Holographic dictionary for non-relativistic theories

Ioannis Papadimitriou

SISSA and INFN - Sezione di Trieste, Italy

Review talk at the Heraklion Workshop on Theoretical Physics

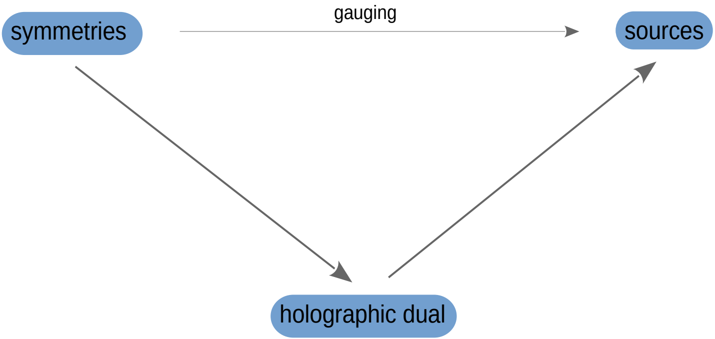
Crete Center for Theoretical Physics, Heraklion, 5-10 September 2016

9 September 2016

Non-relativistic holography

- An important tool to study strongly interacting non-relativistic systems, such as **quantum phase transitions**.
- Useful playground for better understanding holography in **non-AdS backgrounds**.
- It may also provide insights into **emergent symmetries** (e.g Poincaré symmetry or supersymmetry) in non-relativistic systems.

Bottom up holography



Outline

Outline

Non-relativistic vacuum symmetries

- A non-relativistic field theory in d spatial dimensions with Galilean symmetry is invariant under the coordinate transformations

$$
\begin{aligned}
H: &\qquad t \rightarrow t' = t + t_0 \\
P^a: &\qquad x^a \rightarrow x'^a = x^a + x_0^a \\
L_b^a: &\qquad x^a \rightarrow x'^a + L_b^a x^b, \qquad L_b^a \in SO(d) \\
C^a: &\qquad x^a \rightarrow x'^a = x^a - v^a t
\end{aligned}
$$

- This algebra is a contraction of the of the Poincaré group in $d+1$ dimensions
- It admits a central extension:

$$[C^a, P^b] = M\delta^{ab}$$

where M is the non-relativistic mass or the particle number

- A homogeneous and isotropic field theory with Lifshitz symmetry is invariant under the transformations

$$\begin{aligned} H : &\quad t \to t' = t + t_0 \\ P^a : &\quad x^a \to x'^a = x^a + x_0^a \\ L^a_b : &\quad x^a \to x'^a + L^a_b x^b, \qquad L^a_b \in SO(d) \\ D_z : &\quad x^a \to x'^a = \lambda x^a, \qquad t \to t' = \lambda^z t \end{aligned}$$

- This algebra is not a contraction or a subgroup of the conformal group and it does not admit Galilean boosts or a central extension

- It can be generalized to theories which are spatially homogeneous (P^a) but not isotropic ($\not{L}^a_b$) by allowing for different dynamical exponents in each spatial direction

$$D_{z_a} : \qquad x^a \to x'^a = \lambda^{z_a} x^a, \qquad t \to t' = t$$

- Another non-relativistic and scale invariant symmetry algebra is the Schrödinger algebra $\text{Sch}_d(z)$, which can be realized in **two** ways, either in $d+2$ dimensions, with d spatial (x^a) and two lightcone ($x^\pm$) coordinates, or in $d+1$ dimensions, with d spatial (x^a) and a time (t) coordinate.

- In $d+2$ dimensions, besides the spatial translations P^a and spatial rotations L^a_b, it consists of the transformations

$$
\begin{aligned}
H: &\qquad x^+ \to x'^+ = x^+ + x_0^+ \\
M: &\qquad x^- \to x'^- = x^- + x_0^- \\
C^a: &\qquad x^a \to x'^a = x^a - v^a x^+, \qquad x^- \to x'^- = x^- - v^a x^a \\
D_z: &\qquad x^a \to x'^a = \lambda x^a, \qquad x^+ \to x'^+ = \lambda^z x^+, \qquad x^- \to x'^- = \lambda^{2-z} x^-
\end{aligned}
$$

- For any z this is a subalgebra of the conformal algebra $so(d+2,2)$

- For $z=2$ it can be extended to include spatial conformal transformation:

$$
K: \quad x^a \to x'^a = \frac{x^a}{1+kx^+}, \quad x^+ \to x'^+ = \frac{x^+}{1+kx^+}, \quad x^- \to x'^- = \frac{x^- + kx\cdot x/2}{1+kx^+}
$$

where $x\cdot x = 2x^+x^- + x^a x^a$

- In $d+1$ dimensions the Shrödinger algebra is realized as the centrally extended Galilean algebra generated by H, P^a, L^a_b, C^a, M, together with the Lifshitz Dilations D_z and the special conformal symmetry K when $z=2$.
- This can be related to the realization in $d+2$ dimensions by identifying x^+ with time t and the lightcone momentum along x^- with the mass M.
- In non-relativistic theories the mass M is discrete and so x^- must be compactified, leading to problems related to DLCQ.
- An very similar situation arises with the Virasoro $\oplus\ \widehat{u}(1)$ algebra obtained from Compère-Song-Strominger boundary conditions [CSS 2013] imposed on either 3D gravity with $\Lambda<0$, or on the 2D dilaton gravity theory obtained by circle reduction [M. Cvetič, I. P. 2016].
- Null reductions do not always give the correct holographic dictionary in lower dimensions!

