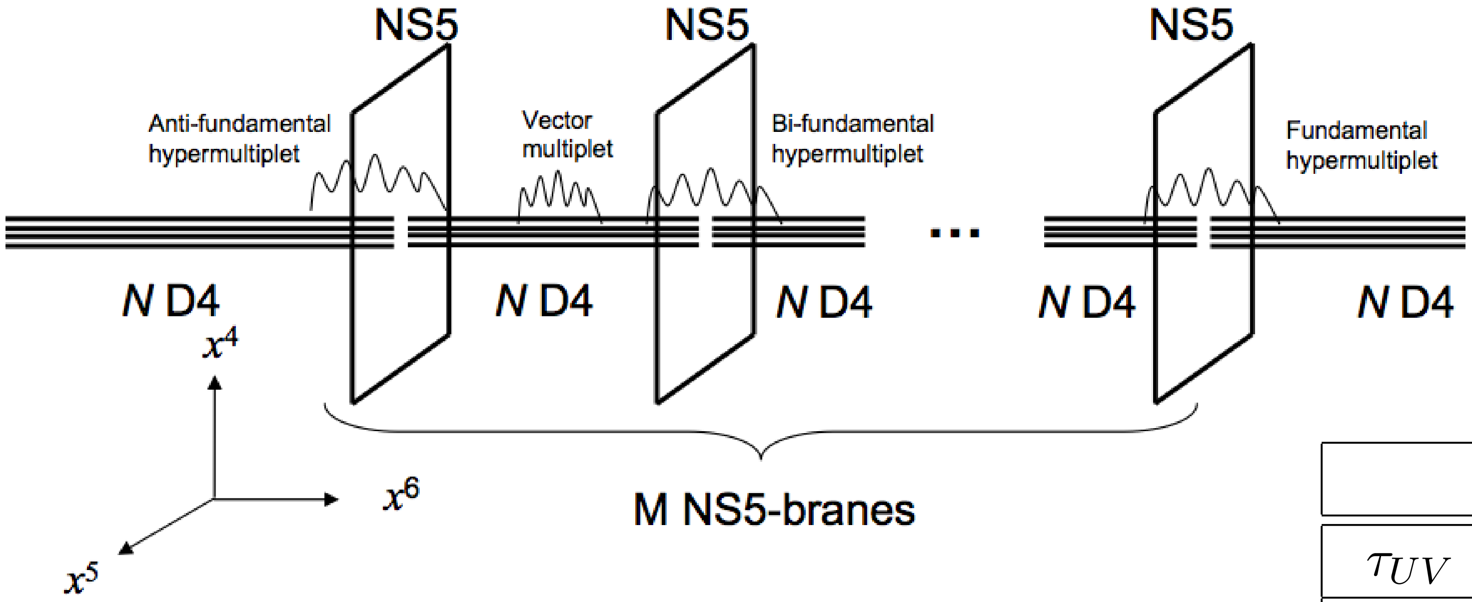


Semi-Infinite D4-branes: Flavor (quark in the Fundamental)

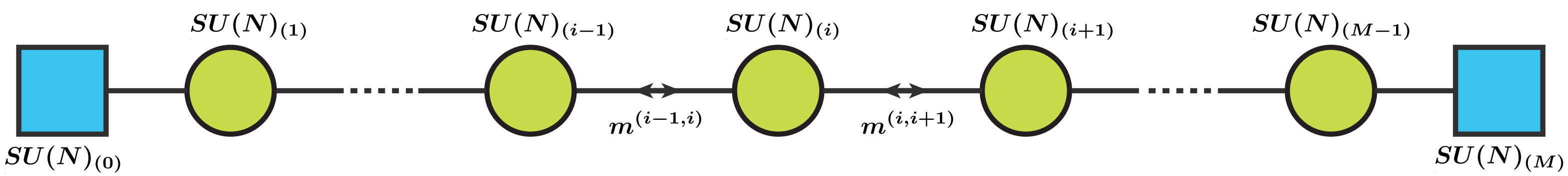




The Hanany Witten set up



	$SU(N)^{M-1}$
τ_{UV}	$M-1$
m_{f}	$2N$
m_{bif}	$M-2$
a	$(N-1)(M-1)$



IIA	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	(x^{10})
M NS5-branes	–	–	–	–	–	–	.	.	.	.	.
N D4-branes	–	–	–	–	.	.	–	.	.	.	–

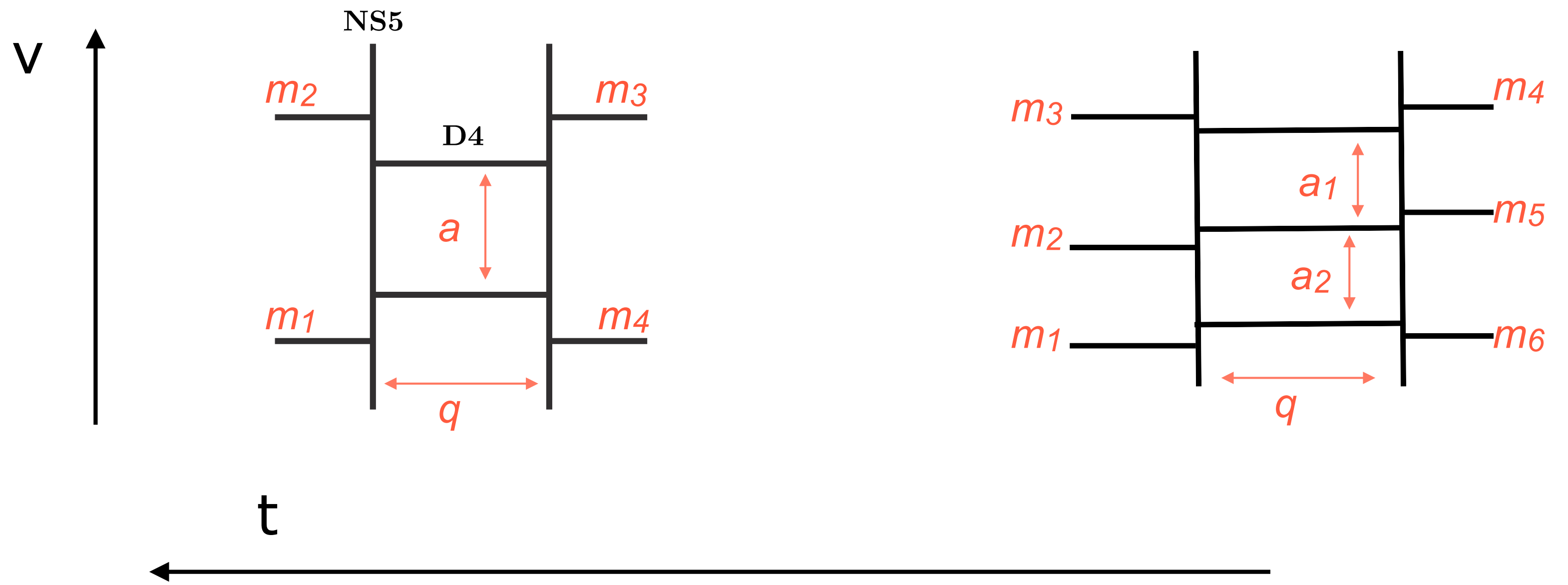
Parameterization

IIA	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	(x^{10})
M NS5-branes	—	—	—	—	—	—	.	.	.	.	.
N D4-branes	—	—	—	—	.	.	—	.	.	.	—

$$v = x^4 + ix^5$$

$$s = x^6 + ix^{10}$$

$$t = e^{-s}$$



$$q = e^{2\pi i\tau}$$

- *Deformations of the web that do* ***not*** *change its asymptotic form = # of faces = Coulomb branch (a's)*

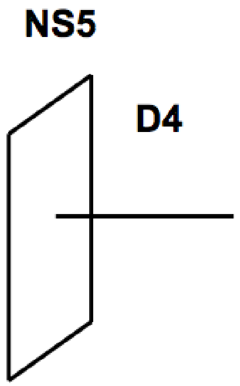
- *Deformations that do change the asymptotic form = # of external branes - 3 = parameters that define the theory: masses and couplings (m's and g's)*

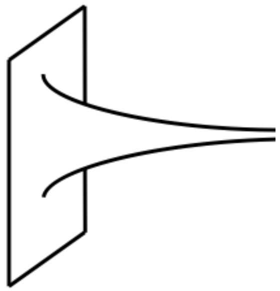
Uplift to M-theory

[Witten 1997]

The above is the NS5/D4 is the classical configuration.

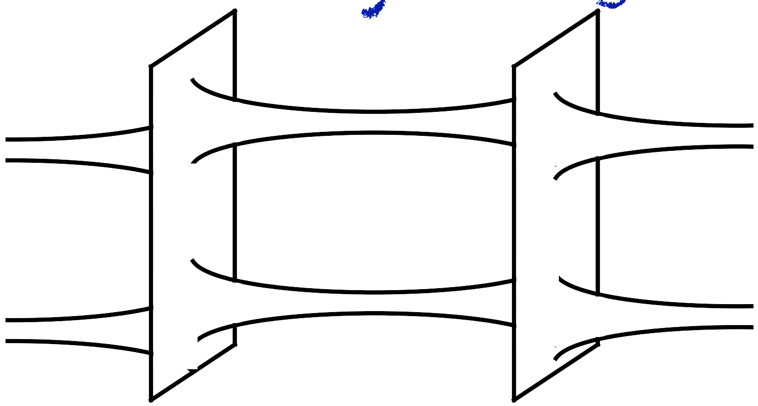
Take in account tension of the branes: include quantum effects





In M-theory both NS5/D4 come from M5.

Uplift to M-theory: a single M5 brane with non trivial topology



2D surface F(t,v)=0 in the 4D space $\{x^4, x^5, x^6, x^{10}\}=\{v,t\}$.