Gravity duals

- The Lifshitz algebra can be realized as the algebra of isometries of the metric

$$ds^2_{d+2} = \frac{dr^2}{r^2} - \frac{dt^2}{r^{2z}} + \frac{dx^a dx^a}{r^2}$$

  where Lifshitz dilations act as

$$D_z: \qquad r \to r' = \lambda r, \qquad t \to t' = \lambda^z t, \qquad x^a \to x'^a = \lambda x^a$$

- The null energy condition requires $z \geq 1$
- This metric does not have any curvature singularities, but it is geodesically incomplete and infalling observers experience large tidal forces as $r \to \infty$.
- Moreover, the initial value problem seems rather problematic in such spacetimes [Copsey, Mann 2010; Keeler, Knodel, Liu 2010, 2014; Horowitz, Way 2011; Harrison, Kachru, Wang 2012; Knodel, Liu 2013]

- The simplest theory that admits Lifshitz backgrounds is [M. Taylor 2008]

$$S = \frac{1}{2\kappa^2}\int d^{d+2}x\sqrt{-g}\Big(R + \frac{d(d+1)}{\ell^2} - \frac{1}{4}F^2 - \frac{1}{2}m^2A^2\Big)$$

where

$$m^2 = \frac{d^2(d+1)z}{z^2+z(d-1)+d^2}, \qquad \ell^2 = \frac{z^2+z(d-1)+d^2}{d(d+1)}$$

- The metric and gauge field take the form

$$ds^2 = dr^2 - e^{2zr/\ell}dt^2 + e^{2r/\ell}dx^a dx^a, \qquad A = \sqrt{\frac{2(z-1)}{z}}e^{zr/\ell}dt$$

- This model is equivalent to that considered in [Kachru, Liu, Mulligan 2008] involving a massless gauge field and a d-form, coupled through a Chern-Simons term.

- Lifshitz backgrounds and Lifshitz black holes also arise as solutions of various, generically non-unitary, higher derivative theories.

Lifshitz backgrounds with running scalars

- Backgrounds with a Lifshitz metric can also be supported by a massless gauge field in the presence of a running scalar, which breaks the scale invariance.

- In particular, the model [M. Taylor 2008]

$$S = \frac{1}{2\kappa^2}\int d^{d+2}x\sqrt{-g}\Big(R - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\lambda\phi}F^2\Big)$$

$$\Lambda = -\frac{1}{2}(d+z)(d+z-1), \qquad \lambda^2 = \frac{2d}{z-1}$$

admits the solution

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}dx^a dx^a, \qquad e^{\lambda\phi} = \mu e^{-2dr}$$

$$F_{rt} = \sqrt{2(z-1)(d+z)/\mu}\; e^{(d+z)r}$$

- The limit $z \to 1$ is not smooth since λ diverges in this limit.

Hyperscaling violating Lifshitz backgrounds

- A more general class of backgrounds with the same isometries as the running dilaton Lifshitz solutions are the hyperscaling violating Lifshitz (hvLf) backgrounds introduced in [Goutereaux, Kiritsis 2011, 2012; Huijse, Sachdev, Swingle 2011].

- The metric takes the form

$$ds^2_{d+2} = \ell^2 u^{-2(d-\theta)/d} \left(du^2 - u^{-2(z-1)} dt^2 + dx^a dx^a \right)$$

  with dynamical exponent $z \neq 1$ and hyperscaling violation exponent $\theta \neq 0$

- The scaling transformation

$$x^a \to \lambda x^a \ , \ t \to \lambda^z t \ , \ u \to \lambda u$$

  is a **conformal** isometry of this metric since

$$ds^2_{d+2} \to \lambda^{2\theta/d} ds^2_{d+2}$$

- The null energy condition requires

$$(d-\theta)(d(z-1)-\theta)\geq 0, \quad (d-\theta+z)(z-1)\geq 0$$

whose general solution is [Chemissany, IP 2014]

I	$z\leq 0$	$\theta\geq d$
II	$0<z\leq 1$	$\theta\geq d+z$
IIIa	$1\leq z\leq 2$	$\theta\leq d(z-1)$
IIIb		$d\leq\theta\leq d+z$
IVa	$2<z\leq\frac{2d}{d-1}$	$\theta\leq d$
IVb		$d(z-1)\leq\theta\leq d+z$
V	$z>\frac{2d}{d-1}$	$\theta\leq d$

- For $\theta\geq d+z$ (cases I and II) the on-shell action does not diverge and hence there is no well defined asymptotic expansion/holographic dictionary (cf. D6 branes).

Theories admitting hvLf backgrounds

- Like Lifshitz backgrounds with a running dilaton, hvLf backgrounds generically are supported by matter fields that include such a dilaton field.
- This allows us to go to the **dual frame** where the metric is asymptotically Lifshitz, but may not be in the Einstein frame.
- In the dual frame, hvLf backgrounds are Lifshitz solutions of the action

$$S_\xi = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}e^{d\xi\phi}\left(R[g]-\alpha_\xi(\partial\phi)^2-Z_\xi(\phi)F^2-W_\xi(\phi)B^2-V_\xi(\phi)\right)$$

with a running dilaton and hypesrscaling violating exponent

$$\theta=-d\mu\xi$$

- In this action $B_\mu = A_\mu - \partial_\mu\omega$, where ω is a Stückelberg scalar transforming non-trivially under $U(1)$ gauge transformations, i.e.

$$A_\mu \to A_\mu + \partial_\mu\Lambda, \quad \omega\to\omega+\Lambda$$

such that B_μ is gauge invariant.

- This model admits Lifshitz solutions of the form

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2, \quad A = \frac{\mathcal{Q}}{\epsilon Z_o}e^{\epsilon r}dt, \quad \phi = \mu r, \quad \omega = const.$$

provided the functions and parameters in the Lagrangian satisfy the relations

$$V_\xi = V_o e^{2(\rho+\xi)\phi}, \qquad Z_\xi = Z_o e^{-2(\xi+\nu)\phi}, \qquad W_\xi = W_o e^{2\sigma\phi}$$

as well as

$$\rho = -\xi, \quad \nu = -\xi + \frac{\epsilon - z}{\mu}, \quad \sigma = \frac{z-\epsilon}{\mu},$$
$$\epsilon = \frac{(\alpha_\xi + d^2\xi^2)\mu^2 - d\mu\xi + z(z-1)}{z-1}, \quad \mathcal{Q}^2 = \frac{1}{2}Z_o(z-1)\epsilon,$$
$$W_o = 2Z_o\epsilon(d+z+d\mu\xi-\epsilon), \quad V_o = -d(1+\mu\xi)(d+z+d\mu\xi) - (z-1)\epsilon$$

- In the Einstein frame these solutions are hyperscaling violating Lifshitz backgrounds with $\theta = -d\mu\xi$.
- These solutions are characterized by three independent parameters, z, θ and μ, with μ related to the beta function of a scalar operator in the dual theory.

- The Schrödinger algebra $\mathrm{Sch}_d(z)$ can be realized geometrically as the isometry algebra of the metric [K. Balasubramanian, J. McGreevy 2008; D. Son 2008]

$$ds^2_{d+3} = -\frac{b^2(dx^+)^2}{r^{2z}} + \frac{1}{r^2}\Big(dr^2 + dx^a dx^a + 2dx^+ dx^-\Big)$$

- Dilatations act as

$$D_z: \qquad r \to \lambda r, \qquad x^a \to \lambda x^a, \qquad x^+ \to \lambda^z x^+, \qquad x^- \to \lambda^{2-z} x^-$$

- The parameter b can be removed by a rescaling of $x^\pm$, but it is useful to keep this parameter explicitly. For $b = 0$ this metric the Poincaré metric on AdS_{d+3}, while $b \neq 0$ corresponds to a (non-relativistic) deformation of the dual relativistic CFT.

Theories admitting Schrödinger backgrounds $(z \neq 0)$

- For $z \neq 0$, Schrödinger backgrounds are solutions of the action [D. Son 2008]

$$S = \frac{1}{2\kappa^2} \int d^{d+2}x \sqrt{-g} \Big(R + d(d+1) - \frac{1}{4} F^2 - \frac{1}{2} z(d+z-1) A^2 \Big)$$

where the massive gauge field necessary to support the geometry takes the form

$$A = \sqrt{\frac{2(z-1)}{z}} \, \frac{b}{r^z} dx^+$$

- For specific values of z Schrödinger backgrounds can be embedded in string theory [Maldacena, Martelli, Tachikawa 2008; Herzog, Rangamani, Ross 2008; Kraus, Perlmutter 2011] and can be realized as solutions of topologically massive gravity in three dimensions [Guica, Skenderis, Taylor, van Rees 2010].

Theories admitting Schrödinger backgrounds $(z = 0)$

- Schrödinger backgrounds with $z = 0$ cannot be realized as solutions of the above massive vector model since the vector field becomes singular.

- A minimal action admitting Schrödinger solutions with $z = 0$ is

$$S = \frac{1}{2\kappa^2} \int d^{d+2}x\sqrt{-g}\Big(R + d(d+1) - \frac{1}{2}(\partial\phi)^2\Big)$$

  where the scalar field takes the form $\phi = \sqrt{d-1}\, bx^+$.

- A slightly more general model which can be embedded in string theory and M-theory is [Donos, Gauntlett 2010; Cassani, Faedo 2011; Halmagyi, Petrini, Zaffaroni 2011; Petrini, Zaffaroni 2012]

$$S = \frac{1}{2\kappa^2} \int d^{d+2}x\sqrt{-g}\Big(R + d(d+1) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\Big)$$

- The holographic dictionary for general AlAdS solutions of this model (which include $z = 0$ Schrödinger solutions) was constructed in [IP 2011].

$z = 2$ Lifshitz from null reduction of Sch$_d(0)$

- Lifshitz backgrounds with $z = 2$ can be obtained by a **null** reduction of $z = 0$ Schrödinger backgrounds [K. Balasubramanian, K. Narayan 2010; R. C. Costa, M. Taylor 2010; K. Narayan 2011].

- Reducing over the null direction x^+ leads to a theory with a massive vector field and the $z = 2$ Lifshitz background

$$ds^2_{d+2} = \frac{1}{r^2}\left(dr^2 + dx^a dx^a - \frac{(dx^-)^2}{b^2 r^2}\right), \qquad A = \frac{1}{b^2 r^2} dx^-$$

- The fact that the reduction is null is problematic for various reasons. The most important in the context of holography is that the lower dimensional theory is not a consistent truncation, since an additional constraint must be imposed in order to solve the higher dimensional equations [R. C. Costa, M. Taylor 2010].

- The presence of a **second class constraint** in the lower dimensional theory seems to be a generic property of null reductions and has significant implications for the relation of the holographic dictionaries before and after the reduction.

Outline

What is a holographic dictionary?

Identification of physical **observables** on the two sides of a holographic duality

- Local observables $\mathcal{O}(x)$ and their sources $J(x)$ comprise a **symplectic manifold** (cf. Local Renormalization Group [H. Osborn 1994]) with symplectic form

$$\Omega \sim \int d^d x \, \delta\mathcal{O}(x) \wedge \delta J(x)$$

- QFT observables require **renormalization**.
- (Renormalized) observables obey general **Ward identities**, which may exhibit quantum **anomalies**.

symplectic space of **bulk asymptotic solutions** $\equiv$ symplectic space of **renormalized local observables**

The space of asymptotic solutions

Non-linear asymptotic solutions

- In order to have a familiar example as a reference point let us consider EH gravity:

$$S=-\frac{1}{2\kappa^2}\int_{\mathcal{M}}d^{d+1}x\sqrt{g}(R-2\Lambda)-\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}}d^dx\sqrt{\gamma}2K$$

- Any solution can (at least locally) be written in the Fefferman-Graham gauge

$$ds^2=dr^2+\gamma_{ij}(r,x)dx^idx^j$$

and takes asymptotically as $r\to+\infty$ the form

$$\gamma_{ij}(r,x)=e^{2r}\left(g_{(0)ij}(x)+e^{-2r}g_{(2)ij}(x)+\cdots+e^{-dr}\left(-2rh_{(d)ij}(x)+g_{(d)ij}(x)\right)+\cdots\right)$$

- The arbitrary functions $g_{(0)ij}(x)$ and $g_{(d)ij}(x)$ satisfy the constraints

$$D_{(0)}{}^{i}\mathcal{T}_{ij}(x)=0,\quad \mathcal{T}_i^i(x)=\mathcal{A}[g_{(0)}]$$

where $\mathcal{T}_{ij}\sim g_{(d)ij}$, and $\mathcal{A}[g_{(0)}]$ is the conformal anomaly.