Point particles in $x^{0,1,2,3}$ will come from M2 branes ending on the M5

their mass ~ volume of the M2

$$\omega = ds \wedge dv = d\log t \wedge dv = d\left(v\, d\log t\right) = d\lambda_{SW}$$

Seiberg-Witten theory

Seiberg-Witten theory

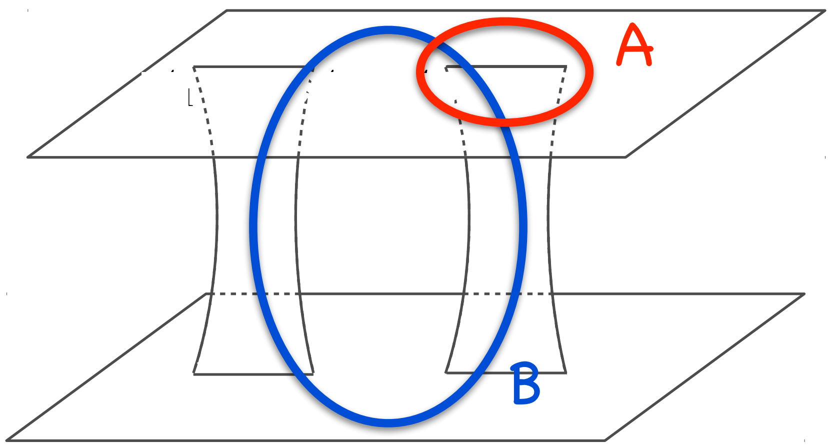
$SU(N)^M \to U(1)^{M(N-1)}$

Coulomb branch low energy effective action $\int d^4\theta \mathcal{F}(\mathcal{W}) + c.c. + \int d^4\theta d^4\bar{\theta} \mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$

is encoded in an auxiliary algebraic curve Σ plus a meromorphic differential:

$$F(t, v) = 0$$

$$\omega = ds \wedge dv = d\log t \wedge dv = d(v\, d\log t) = d\lambda_{SW}$$



$$\lambda_{SW} = v\frac{dt}{t}$$

$$a_i = \oint_{A_i} \lambda_{SW}$$

$$a_D{}^i = \oint_{B^i} \lambda_{SW} = \frac{\partial \mathcal{F}(a)}{\partial a_i}$$

The spectrum of BPS dyons:

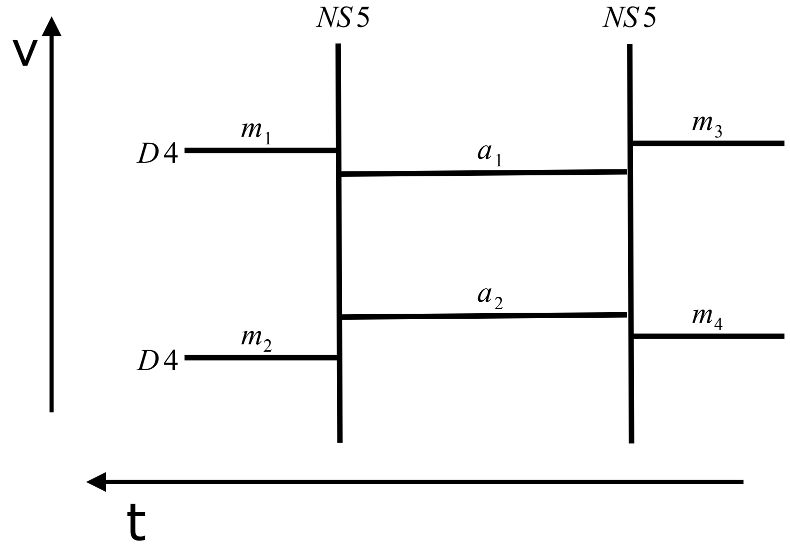
$$m_{BPS}^2 = |na + ma_D|^2$$

n: electric charge,

m: magnetic

$$\tau_{IR} = \frac{\partial a_D}{\partial a}$$

The SW Curve



Simple example: SU(2) with 4 flavors

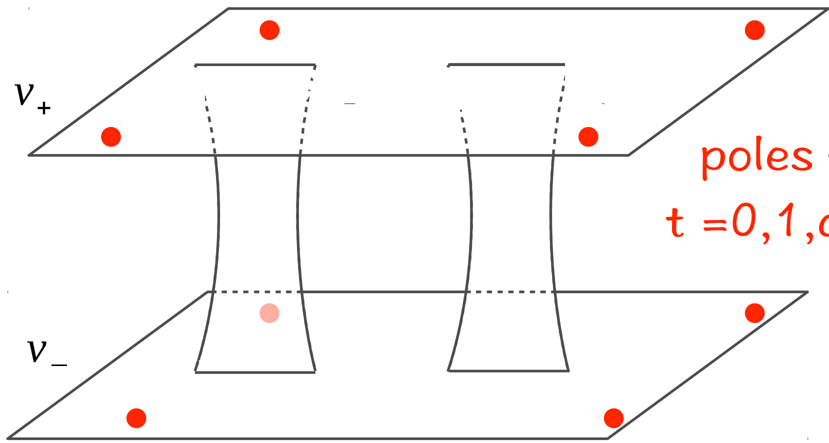
$$(t-1)(t-q)v^2 - P_1(t)v + P_2(t) = 0$$

$$(v-m_1)(v-m_2)t^2 + \left(-(1+q)v^2 + qMv + u\right)t + q(v-m_3)(v-m_4) = 0$$

coupling constant $q = e^{2\pi i \tau}$ $u = \mathrm{tr}\,\phi^2$ $M = m_1 + m_2 + m_3 + m_4$

Two sheets connected by branch cuts:

$$v_\pm = \frac{P_1(t) \pm \left(P_1(t)^2 - 4(t-1)(t-q)P_2(t)\right)^{1/2}}{2(t-1)(t-q)}$$



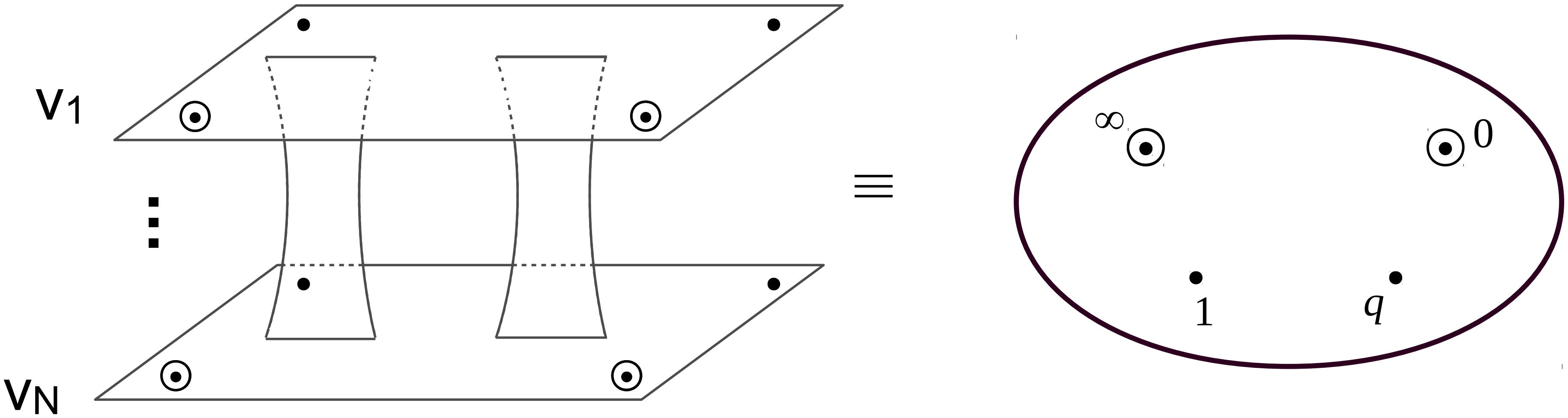
poles at $t = 0, 1, q, \infty$

Bring the curve in the form:

$$x^2 = \phi_2(z) = \frac{P(z)}{(z-z_1)^2(z-z_2)^2(z-z_3)^2(z-z_4)^2}$$

← polynomial

Gaiotto's from



SW or IR curve Σ

torus with four punctures

Gaiotto or UV curve $C_{0,4}$

a sphere with four puncture

Σ is an N sheeted cover of $C_{0,4}$

$$x^N = \sum_{\ell=2}^{N} \phi_\ell(z)\, x^{N-\ell}$$

with $\phi_l(z)$ meromorphic (only poles) functions on the z plane (sphere)

Anticipating a 4D/2D relation

Close to the poles:

$$\phi_2(z) \sim \frac{m_i^2}{(z-z_i)^2}$$

Recall 2D Ward Identities:

$$\langle T(z) \prod_i V_i(z_i) \rangle = \sum_j \left[\frac{h_j}{(z-z_j)^2} + \frac{\partial_j}{z-z_j} \right] \langle \prod_i V_i(z_i) \rangle$$

Match the double poles for: $h_i = -m_i^2$

$$\phi_2(z) = -\frac{\langle T(z) \prod_i V_i(z_i) \rangle}{\langle \prod_i V_i(z_i) \rangle}$$

Do a free field computation:

Match all poles!

$$\phi_\ell(z) = -\frac{\langle W_\ell(z) \prod_i V_i(z_i) \rangle}{\langle \prod_i V_i(z_i) \rangle}$$

S duality
Gaiotto's class S

S-duality

N=4 SYM has EM/S-duality: SL(2,Z) symmetry [Montonen,Olive 1977]

The theory with color group G and coupling constant τ

=

The theory with color group ${}^{L}G$ and coupling constant $-1/\tau$

Take the 6D (2,0) SCFT and compactify on a torus. [Vafa 1997]

S-duality is interpreted as
the modular group of the torus.