Symplectic structure and the holographic dictionary

- The space of asymptotic solutions is parameterized by $g_{(0)ij}(x)$ and $g_{(d)ij}(x)$.
- This space is equipped with the symplectic form

$$\Omega = \int d^d x \; \delta\pi_{(d)}{}^{ij} \wedge \delta g_{(0)ij}$$

where $\pi_{(d)}{}^{ij} \equiv \sqrt{g_{(0)}}\;\mathcal{T}^{ij} \sim \sqrt{g_{(0)}}\; g_{(d)}^{ij}$

- This symplectic form leads to the Poisson bracket

$$\left\{\pi_{(d)}{}^{ij}(x), g_{(0)kl}(x')\right\} = \delta^i_{(k}\delta^j_{l)}\delta^{(d)}(x-x')$$

which is allows us to determine the **asymptotic symmetry algebra**.

- The symplectic structure of the space of asymptotic solutions provides a **definition** of the holographic dictionary.
- This **may not be unique**: boundary conditions (equivalently holographic dictionaries) classified by symplectic transformations.

Generalized PBH transformations

- The parameterization of the space of asymptotic solutions in terms of $g_{(0)ij}(x)$ and $\pi_{(d)}{}^{ij}(x)$ is **gauge redundant**.
- There are bulk diffeomorphisms (generally bulk local transformations) which preserve the FG gauge, but transform $g_{(0)ij}(x)$ and $g_{(d)ij}(x)$.
- Such Penrose-Brown-Henneaux transformations correspond to the gauging of the global symmetries in the dual theory in the presence of arbitrary sources and lead to the **Ward identities**.
- For AdS gravity they comprise of bulk diffeomorphisms $\delta_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$ with

$$\xi^r = \sigma(x), \qquad \xi^i = \xi^i_o(x) + \partial_j\sigma(x)\int_r^\infty dr'\gamma^{ji}(r',x)$$

where $\sigma(x)$ and $\xi^i_o(x)$ are arbitrary.

- Under PBH diffeomorphisms the $g_{(0)ij}(x)$ and $\pi_{(d)}{}^{ij}(x)$ transform as

$$\delta_\xi g_{(0)ij} = D_{(0)i}\xi_{oj} + D_{(0)j}\xi_{oi} + 2\sigma g_{(0)ij}$$
$$\delta_\xi \pi_{(d)}{}^{ij} = D_{(0)k}\left(\pi_{(d)}{}^{ij}\xi_o^k - \pi_{(d)}{}^{ik}D_{(0)k}\xi_o^j - \pi_{(d)}{}^{jk}D_{(0)k}\xi_o^i\right) + 2\sigma(x)\pi_{(d)}{}^{ij} + \frac{\delta}{\delta g_{(0)ij}}\int d^dx\sqrt{g_{(0)}}\mathcal{A}\sigma$$

- These transformations are generated through the Poisson bracket by a constraint function on the space of asymptotic solutions

$$\mathcal{C}[\xi_o,\sigma] = \int d^dx\sqrt{g_{(0)}}\Big(\xi_o^i(x)D_{(0)}{}^j\mathcal{T}_{ij} + \sigma(x)\left(\mathcal{T}_i^i - \mathcal{A}\right)\Big)$$

$$\Big\{\mathcal{C}[\xi_o,\sigma], g_{(0)ij}(x)\Big\} = \delta_\xi g_{(0)ij}(x), \quad \Big\{\mathcal{C}[\xi_o,\sigma], \pi_{(d)}{}^{ij}(x)\Big\} = \delta_\xi \pi_{(d)}{}^{ij}(x)$$

- The Poisson brackets of the constraints lead to the **asymptotic symmetry algebra** once boundary conditions have been chosen.

Equivalence classes and boundary counterterms

- There are two ways to remove the gauge redundancy of the space of asymptotic solutions: either by solving the constraints explicitly or by considering instead the **equivalence classes** $[g_{(0)}]$ and $[\pi_{(d)}]$ under PBH transformations.

- The local boundary counterterms are required in order to formulate the variational problem in terms of these equivalence classes [I.P., Skenderis 2005].

- A generic variation of the renormalized action takes the form

$$\delta S_{\rm ren} = \int d^dx \sqrt{g_{(0)}}\, \pi_{(d)}{}^{ij}(x) \delta g_{(0)ij}(x)$$

- For variations that coincide with PBH transformations this expression is proportional to the conformal anomaly and, hence, provided the conformal anomaly is numerically zero, $S_{\rm ren}[g_{(0)}]$ is a **class function**.

- This imposes more stringent conditions on the boundary counterterms than the simple requirement of finiteness (e.g. they should preserve the constraints).

- Formulating the variational problem in terms of equivalence classes also leads to well defined conserved charges and thermodynamics [I.P., Skenderis 2005; O. S. An, Cvetič, I.P 2016; I.P., Skenderis to appear]

Algorithmic construction of the holographic dictionary

Generalized holography

- Any holographic duality can be understood in terms of the space of asymptotic solutions of the bulk theory and has a qualitatively similar structure to AdS gravity.
- Contrary to AdS gravity, in general the space of asymptotic solutions, including the **symplectic structure**, the **constraints** and the **boundary counterterms**, must be constructed from scratch.
- There is an algorithmic procedure that allows one to do this for any bulk theory and any boundary conditions.

Step I: Radial Hamiltonian formulation

- We start by decomposing all tensor fields in radial and transverse components as

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j$$

where N, N_i and γ_{ij} are induced fields on the radial slice $\Sigma_r \cong \partial\mathcal{M}$.

- This decomposition need only hold in an open neighborhood of the boundary $\partial\mathcal{M}$ – no requirement of global hyperbolicity!