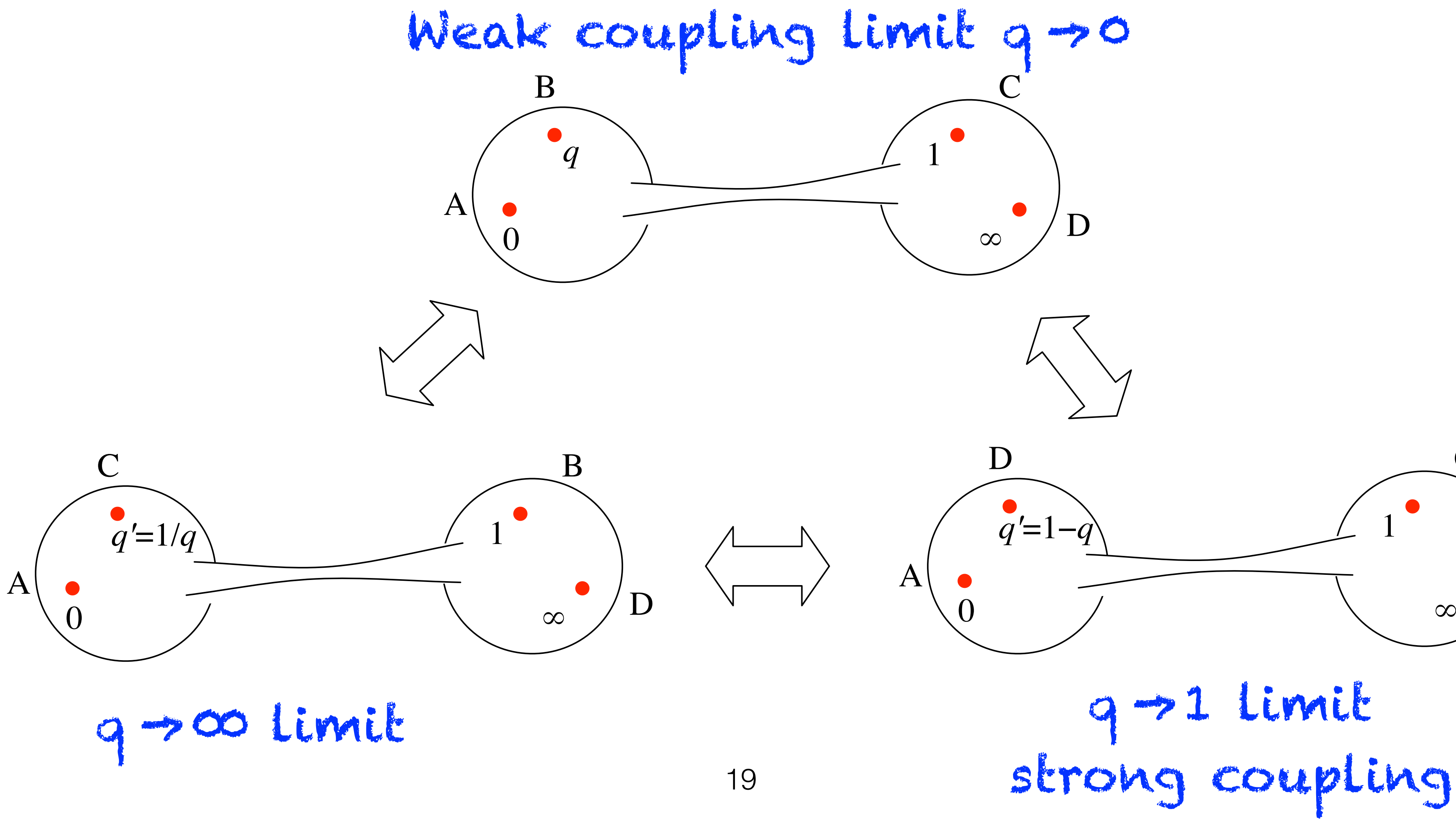
Theorem:

[Cordova,Dumitrescu,Intriligator]

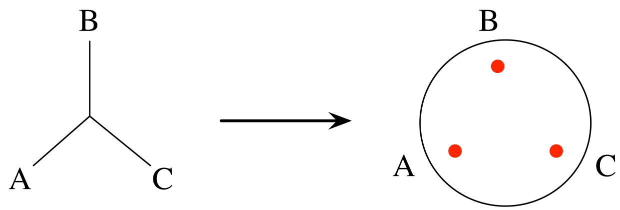
6D (2,0) SCFT has no marginal deformations
All coupling constants come form the moduli of the surface.

S-duality SU(2) SCQCD [Seiberg,Witten 1994]

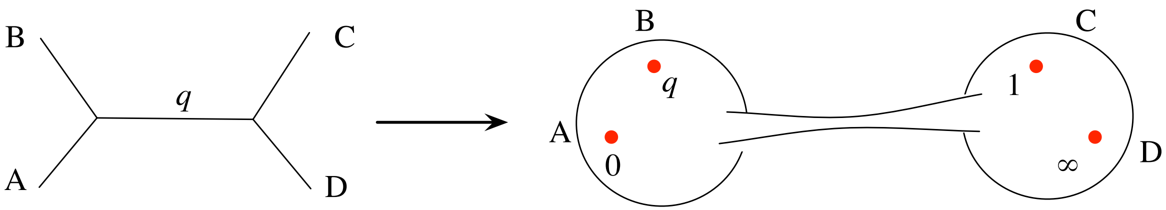
Self-dual under S-duality with punctures exchanged mass parameters mixed



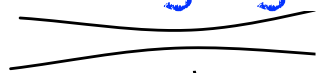
Free trinions, gauging and S-duality



Free hyper
(trinion)

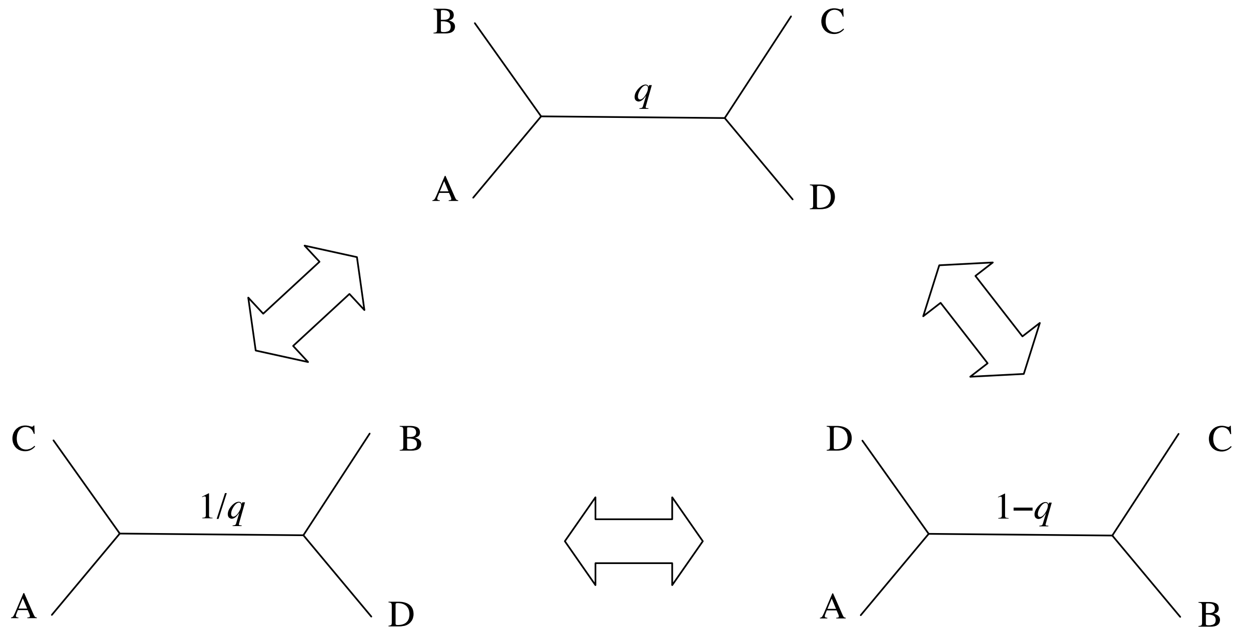


Gauging



Trivalent diagrams
for the UV curves

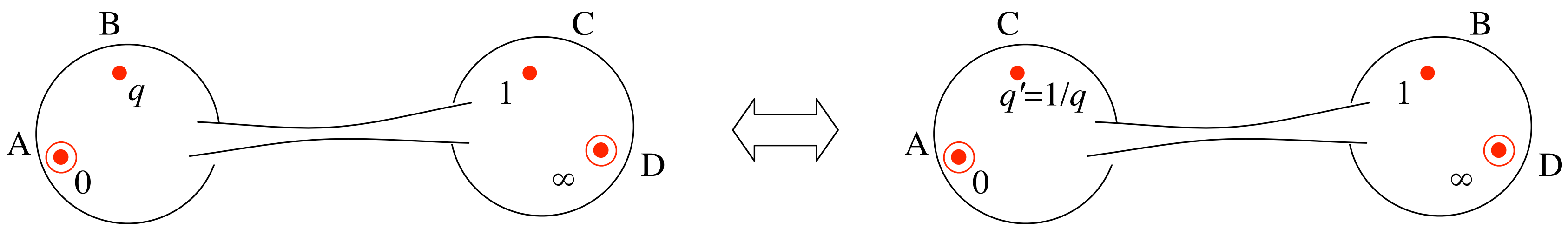
S-duality looks like
crossing equation!



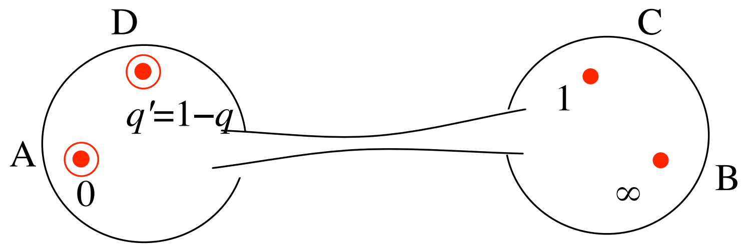
S-duality of SU(3) SCQCD

[Argyres,Seiberg 2007]
[Argyres,Wittig 2007]

Self-dual under S-duality with B and C punctures exchanged

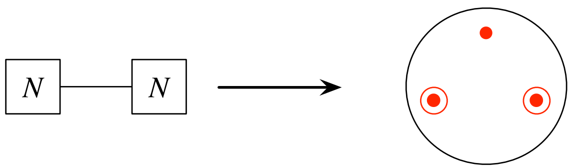


But, very different when B and D exchanged:
E_6 global symmetry (no-Lagrangian description)

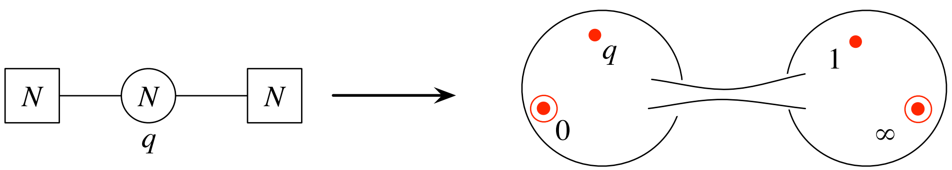


There are two different types of punctures. One with SU(3) and one with U(1) symmetry

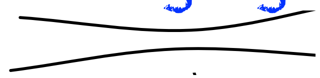
Free hyper (trinion)

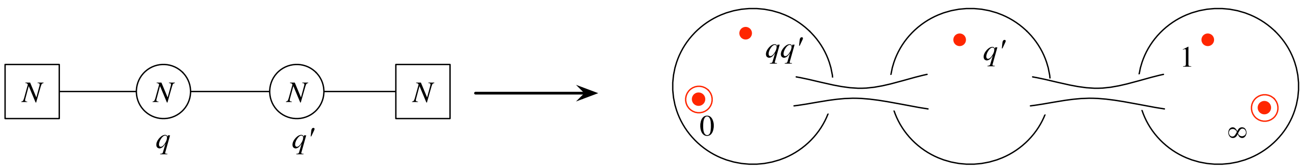


[Gaiotto 2009]



Gauging





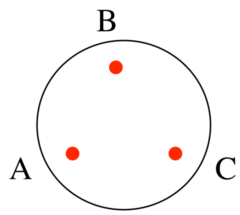
There are different types of punctures classified by Young diagrams.

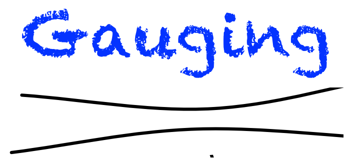
Class S of Gaiotto

[Gaiotto 2009]

Class S *of 4D N=2 SCFT $T_{g,n}$ by a compactification of the 6D (2,0) SCFT on Riemann surface $C_{g,n}$ of genus g and with n punctures.*

Riemann surface decomposition
in **pairs of pants** and **tubes**.

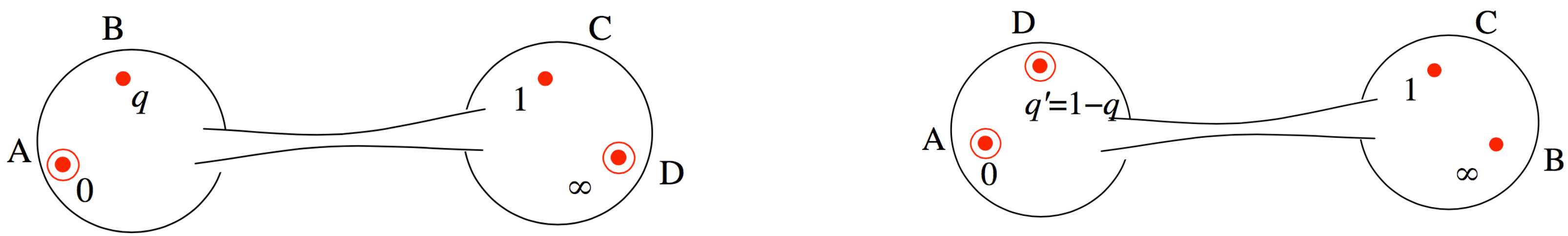




Building blocks of Gauge theories: **Trinion theories** and **color factors**

Theorem:
[Cordova,Dumitrescu,Intriligator]

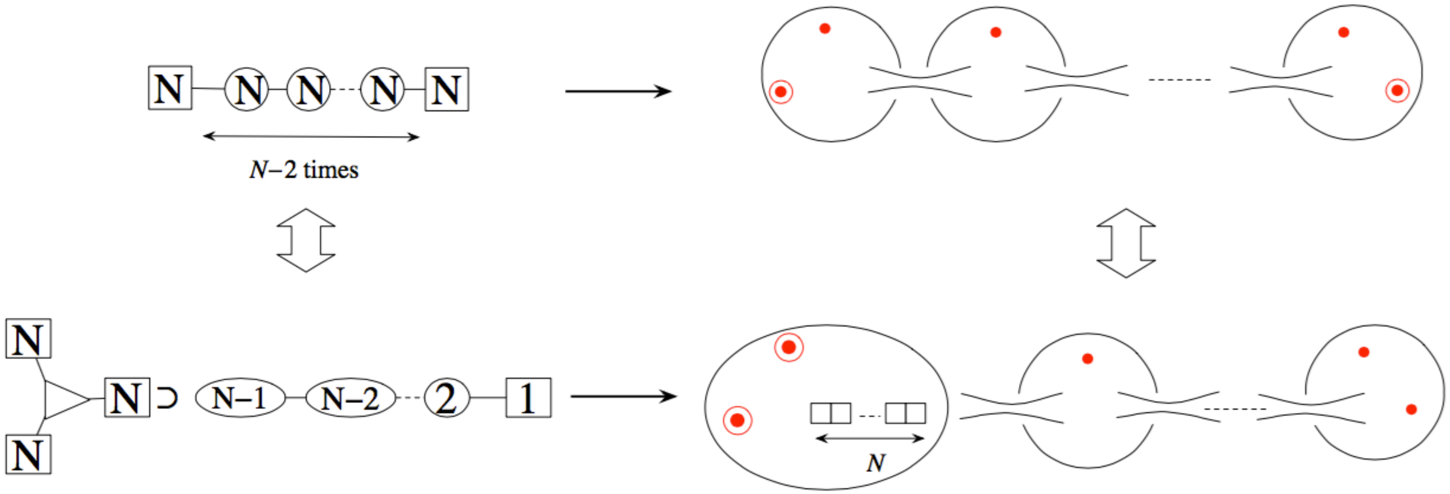
6D (2,0) SCFT has no marginal deformations
All coupling constants come form the moduli of the surface.



Generalized S-duality = modular transformations

Non-Lagrangian T_N theories

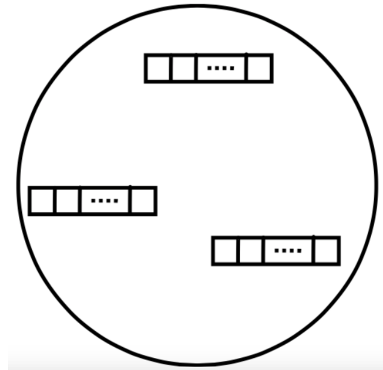
Argyres Seiberg duality: we discovered the T_3 theory with E_6 symmetry.



Generalized Argyres Seiberg duality:

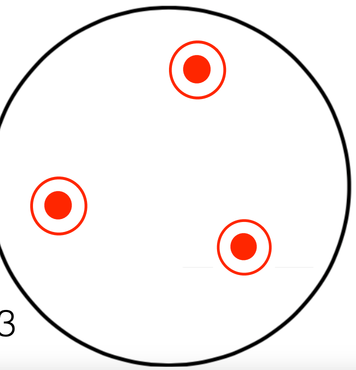
we discover the T_N theories

[Gaiotto 2009]



3 "full" punctures

- *Isolated fixed points*
- *No Lagrangian description*
- *$SU(N)^3$ global symmetry*



Supersymmetric Localization

Supersymmetric Localization

1 $Z = \int D\phi \, e^{-S(\phi)}$ there is a grassmann odd symmetry of the action

Such that: $\mathcal{Q}^2 = \text{Bosonic symmetries}$ *And is not anomalous!*

2 Find: $\delta^2 V(\phi) = 0$ and $\Big(\delta V(\phi)\Big)_B \geq 0$

3 The PI: $Z(t) = \int D\phi \, e^{-S(\phi) - t\delta V(\phi)}$ *does not depend on t!*

$$\frac{dZ(t)}{dt} = \int D\phi \, e^{-S(\phi) - t\delta V(\phi)} \Big(-\delta V(\phi) \Big) = \int D\phi \, V(\phi) \, \delta \Big(e^{-S(\phi) - t\delta V(\phi)} \Big) = 0$$

$Z(0) = Z(\infty)$ where the PI localizes if $\Big(\delta V(\phi)\Big)_B = 0$

Supersymmetric Localization

When δV_B=0 for a discreet set of ϕ_i, the PI localizes on the saddle points:

$$Z = \int D\phi\, e^{-S(\phi)-t\delta V(\phi)} \simeq \sum_i e^{-S(\phi_*^i)} \frac{\det\left(\delta V_F^{(2)}(\phi_*^i)\right)}{\sqrt{\det\left(\delta V_B^{(2)}(\phi_*^i)\right)}}$$

When the zeros of δV are a parameter family (Moduli space)

$$Z \simeq \sum_k \int_{\mathcal{M}_k} Z_{\text{tree}}[\phi_*(\rho)] Z_{\text{1-loop}}[\phi_*(\rho)]$$

The task is to calculate the determinants! [Pestun 2007]

And do the integrals over the Moduli space! [Nekrasov 2002]

For a BPS observable: $\mathcal{Q}\,\mathcal{O}(\phi) = 0$ also $\langle\mathcal{O}\rangle(t) = \int D\phi\, \mathcal{O}(\phi)\, e^{-S(\phi)-t\delta V(\phi)}$

Supersymmetric Localization [Pestun 2007]

[Hama,Hosomichi 2012]

Coulomb Branch Localization

$$\langle \mathcal{O} \rangle_{\mathbb{S}^4_{r_1,r_2}} = \int da\, \mathcal{O}(a)\, \left| \mathcal{Z}_{\text{pert}}(a)\, \mathcal{Z}_{\text{inst}}(a) \right|^2$$

On sphere or Ellipsoid

$$\epsilon_{1,2} = \frac{1}{r_{1,2}}$$

$$\langle \phi \rangle = \text{diag}\,(a_1, \ldots, a_N)$$

$$\Upsilon(x) \sim \text{Reg}\left[\prod_{n_1,n_2 \geq 0} (x + b\, n_1 + b^{-1} n_2)\,(-x + b\,(n_1+1) + b^{-1}(n_2+1)) \right]$$

satisfies the shift relations $\begin{cases} \Upsilon(x+b) = \gamma(bx)\, b^{1-2bx}\, \Upsilon(x) \\ \Upsilon(x+b^{-1}) = \gamma(b^{-1}x)\, b^{2b^{-1}x-1}\, \Upsilon(x) \end{cases}$.