- Inserting this decomposition in the action leads to the radial Lagrangian

$$L = -\frac{1}{2\kappa^2}\int_{\Sigma_r} d^d x\sqrt{\gamma}N\left(R[\gamma] - 2\Lambda + K^2 - K^i_j K^j_i\right)$$

and the canonical momentum

$$\pi^{ij} = \frac{\delta L}{\delta\dot{\gamma}_{ij}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(K\gamma^{ij} - K^{ij})$$

- The Hamiltonian takes the form

$$H = \int_{\Sigma_r} d^d x \pi^{ij} \dot{\gamma}_{ij} - L = \int_{\Sigma_r} d^d x \left(N\mathcal{H} + N_i \mathcal{H}^i \right)$$

  where

$$\mathcal{H} = 2\kappa^2 \gamma^{-\frac{1}{2}} \left(\pi^i_j \pi^j_i - \frac{1}{d-1}\pi^2 \right) + \frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - 2\Lambda\right)$$

$$\mathcal{H}^i = -2D_j \pi^{ij}$$

- Hamilton's equations for N and N_i lead to the **first class** constraints

$$\mathcal{H} = 0, \quad \mathcal{H}^i = 0$$

- The symplectic form

$$\Omega = \int_{\Sigma_r} d^d x \; \delta\pi^{ij} \wedge \delta\gamma_{ij}$$

  is **independent of the radial coordinate** and leads to the Poisson bracket

$$\left\{ \gamma_{ij}(r,x), \pi^{kl}(r,x') \right\} = \delta_i^{(k}\delta_j^{l)}\delta^{(d)}(x - x')$$

- The constraints generate bulk diffeomorphisms through the Poisson bracket:

$$\mathcal{C}[\xi] = \int_{\Sigma_r} d^d x \left(\xi\mathcal{H} + \xi^i \mathcal{H}_i\right),$$

$$\{\mathcal{C}[\xi], \gamma_{ij}\} = \delta_{\tilde{\xi}}\gamma_{ij}, \qquad \left\{\mathcal{C}[\xi], \pi^{ij}\right\} = \delta_{\tilde{\xi}}\pi^{ij}, \qquad \tilde{\xi}^\mu = \left(\xi/N, \xi^i - \xi N^i/N\right)$$

- The algebra of constraints closes, but with **field dependent** parameter:

$$\left\{\mathcal{C}[\xi], \mathcal{C}[\xi']\right\} = \mathcal{C}[\xi'']$$

$$\xi''^\mu = \left(\xi^i\partial_i\xi' - \xi'^i\partial_i\xi, \quad \xi^i\partial_i\xi^j - \xi'^i\partial_i\xi^j - (\xi D^j\xi' - \xi' D^j\xi)\right)$$

Isomorphism of symplectic spaces

- A sequence of symplectomorphisms:

phase space on radial slice	$\cong$	symplectic space of renormalized local observables	$\cong$	symplectic space of renormalized local QFT observables

- The symplectic map between phase space and the space of asymptotic solutions is **not diagonal** (e.g. both γ_{ij} and π^{ij} asymptotically proportional to $g_{(0)}$, with $g_{(d)}$ entering in subleading terms)

- **Holographic renormalization**: canonical transformation that diagonalizes this symplectic map [I.P. 2010]

Step II: Recursive solution of the constraints

- The canonical transformation that diagonalizes the symplectic map can be obtained by solving (suitable) **Hamilton-Jacobi equations**.
- In most cases (see exceptions below) the relevant HJ equations are obtained by writing the canonical momenta as gradients

$$\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}}$$

of Hamilton's principal function $\mathcal{S}[\gamma]$ and inserting these in the constraints:

$$2\kappa^2 \gamma^{-\frac{1}{2}} \left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{ij}\gamma_{kl} \right) \frac{1}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}} \frac{1}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - 2\Lambda\right)$$

$$D_j \left(\frac{1}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}} \right) = 0$$

Different types of solutions of the HJ equations

- Exact solutions using an ansatz lead to RG flows.
- Derivative expansion around RG flows that correspond to a VEV determines the effective action.
- Perturbative solution for $\gamma_{ij} = \gamma^B_{ij} + h_{ij}$ computes correlation functions.
- Covariant **asymptotic** solution determines the boundary counterterms and the asymptotic expansions of the fields.

Asymptotic solutions of the HJ equations

- In simple cases the general asymptotic solution of the HJ equations can be obtained by making a local ansatz for $\mathcal{S}[\gamma]$, e.g. in the form of a covariant derivative expansion [de Boer, Verlinde, Verlinde 1999; Martelli, Mück 2002; Elvang, Hadjiantonis 2016].
- However, such asymptotic solutions can be obtained through a recursive algorithm in the form of a covariant expansion in eigenfunctions of suitable functional operators [I.P., Skenderis 2004; I.P. 2011; Chemissany, I.P. 2014].
- These recursive algorithms do not require any ansatz, which is particularly useful when classifying all possible terms that can appear is not straightforward (especially in non-relativistic theories).

Recursive solution of the HJ equations

- For pure AdS gravity a solution is sought in the form

$$\mathcal{S} = \underbrace{\mathcal{S}_{(0)} + \mathcal{S}_{(2)} + \cdots + \log e^{-2r}\widetilde{\mathcal{S}}_{(d)}}_{-S_{\text{ct}}} + \mathcal{S}_{(d)} + \cdots$$

where

$$\delta_D \mathcal{S}_{(2n)} = (d-2n)\mathcal{S}_{(2n)}, \quad \delta_D \widetilde{\mathcal{S}}_{(d)} = 0, \quad \delta_D \mathcal{S}_{(d)} = -2\widetilde{\mathcal{S}}_{(d)}$$

and δ_D is the dilatation operator [I.P., Skenderis 2004]

$$\delta_D = \int d^d x\, 2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}}$$

- This is an asymptotic expansion since $\partial_r \sim \delta_D$
- For AdS_5 the counterterms are

$$S_{\text{ct}} = \frac{1}{\kappa^2}\int d^4x\sqrt{\gamma}\left(3 + \frac{1}{4}R[\gamma] - \log e^{-2r}\frac{1}{16}\left(R^{ij}R_{ij} - \frac{1}{3}R^2\right)\right)$$

- The renormalized on-shell action is

$$S_{\text{ren}} = \lim_{r\to\infty}(S + S_{\text{ct}}) = \mathcal{S}_{(d)}$$

and corresponds to an **integration 'constant'** of the asymptotic solution.