The perturbative part (1-loop det) read off from the quiver:

$$|\mathcal{Z}_{\text{pert}}(a)|^2 = \prod_i \mathcal{Z}^{\text{vect}}_{\text{pert}}(a_i)\mathcal{Z}^{\text{hyper}}_{\text{pert}}(a_i, a_{i+1}, m)$$

Assign to color factors

$$\mathcal{Z}^{\text{vect}}_{\text{1-loop}} = \prod_{i<j=1}^{N} \Upsilon(a_i - a_j)\Upsilon(a_j - a_i)$$

and hypers:

$$\mathcal{Z}^{\text{hyper}}_{\text{1-loop}} = \frac{1}{\prod_{i,j=1}^{N} \Upsilon\left((a_i^{(1)} - a_j^{(2)}) + m - \frac{\epsilon_+}{2} \right)}$$

Supersymmetric Localization [Pestun 2007]

[Hama,Hosomichi 2012]

Coulomb Branch Localization

On sphere or Ellipsoid

$$\langle \mathcal{O} \rangle_{\mathbb{S}^4_{r_1,r_2}} = \int da\, \mathcal{O}(a)\, |\mathcal{Z}_{\text{pert}}(a)\, \mathcal{Z}_{\text{inst}}(a)|^2$$

$$\epsilon_{1,2} = \frac{1}{r_{1,2}}$$

$$\langle \phi \rangle = \text{diag}\,(a_1, \ldots, a_N)$$

The instanton part is also read off from the quiver:

$$\mathcal{Z}_{\text{inst}}(a) = \prod_i \mathcal{Z}^{\text{vect}}_{\text{inst}}(a_i) \mathcal{Z}^{\text{hyper}}_{\text{inst}}(a_i, a_j, m)$$

$$\mathcal{Z}_{\text{inst}} = \sum_k q^k \mathcal{Z}_k$$

[Nekrasov 2002]

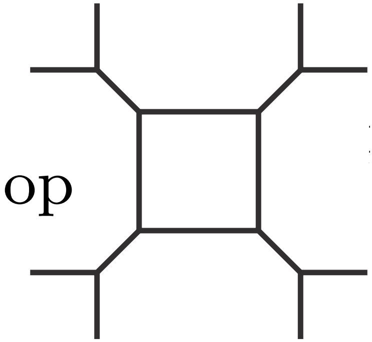
$$Z^{\text{vect}}_k(a, \epsilon_1, \epsilon_2) \simeq \oint \prod_{i=1}^{k} \frac{\mathrm{d}\phi_i}{2\pi i} \prod_{i \neq j} \frac{(\phi_i - \phi_j)(\phi_i - \phi_j - \epsilon_1 - \epsilon_2)}{(\phi_i - \phi_j - \epsilon_1)(\phi_i - \phi_j - \epsilon_2)} \prod_{i=1}^{k} \prod_{l=1}^{N} \frac{1}{(\phi_i - a_l - \epsilon_+)(\phi_i - a_l + \epsilon_+)}$$

Ex. of a BPS observable circular Wilson loop:

$$W = P e^{\oint iA + \phi ds} \longrightarrow W(a) = \frac{1}{N} \sum_i e^{2\pi a_i}$$

From 5-brane web diagrams to Partition functions with topological strings

Read off the holomorphic half of the partition function of a theory from its web diagram: $\mathcal{Z}_{\mathbb{S}^4 \times \mathbb{S}^1} \propto \mathcal{Z}_{\text{top}}$



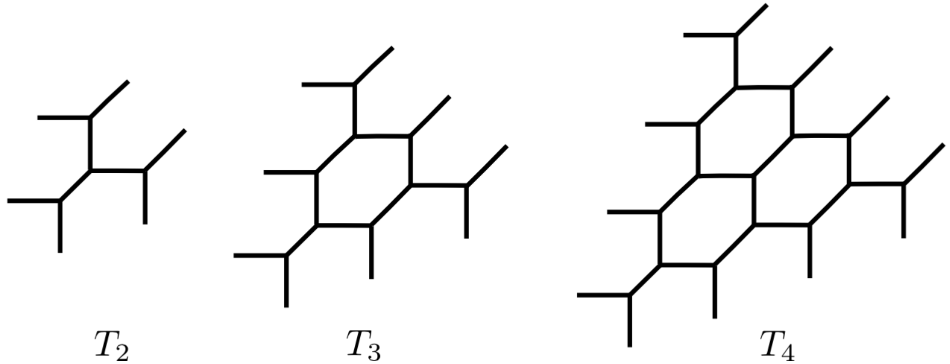
Use topological strings: refined topological vertex formalism

[Awata,Kanno]

[Iqbal,Kozcaz,Vafa]

[Taki]

$$\mathcal{Z}_{\text{top}} = \sum_R (\text{three-vertices}) \times (\text{oriented lines})$$



[Bao,Mitev,EP,Taki,Yagi]

[Hayashi,Kim,Nishinaka]

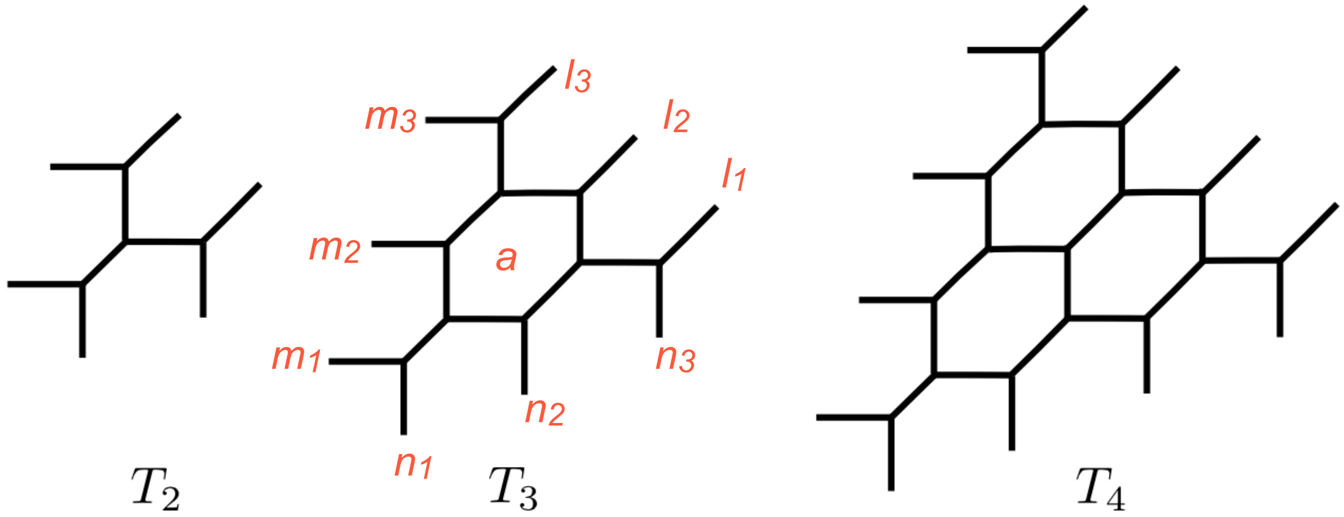
T_N: No Lagrangian description: no Nekrasov Localization! Have to use topological strings

T_N Web diagrams

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5-branes	—	—	—	—	—	—	.	.	.	.
D5-branes	—	—	—	—	.	—	—	.	.	.

[Benini,Benvenuti,Tachikawa]

The low energy dynamics of 5D TN theories is encoded in:



5D theories on S^1

Also the SW curves match!

- *Deformations of the web that do **not** change its asymptotic form = # of faces = Coulomb branch (a's) = (N-1)(N-2)/2*
- *Deformations that change the asymptotic form of the web = # of external branes - 3 = parameters that define the theory: masses and couplings (m's and g's) = 3N-3 (No coupling)*
- *$SU(N)^3$ global symmetry*

Extra degrees of freedom

When the web diagram has parallel external legs, Z_{top} includes extra degrees of freedom.

[Bao,Mitev,EP,Taki,Yagi]

[Hayashi,Kim,Nishinaka]

[Bergman-Gomez-Zafrir]…

$$\mathcal{Z}_{\mathbb{S}^4\times\mathbb{S}^1} = \frac{\mathcal{Z}_{\text{top}}}{\mathcal{Z}_{\text{extra}}}$$

$$|\mathcal{Z}^{=}_{\text{extra}}|^2 = \Upsilon(m_1 - m_2)$$

- They depend on the ***distance*** between the parallel external legs.
- These extra d.o.f. do not transform as a correct representation of 5D Poincare. They are 6D d.o.f.

For the T_N:

$$\left|\mathcal{Z}^{T_N}_{\text{extra}}\right|^2 = \prod_{i<j=1}^{N} \Upsilon(m_i - m_j)\Upsilon(n_i - n_j + \frac{\epsilon_+}{2})\Upsilon(l_i - l_j)$$

4D partition functions

and

2D correlators

2D CFT Review

$\Upsilon(x) \sim \mathrm{Reg}\left[\prod_{n_1,n_2\geq 0}(x+b\,n_1+b^{-1}n_2)(-x+b\,(n_1+1)+b^{-1}(n_2+1))\right]$

satisfies the shift relations $\begin{cases}\Upsilon(x+b)=\gamma(bx)\,b^{1-2bx}\,\Upsilon(x)\\ \Upsilon(x+b^{-1})=\gamma(b^{-1}x)\,b^{2b^{-1}x-1}\,\Upsilon(x)\end{cases}$.

Two and three point functions of primaries are fixed by conformal symmetry.

*Up to the **3-pt structure constants**:*

✱ ***Liouville** CFT DOZZ 3pt:* [Dorn,Otto 94] [Zamolodchikov^2 94] [Teschner 95]

$$C(\alpha_1,\alpha_2,\alpha_3)=\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-\sum_{i=1}^3\alpha_i}{b}}\frac{\Upsilon'(0)\prod_{i=1}^3\Upsilon(2\alpha_i)}{\Upsilon(\sum_{i=1}^3\alpha_i-Q)\prod_{j=1}^3\Upsilon(\sum_{i=1}^3\alpha_i-2\alpha_j)}$$

✱ ***Toda** CFT (higher spin W_N, N>2) the state of the art:* [Fateev,Litvinov 2005]

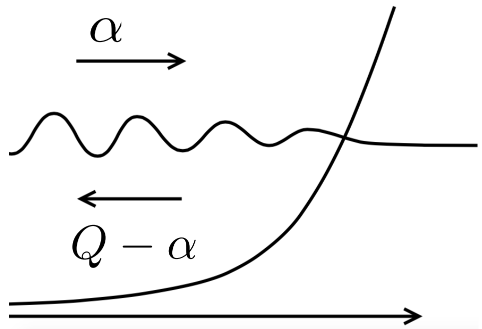
$$C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\varkappa\omega_{N-1})=\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}{b}}\times$$

$$\times\frac{\Upsilon'(0)^{N-1}\Upsilon(\varkappa)\prod_{e>0}\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_1,e))\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_2,e))}{\prod_{i,j=1}^N\Upsilon(\frac{\varkappa}{N}+(\boldsymbol{\alpha}_1-\mathcal{Q},h_i)+(\boldsymbol{\alpha}_2-\mathcal{Q},h_j))}$$

Primary with null vector at level 1

Weyl Reflections and symmetry enhancement

$$S = \frac{1}{4\pi}\int d^2z\sqrt{g}\left(g^{ab}\partial_a\phi\partial_b\phi + QR\phi + 4\pi\mu e^{2b\phi}\right)$$



$$V_\alpha = e^{\alpha\phi} \qquad V_{Q-\alpha} = R(\alpha)V_\alpha \qquad h = \alpha\,(Q-\alpha)$$

It is possible to factor the structure constants to Weyl covariant and invariant part.

$$C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = C^{\text{inv}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)C^{\text{cov}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)$$

[Fateev,Litvinov]

For Liouville (N=2):

$$C^{\text{cov}}(\alpha_1,\alpha_2,\alpha_3) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-\sum_{i=1}^3\alpha_i}{b}}\prod_{i=1}^{3}\Upsilon(2\alpha_i)$$

The invariant part is invariant under SU(4)

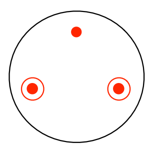
$$C^{\text{inv}} = \frac{\Upsilon'(0)}{\prod_{i=1}^4\Upsilon(u_i+\frac{Q}{2})} \quad \text{with} \quad \sum_{i=1}^{4}u_i = 0$$

*N=3 invariant part has **E₆***

Symmetry Enhancement!!

[Mitev,EP]

$$C^{\text{inv}}(\varkappa\omega_{N-1},\alpha_2,\alpha_3) = \frac{1}{\prod_{i,j=1}^{N}\Upsilon(m_{ij})} = \mathcal{Z}^{S^4}_{N^2}\ \text{free hypers}$$



The AGT-W correspondence

[Alday,Gaiotto,Tachikawa] [Wyllard]

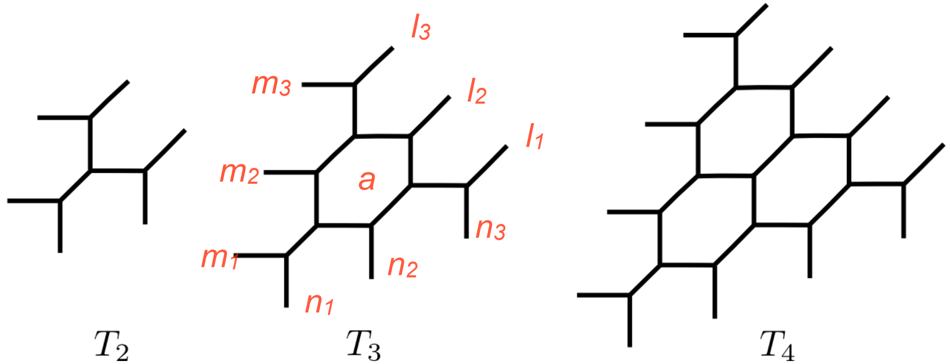
A relation between:

- 4D N=2 theories $T_{g,n}$ of class S with SU(2)/SU(N) factors
- 2D Liouville/Toda CFT

$$\mathcal{Z}_{\mathbb{S}^4}\left[\mathcal{T}_{g,n}\right] = \int [da]\, \mathcal{Z}_{\text{pert}} ||\mathcal{Z}_{\text{inst}}||^2 = \int d\alpha\, C \cdots C\, ||\mathcal{F}_{\alpha}^{\alpha_{\text{ext}}}||^2 = \langle \prod_{i=1}^{n} V_{\alpha_i} \rangle_{\mathcal{C}_{g,n}}$$

4D gauge theory	2D CFT
instanton partition function	conformal block
perturbative part	3-point function (Liouville)
coupling constants	cross ratios
masses	external momenta
Coulomb moduli	internal momenta
generalized S-duality	crossing symmetry
Omega background $\epsilon_1 = b,\ \epsilon_2 = b^{-1}$	Coupling constant/central charge

T_N partition functions and Toda 3pt functions



1 Can compute (read off) $\mathcal{Z}_{T_N}$ from web diagrams [Benini,Benvenuti,Tachikawa] using the ***topological vertex formalism*** as long as we remove ***extra d.o.f*** $\mathcal{Z}_{\text{extra}}$ ***!*** [Bao,Mitev,EP,Taki,Yagi] [Hayashi,Kim,Nishinaka]

2 Proposal for the 3pt functions of Toda: ***3 generic primaries*** [Mitev,EP]

$$C^{\text{our}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = C^{\text{inv}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)C^{\text{cov}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)$$

$m_i = (\boldsymbol{\alpha}_1 - \mathcal{Q}, h_i)$

$n_i = -(\boldsymbol{\alpha}_2 - \mathcal{Q}, h_i)$

$l_i = -(\boldsymbol{\alpha}_3 - \mathcal{Q}, h_{N+1-i})$

$$C^{\text{cov}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \mathcal{Z}_{\text{extra}}$$

$$C^{\text{inv}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \mathcal{Z}_{T_N}$$

- ☑ N=2 Liouville case
- ☑ symmetries, zeros

3 Very strong check: [Fateev,Litvinov]

$$C^{\text{our}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\varkappa\omega_{N-1}) = C^{\text{F.L.}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\varkappa\omega_{N-1})$$

[Isachenkov,Mitev,EP]

Conclusions and Vision

Ideal talk Conclusions

The web knows it all!

- ☑ *SW theory*
- ☑ *String theory/M-theory constructions*
- ☑ *Partition functions (Topological strings)*
- ☐ *Different BPS observables!*
- ☑ *AGT (4D partition functions = 2D correlators) and Toda 3pt*
- ☐ *Superconformal Index (yet an other 4D/2D relation)* [Gadde,EP,Rastelli,Razamat]
- ☐ *Bootstrap, Correlation functions Schur Operators (Higgs branch)* [Beem,Lemos,Liendo,Peelaers,Rastelli,van Rees]
- ☐ *Correlation functions of Chiral Operators (Coulomb branch)* [Baggio,Niarchos,Papadodimas] [Gerchkovitz,Gomis,Ishtiaque,Komargodski,Pufu]
- ☐ *AdS/CFT (Integrability), Exact anomalous dimensions* [Mitev,EP]

Vision for the future (what I like)

- N=1 theories
  - Superconformal Index = 2D TQF [Gaiotto,Razamat 2015]
  - 4D partition functions = 2D correlators ?
- N=2
  - chiral correlators/tt* equations
  - Precision AdS/CFT (Integrability) [Zarembo Review 2016]
  - Magical novel relations
- Increase the list of Exact observables!!
- N=3 theories
- Classification of N=2 theories (non Lagrangian) [Argyres,Lotito,Lu,Martone]

Class S_k

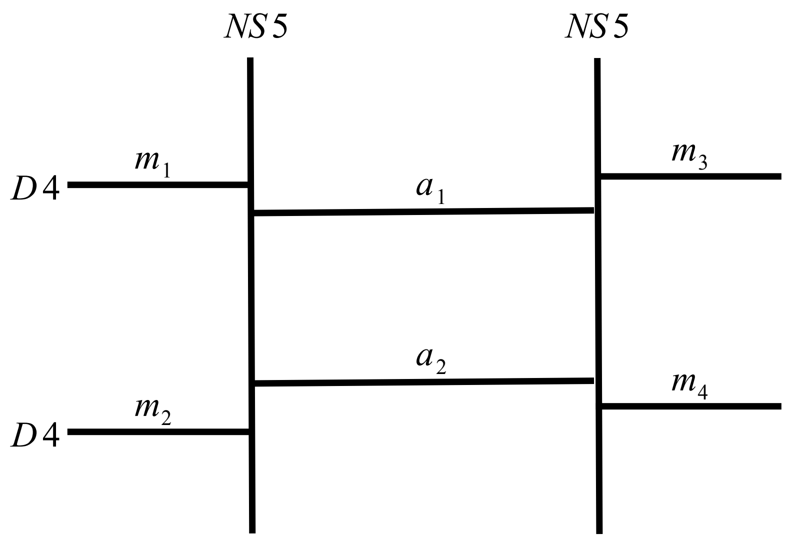
[Gaiotto,Razamat 2015]