- S_{ct} is the generating function of the canonical transformation that diagonalizes the symplectic map between phase space and the space of asymptotic solutions:

$$\begin{pmatrix}\pi^{ij}\\ \gamma_{ij}\end{pmatrix} \to \begin{pmatrix}\Pi^{ij}\\ \gamma_{ij}\end{pmatrix} = \begin{pmatrix}\pi^{ij} + \frac{\delta S_{\text{ct}}}{\delta\gamma_{ij}}\\ \gamma_{ij}\end{pmatrix} \sim \begin{pmatrix}e^{-2r}\pi_{(d)}{}^{ij}\\ e^{2r}g_{(0)ij}\end{pmatrix} \sim \begin{pmatrix}e^{-2r}\sqrt{g_{(0)}}\, g_{(d)}{}^{ij}\\ e^{2r}g_{(0)ij}\end{pmatrix}$$

- It preserves the symplectic form:

$$\Omega = \int d^dx\, \delta\pi^{ij}\wedge\delta\gamma_{ij} = \int d^dx\, \delta\Pi^{ij}\wedge\delta\gamma_{ij} = \int d^dx\, \delta\pi^{ij}_{(d)}\wedge\delta g_{(0)ij}$$

- $\pi^{ij}_{(d)} \sim \delta\mathcal{S}_{(d)}/\delta g_{(0)ij}$ corresponds to the renormalized stress tensor.

Step III: Asymptotic solutions from flow equations

- Combining the Hamiltonian and Hamilton-Jacobi expressions for the canonical momenta leads to first order **flow equations**:

$$\dot{\gamma}_{ij} = 4\kappa^2 \left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{kl}\gamma_{ij} \right) \frac{1}{\sqrt{\gamma}} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}}$$

- Inserting different types of HJ solutions $\mathcal{S}$ **automatically** leads to different types of solutions of the second order eoms.
- Conversely, for **any** solution of the second order equations there is a solution $\mathcal{S}$ of the HJ equations such that this flow equation holds, at least locally in field space.
- Inserting the general asymptotic solution for $\mathcal{S}$ gives the asymptotic solution of the bulk fields (Fefferman-Graham expansions).
- Inserting an ansatz without transverse derivatives gives an exact RG flow.
- Inserting a derivative expansion for $\mathcal{S}$ around a solution with a length scale gives low energy effective dynamics (hydrodynamics, Goldstone modes).

Exceptional case studies

- The canonical transformation that diagonalizes the symplectic map is not always generated by a local functional of the induced fields.
- An example is p-form fields in AdS$_{d+1}$, whose equaiton of motion takes the form

$$\ddot{A}_{i_1\ldots i_p} + (d-2p)\ell^{-1}\dot{A}_{i_1\ldots i_p} \sim 0$$

and so the general asymptotic solution is

$$A_{i_1\ldots i_p} \sim A_{(0)i_1\ldots i_p}(x) + \cdots + e^{-(d-2p)r/\ell}A_{(d-2p)i_1\ldots i_p}(x) + \cdots$$

- For $p < d/2$ the canonical transformation that diagonalizes the symplectic map is

$$S_{\rm ct}[\gamma, A] = -\frac{1}{2\kappa^2(d-2p-2)(p+1)!}\int_{\Sigma_{r_o}} d^dx\,\sqrt{-\gamma}F_{i_1 i_2\ldots i_{p+1}}F^{i_1 i_2\ldots i_{p+1}} + \cdots$$

- For $p \geq d/2$ the canonical transformation that diagonalizes the symplectic map is

$$-\int_{\Sigma_{r_o}} d^d x\, \pi^{i_1\ldots i_p} A_{i_1\ldots i_p} + S_{\rm ct}[\gamma,\pi]$$

with

$$S_{\rm ct}[\gamma,\pi] = \frac{p!\kappa^2}{d-2p}\int_{\Sigma_{r_o}} d^d x\, \frac{1}{\sqrt{-\gamma}}\pi^{i_1 i_2\ldots i_p}\pi_{i_1 i_2\ldots i_p} + \cdots$$

- The form of the counterterms for $p \geq d/2$ can be obtained by Hodge dualizing the standard counterterms for the Hodge dual $(d-p-1)$-form fields.

- This case covers gauge fields in AdS_3 and AdS_2 and it is important for AdS_2 holography [Cvetič, I.P. 2016], including correlation functions in a holographic Kondo model [Erdmenger, Hoyos, O'Bannon, I.P., Probst, Wu to appear].

- Hodge dualizing is not always possible (e.g. AdS_2) but the counterterms $S_{\rm ct}[\gamma,\pi]$ can be obtained by solving a Legendre transformed Hamilton-Jacobi equation.

Second class constraints and irrelvant deformations

- Another subtlety that is particularly relevant for non-relativistic holography is the presence of (asymptotic) **second class constraints** on phase space.

- Such constraints are of the form

$$C[\gamma, A, \varphi; \pi_\gamma, \pi_A, \pi_\varphi] \approx 0$$

  where C can be either an algebraic function, or it can involve derivatives along Σ_r.

- Fluctuations orthogonal to the constraints correspond to (generically irrelevant) deformations by an operator whose source is **composite** in terms of the original symplectic variables.

- The “no-tadpole” condition for such composite fluctuations leads to exotic scaling dimensions (e.g. composite scalar in Lifshitz).

- The counterterms take the form of a Taylor expansion in C, with the coefficient of C^n determined by the n-point function of the composite operator.

- Null (Kaluza-Klein or Scherk-Schwarz) reductions typically lead to such second class constraints, but the precise relation is still to be determined.

Outline

Lifshitz dictionary from (massive) vector models

- Both Lifshitz and hvLf backgrounds in the Einstein frame can be obtained from asymptotically Lifshitz solutions of the form

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2, \quad A = \frac{\mathcal{Q}}{\epsilon Z_o}e^{\epsilon r}dt, \quad \phi = \mu r, \quad \omega = const.$$

in the **dual frame**.

- The action in the dual frame is

$$S_\xi = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}e^{d\xi\phi}\left(R[g] - \alpha_\xi(\partial\phi)^2 - Z_\xi(\phi)F^2 - W_\xi(\phi)B^2 - V_\xi(\phi)\right)$$

$$V_\xi \sim V_o e^{2(\rho+\xi)\phi}, \qquad Z_\xi \sim Z_o e^{-2(\xi+\nu)\phi}, \qquad W_\xi \sim W_o e^{2\sigma\phi}$$

$$\rho = -\xi, \quad \nu = -\xi + \frac{\epsilon - z}{\mu}, \quad \sigma = \frac{z-\epsilon}{\mu}, \quad \theta = -d\mu\xi$$

$$\epsilon = \frac{(\alpha_\xi + d^2\xi^2)\mu^2 - d\mu\xi + z(z-1)}{z-1}, \quad \mathcal{Q}^2 = \frac{1}{2}Z_o(z-1)\epsilon,$$

$$W_o = 2Z_o\epsilon(d + z + d\mu\xi - \epsilon), \quad V_o = -d(1+\mu\xi)(d+z+d\mu\xi) - (z-1)\epsilon$$

- ADM decomposition

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j$$

- Radial ADM Lagrangian:

$$L = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-\gamma}N\left(\left(1+\frac{d^2\xi^2}{\alpha_\xi}\right)K^2 - K^{ij}K_{ij} - \frac{\alpha_\xi}{N^2}\left(\dot{\phi} - N^i\partial_i\phi - \frac{d\xi}{\alpha_\xi}NK\right)^2\right.$$
$$-\frac{2}{N^2}Z_\xi(\phi)(F_{ri} - N^k F_{ki})(F_r{}^i - N^l F_l{}^i) - \frac{1}{N^2}W_\xi(\phi)\left(A_r - N^i A_i - \dot{\omega} + N^i\partial_i\omega\right)^2$$
$$\left.+R[\gamma] - \alpha_\xi\partial_i\phi\partial^i\phi - Z_\xi(\phi)F_{ij}F^{ij} - W_\xi(\phi)B_iB^i - V_\xi(\phi) - 2\Box_\gamma\right)e^{d\xi\phi}$$

- Hamiltonian:

$$H = \int d^{d+1}x \left(\dot{\gamma}_{ij}\pi^{ij} + \dot{A}_i\pi^i + \dot{\phi}\pi_\phi + \dot{\omega}\pi_\omega\right) - L$$
$$= \int d^{d+1}x \left(N\mathcal{H} + N_i\mathcal{H}^i + A_r\mathcal{F}\right)$$

where

$$\mathcal{H} = -\frac{\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(2\pi^{ij}\pi_{ij} - \frac{2}{d}\pi^2 + \frac{1}{2\alpha}\left(\pi_\phi - 2\xi\pi\right)^2 + \frac{1}{4}Z_\xi^{-1}(\phi)\pi^i\pi_i + \frac{1}{2}W_\xi^{-1}(\phi)\pi_\omega^2\right)$$
$$+ \frac{\sqrt{-\gamma}}{2\kappa^2}\left(-R[\gamma] + \alpha_\xi\partial^i\phi\partial_i\phi + Z_\xi(\phi)F^{ij}F_{ij} + W_\xi(\phi)B^iB_i + V_\xi(\phi) + 2\Box_\gamma\right)e^{d\xi\phi}$$

$$\mathcal{H}^i = -2D_j\pi^{ji} + F^i{}_j\pi^j + \pi_\phi\partial^i\phi - B^i\pi_\omega$$

$$\mathcal{F} = -D_i\pi^i + \pi_\omega$$

Canonical momenta

- From off-shell Lagrangian:

$$
\begin{aligned}
\pi^{ij} &= \frac{\delta L}{\delta \dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(K\gamma^{ij} - K^{ij} + \frac{d\xi}{N}\gamma^{ij}\left(\dot{\phi} - N^k\partial_k\phi\right)\right),\\
\pi^{i} &= \frac{\delta L}{\delta \dot{A}_{i}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}Z_\xi(\phi)\frac{4}{N}\gamma^{ij}\left(F_{rj} - N^k F_{kj}\right),\\
\pi_{\phi} &= \frac{\delta L}{\delta \dot{\phi}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(2d\xi K - \frac{2\alpha_\xi}{N}(\dot{\phi} - N^i\partial_i\phi)\right),\\
\pi_{\omega} &= \frac{\delta L}{\delta \dot{\omega}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}W_\xi(\phi)\frac{2}{N}\left(\dot{\omega} - N^i\partial_i\omega - A_r + N^i A_i\right)
\end{aligned}
$$

- From on-shell action:

$$
\pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}}, \quad \pi^{i} = \frac{\delta \mathcal{S}}{\delta A_{i}}, \quad \pi_{\phi} = \frac{\delta \mathcal{S}}{\delta \phi}, \quad \pi_{\omega} = \frac{\delta \mathcal{S}}{\delta \omega}
$$

- Combining the two expressions for the momenta:

$$\dot{\gamma}_{ij} = -\frac{4\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\left(\gamma_{ik}\gamma_{jl} - \frac{\alpha_\xi + d^2\xi^2}{d\alpha} \gamma_{ij}\gamma_{kl} \right) \frac{\delta}{\delta\gamma_{kl}} - \frac{\xi}{2\alpha}\gamma_{ij}\frac{\delta}{\delta\phi} \right) \mathcal{S},$$

$$\dot{A}_i = -\frac{\kappa^2}{2}\frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} Z_\xi^{-1}(\phi)\gamma_{ij}\frac{\delta}{\delta A_j}\mathcal{S},$$

$$\dot{\phi} = -\frac{\kappa^2}{\alpha}\frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\frac{\delta}{\delta\phi} - 2\xi\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} \right) \mathcal{S},$$

$$\dot{\omega} = -\frac{\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} W_\xi^{-1}(\phi)\frac{\delta}{\delta\omega}\mathcal{S}$$