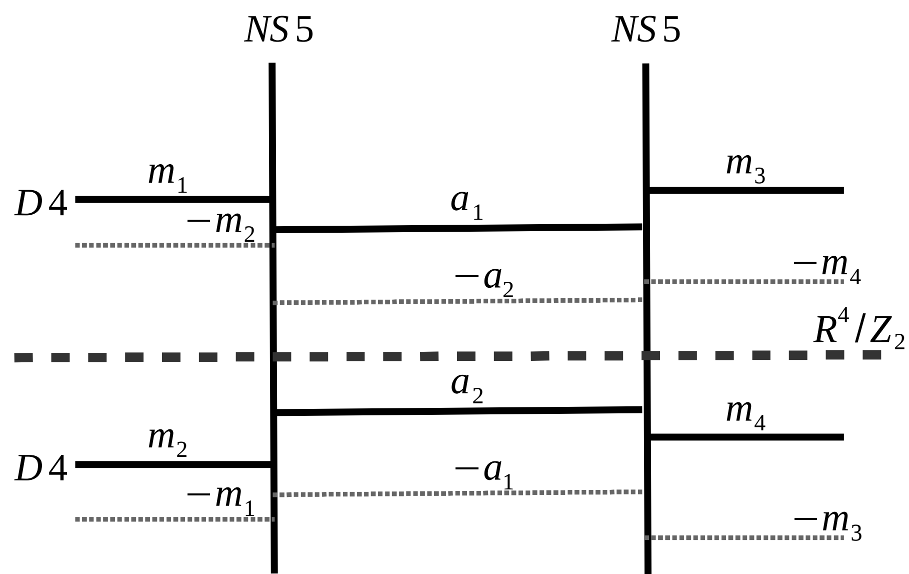
6D **(2,0)** SCFT on Riemann surface: 4D N=2 theories of **class S**

Z_k Orbifold the 6D (2,0) SCFT to 6D (1,0) SCFT

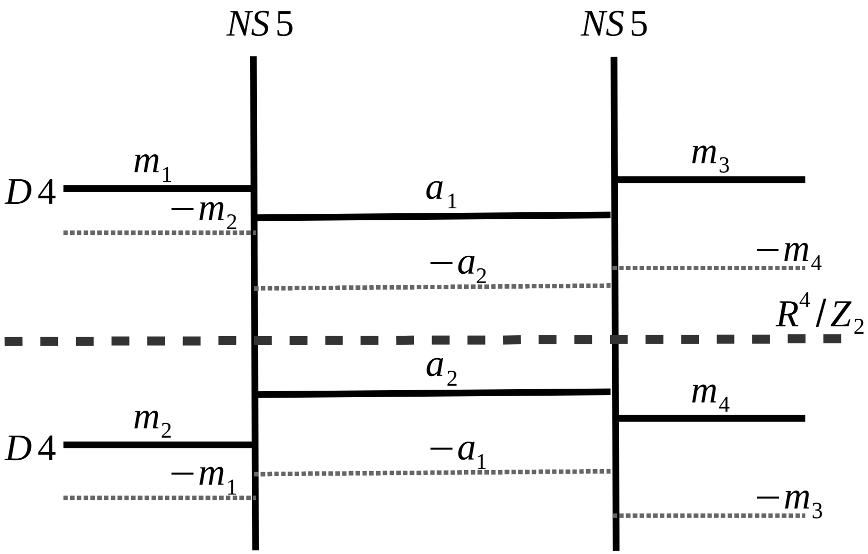
6D **(1,0)** SCFT on Riemann surface: 4D N=1 theories of **class S_k**

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	(x^{10})
M NS5 branes	—	—	—	—	—	—	.	.	.	.	.
N D4-branes	—	—	—	—	.	.	—	.	.	.	—
A_{k-1} orbifold	.	.	.	.	—	—	.	—	—	.	.





Curves of Class Sk [Coman,EP,Taki,Yagi 2015]



$$m_i^{(n)} = e^{\frac{2\pi i n}{k}} m_i \quad n = 1, \ldots, k$$

$$(v - m_i) \longrightarrow \prod_{n=1}^{k} (v - m_i^{(n)}) = (v^k - m_i^k)$$

$$(v^k - m_1^k)(v^k - m_2^k)t^2 + P(v)t + q(v^k - m_3^k)(v^k - m_4^k) = 0$$

The polynomial P(v) also has to respect the orbifold

$$P(v) = -(1+q)v^{2k} + u_k v^k + u_{2k}$$

Is a function of the vevs of the gauge invariant glueballs that parameterize the Coulomb branch

$$\langle \mathrm{tr}\left(\Phi_{(1)} \cdots \Phi_{(k)}\right)\rangle \sim u_k \qquad \langle \mathrm{tr}\left(\Phi_{(1)} \cdots \Phi_{(k)}\right)^2\rangle \sim u_{2k}$$

Novel N=3 theories

[Garcıa-Etxebarria,Regalado15]
[Aharony,Evtikhiev15]
[Aharony,Tachikawa15]
[Argyres,Lotito,Lu,Martone16]

* *Multiplets must be CPT invariant: N=3 vector multiplet becomes N=4*

* *Non-Lagrangian N=3 theories (no description in terms of elementary fields)*

Theorem:
[Cordova,Dumitrescu,Intriligator]

4D N=3 SCFT have no marginal deformations!
They are isolated fixed points.

* *Generalization of Orbifold/Orientifold:* ***S-fold***

R-symmetry and S-duality identification:

N=3 theory = (N=4 U(N) SYM)/r*s

With this r*s operation being a Z_k with k=3,4,6

Chiral correlators

[Baggio,Niarchos,Papadodimas]

[Gerchkovitz,Gomis,Ishtiaque,Komargodski,Pufu]

Coulomb branch operators $\mathcal{O}_I = \left(\mathrm{tr}\phi^\ell \mathrm{tr}\phi^m \cdots\right)$ are chiral $\mathcal{E}_r$ with $\Delta = r$

The Lagrangian = Q^4 descendant of the $\mathcal{E}_{r=2}$

* *Zamolodchikov metric (**metric on theory space**: as we marginally deform)*

$$\langle \mathcal{O}_i(x) \bar{\mathcal{O}}_{\bar{j}}(0) \rangle = \frac{g_{i\bar{j}}}{|x|^{2\Delta}} \qquad S \to S + \frac{\delta\lambda^i}{4\pi^2} \int d^4x\, \mathcal{O}_i(x) + \frac{\delta\overline{\lambda}^i}{4\pi^2} \int d^4x\, \overline{\mathcal{O}}_i(x)$$

$$\mathcal{O}_i = Q^4 \mathrm{tr}\phi_i^2$$

* *Extremal correlations* $\langle \mathcal{O}_{I_1} \cdots \mathcal{O}_{I_k} \bar{\mathcal{O}}_J \rangle$

* *Precision AdS/CFT*

Magical novel relations in N=2 theories

* *A Sum Rule:* $$2a - c = \frac{1}{4}\sum_{i=1}^{\text{rank}}(2r_i - 1)$$ [Shapere,Tachikawa 2008]

Sum over the generators that parametrize the Coulomb branch
e.x. $SU(N)^M$: rank=(N-1)M

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2}(\text{Weyl})^2 - \frac{a}{16\pi^2}(\text{Euler})$$

* *Schur index = BPS index*

The Schur limit of the superconformal index counts operators in the Higgs branch (+ more)

[Rastelli]
[Cordova,Shao 2015]
[Cordova,Gaiotto,Shao 2016]

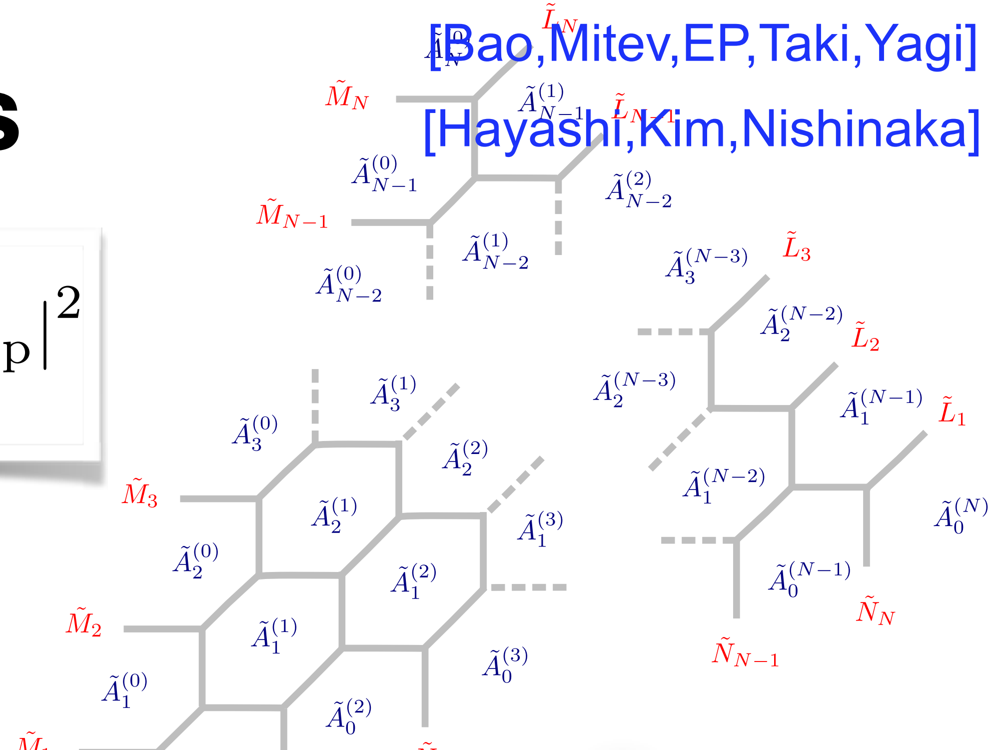
The BPS index = spectrum of BPS particles on the Coulomb branch.

Thank you!

Extra slides

T_N partition functions

[Bao,Mitev,EP,Taki,Yagi]
[Hayashi,Kim,Nishinaka]



$$\mathcal{Z}_{T_N}^{S^4\times S^1} = \frac{1}{|\mathcal{Z}_{\text{extra}}|^2} \oint \prod_{\text{faces}} [da]\, |\mathcal{Z}_{\text{top}}|^2$$

- (N-1)(N-2)/2 integrals = # of faces

$$\mathcal{Z}_{\text{top}} = \mathcal{Z}_{\text{pert}}\,\mathcal{Z}_{\text{inst}}$$

$$\mathcal{M}(Q) = \prod_{i,j=1}^{\infty} (1 - Q\mathfrak{t}^{i-1}\mathfrak{q}^{j})^{-1}$$

$$\mathcal{Z}_N^{\text{pert}} = \prod_{r=1}^{N-1}\prod_{i\le j=1}^{N-r} \frac{\mathcal{M}\left(\frac{\tilde{A}_i^{(r-1)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_{j+1}^{(r-1)}}\right)}{\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r-1)}\tilde{A}_{j-1}^{(r)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_j^{(r)}}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r-1)}}\right)} \prod_{i\le j=1}^{N-r-1} \mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r)}}\right)$$

$$\mathsf{N}^{\beta}_{\lambda\mu}(m) = \prod_{(i,j)\in\lambda} 2\sinh\frac{\beta}{2}\left[m + \epsilon_1(\lambda_i - j + 1) + \epsilon_2(i - \mu_j^t)\right] \times \prod_{(i,j)\in\mu} 2\sinh\frac{\beta}{2}\left[m + \epsilon_1(j - \mu_i) + \epsilon_2(\lambda_j^t - i + 1)\right]$$

$$\mathcal{Z}_N^{\text{inst}} = \sum_{\boldsymbol{\nu}} \prod_{r=1}^{N}\prod_{i=1}^{N-r}\left(\frac{\tilde{N}_r\tilde{L}_{N-r}}{\tilde{N}_{r+1}\tilde{L}_{N-r+1}}\right)^{\frac{|\nu_i^{(r)}|}{2}} \times \prod_{r=1}^{N}\prod_{i\le j=1}^{N-r}\left[\frac{\mathsf{N}^{\beta}_{\nu_i^{(r-1)}\nu_j^{(r)}}\left(a_i^{(r-1)} + a_{j-1}^{(r)} - a_{i-1}^{(r-1)} - a_j^{(r)} - \epsilon_+/2\right)}{\mathsf{N}^{\beta}_{\nu_i^{(r-1)}\nu_{j+1}^{(r-1)}}\left(a_i^{(r-1)} + a_j^{(r-1)} - a_{i-1}^{(r-1)} - a_{j+1}^{(r-1)}\right)} \times \frac{\mathsf{N}^{\beta}_{\nu_i^{(r)}\nu_{j+1}^{(r-1)}}\left(a_i^{(r)} + a_j^{(r-1)} - a_{i-1}^{(r)} - a_{j+1}^{(r-1)} - \epsilon_+/2\right)}{\mathsf{N}^{\beta}_{\nu_i^{(r)}\nu_j^{(r)}}\left(a_i^{(r)} + a_{j-1}^{(r)} - a_{i-1}^{(r)} - a_j^{(r)} - \epsilon_+\right)}\right]$$

- N(N-1)/2 sums still left to perform

Planar spectrum integrability

[EP 2013]
[Mittev,EP 2014+15]

1 *Every N=2 SCFT has a purely gluonic subsector with SU(2,1|2) symmetry that is* ***integrable in the planar limit***

$$H_{\mathcal{N}=2}(g) = H_{\mathcal{N}=4}(\mathbf{g})$$

2 *The Exact Effective coupling (****relative finite renormalization*** *of g)*

$$\mathbf{g}^2 = f(g^2) = g^2 + g^2(Z_{\mathcal{N}=2} - Z_{\mathcal{N}=4})$$

We can compute using localization:

$$W_{\mathcal{N}=2}(g^2) = W_{\mathcal{N}=4}(\mathbf{g}^2)$$

3 *AdS/CFT: effective string tension*

$$f(g^2) = T^2_{eff} = \left(\frac{R^4}{(2\pi\alpha')^2}\right)_{eff}$$

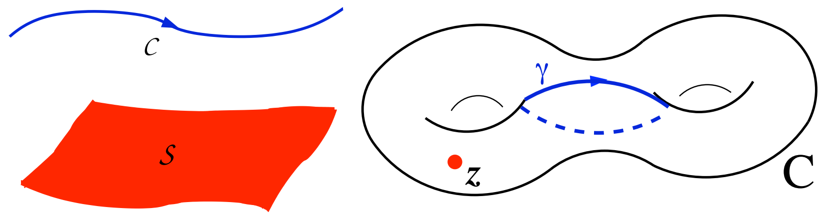
Obtain any observable that classically in the factor $AdS_5 \times S^1$ factor of the geometry by replacing:

$$g^2 \to f(g^2)$$

BPS observables from M-theory [AGGTV 2009]

M2-branes ending on M5-branes or intersecting M5-branes

	M_4	C	operator
M2	2	0	minimal surface operator
	1	1	line operator
	0	2	local operator, change 2d theory
M5	4	0	change 4d theory
	3	1	domain wall
	2	2	surface operator, change 2d theory



For ex. Surface operator is a degenerate primary in CFT:

$$e^{-(b/2)\phi(z)}$$

$$\langle\!\langle\, \mathcal{O} \,\rangle\!\rangle_b^{\text{Liou}} := \frac{\langle\, \mathcal{O}\; e^{2\alpha_4\phi(\infty)} e^{2\alpha_3\phi(1)} e^{2\alpha_2\phi(q)} e^{2\alpha_1\phi(0)} \,\rangle_b^{\text{Liou}}}{\langle\, e^{2\alpha_4\phi(\infty)} e^{2\alpha_3\phi(1)} e^{2\alpha_2\phi(q)} e^{2\alpha_1\phi(0)} \,\rangle_b^{\text{Liou}}}$$

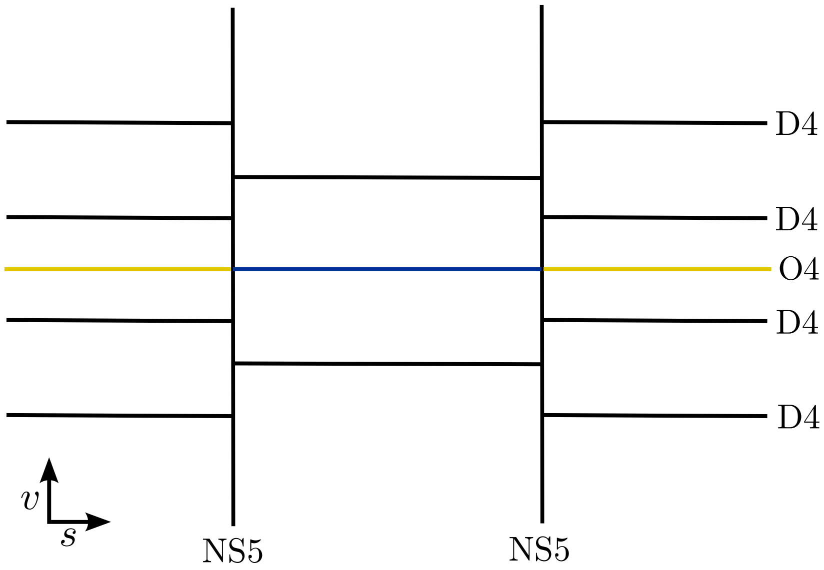
For ex. Line operators in the Liouville side: understood as insertions of

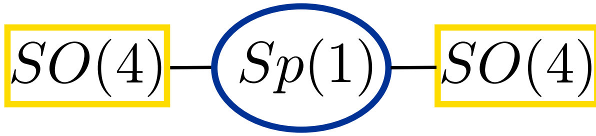
$$\mathsf{L}_\gamma := \operatorname{tr}\left[\mathcal{P}\exp\left(\int_\gamma \mathcal{A}_y\right)\right] \qquad \mathcal{A} := \begin{pmatrix} -\frac{b}{2}\partial_z\phi & 0 \\ \mu e^{b\phi} & \frac{b}{2}\partial_z\phi \end{pmatrix} dz + \begin{pmatrix} \frac{b}{2}\partial_{\bar z}\phi & \mu e^{b\phi} \\ 0 & -\frac{b}{2}\partial_{\bar z}\phi \end{pmatrix} d\bar z$$

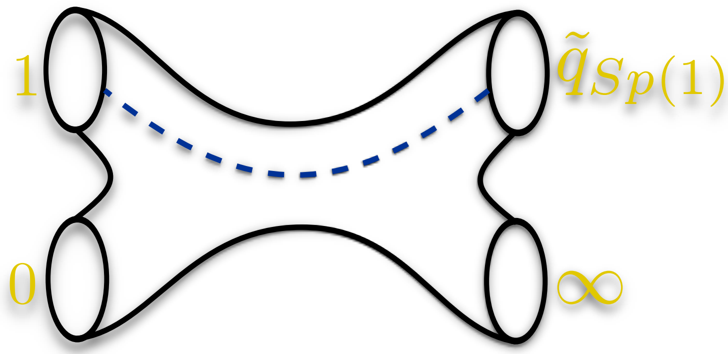
$$\langle\!\langle\, \mathcal{W} \,\rangle\!\rangle_{S^4}^{S} = \langle\!\langle\, \mathsf{L}_{\gamma_s} \,\rangle\!\rangle_b^{\text{Liou}} \qquad \langle\!\langle\, \mathcal{T} \,\rangle\!\rangle_{S^4}^{S} = \langle\!\langle\, \mathsf{L}_{\gamma_t} \,\rangle\!\rangle_b^{\text{Liou}}$$

Orientifolds and SO-Sp quivers

[Hollands,Keller,Song]
[Song PhD thesis]







O⁻4 in yellow and O⁺4 in blue