Recursive solution of the Hamilton-Jacobi equation

- The zero order solution of the HJ equation contains no transverse derivatives:

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2}\int d^{d+1}x\sqrt{-\gamma}U(\phi, A_iA^i)$$

- Inserting this ansatz into the Hamiltonian constraint yields a PDE for $U(X,Y)$, where $X := \phi$, $Y := B_iB^i = A_iA^i$ (cf. superpotential equation)

$$\frac{1}{2\alpha}\left(U_X - \xi(d+1)U + 2\xi Y U_Y\right)^2 + Z_\xi^{-1}(X)YU_Y^2$$
$$-\frac{1}{2d}\left((d+1)U + 2(d-1)YU_Y\right)\left(U - 2YU_Y\right) = \frac{1}{2}e^{2d\xi X}\left(W_\xi(X)Y + V_\xi(X)\right)$$

- This equation for the ‘superpotential’ $U(X,Y)$ determines the zero derivative solution of the Hamilton-Jacobi equation: It can be used to holographically renormalize any homogeneous background of the equations of motion and any exact solution of this PDE leads to exact solutions of the equations of motion via the flow equations.

- Imposing Lifshitz boundary conditions requires that asymptotically the gauge invariant vector field behaves as

$$B_i \sim B_{oi} = \sqrt{\frac{z-1}{2\epsilon}} Z_\xi^{-1/2}(\phi)\mathbb{n}_i$$

where $\mathbb{n}_i$ is the unit normal to the constant t surfaces

- This in turn implies that the superpotential $U(X,Y)$ must satisfy

$$U(X, Y_o(X)) \sim e^{d\xi X}\left(d(1+\mu\xi)+z-1\right)$$
$$U_Y(X, Y_o(X)) \sim -\epsilon e^{d\xi X} Z_\xi(X)$$
$$U_X(X, Y_o(X)) \sim e^{d\xi X}\left(-\mu\alpha_\xi + d\xi(d+z)\right)$$

- Hence, the asymptotic form of the zero order solution of the HJ equation is

$$\mathcal{S}_{(0)} \sim \frac{1}{\kappa^2}\int_{\Sigma_r} d^{d+1}x\sqrt{-\gamma}e^{d\xi\phi}\left(d(1+\mu\xi)+\frac{1}{2}(z-1)-\epsilon Z_\xi(\phi)B_iB^i\right)$$

Taylor expansion of the superpotential

- Since Lifshitz boundary conditions require that $B_i \sim B_{oi}$ asymptotically, the solution of the HJ equation can be expressed as a Taylor series in $B_i - B_{oi}$
- The zero derivative solution $\mathcal{S}_{(0)}$ can be Taylor expanded in

$$Y - Y_o = 2B_o^i(B_i - B_{oi}) + \mathcal{O}(B - B_o)^2$$

where $Y_o \equiv B_o^i B_{oi}$, as

$$U = e^{(d+1)\xi\phi}\left(u_0(\phi) + Y_o^{-1}u_1(\phi)(Y - Y_o(\phi)) + Y_o^{-2}u_2(\phi)(Y - Y_o(\phi))^2 + \cdots\right)$$

- Inserting this expansion in the superpotential equation for $U(X, Y)$ leads to a tower of equations for the functions $u_n(\phi)$

- An additional relation between the functions $u_0(\phi)$ and $u_1(\phi)$ is imposed by the consistency of the Taylor expansion ("no-tadpole" condition), i.e. requiring that

$$\dot{Y} - \dot{Y}_o = \mathcal{O}(Y - Y_o)$$

- In a bottom up approach these equations can be used to *define* the potentials $V(\phi)$, $Z(\phi)$ and $W(\phi)$ in terms of $u_0(\phi)$ and $u_1(\phi)$, with all $u_n(\phi)$ for $n \geq 2$ being determined in terms of these functions.

- Lifshitz boundary conditions require

$$\boxed{\begin{aligned} u_0(\phi) &\sim (z - 1 + d(1 + \mu\xi))\, e^{-\xi\phi} \\ u_1(\phi) &\sim \frac{1}{2}(z-1)e^{-\xi\phi} \end{aligned}}$$

- The function $u_2(\phi)$ satisfies a quadratic (Riccati) equation and determines the scaling behavior of the independent mode $Y - Y_o$, while $u_n(\phi)$ with $n \geq 3$ satisfy linear equations.

Recursive solution of the HJ equation

- To summarize the above analysis, we have shown that the most general zero derivative solution of the HJ equation takes the form

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2}\int_{\Sigma_r} d^{d+1}x\sqrt{-\gamma}U(\phi, B^2)$$

  where for Lifshitz boundary conditions the superpotential $U(X,Y)$ admits a Taylor expansion in $Y - Y_o$. Moreover, this zero derivative solution is the asymptotically leading one, with derivative terms entering only in asymptotically subleading orders.

- In order to systematically determine these asymptotically subleading derivative terms of the solution of the HJ equation, we expand $\mathcal{S}$ in a covariant expansion in eigenfunctions of a suitable operator.

- For backgrounds with asymptotic scaling invariance one can use the dilatation operator [I. P. & Skenderis 2004] but in the presence of an asymptotically running dilaton, meaning that asymptotic scale invariance is broken, this is not sufficient.

- Instead we need an operator such that $\mathcal{S}_{(0)}$ is an eigenfunction for any superpotential $U(\phi, B^2)$.

- In fact there are two mutually commuting such operators:

$$\widehat{\delta} := \int d^{d+1}x \left(2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}} + B_i \frac{\delta}{\delta B_i} \right), \quad \delta_B := \int d^{d+1}x \left(2Y^{-1} B_i B_j \frac{\delta}{\delta\gamma_{ij}} + B_i \frac{\delta}{\delta B_i} \right)$$

which satisfy

$$\widehat{\delta}\mathcal{S}_{(0)} = (d+1)\mathcal{S}_{(0)}, \quad \delta_B \mathcal{S}_{(0)} = \mathcal{S}_{(0)}, \quad [\widehat{\delta}, \delta_B] = 0$$

- This allows us to seek a solution of the HJ equation in the form of a *graded* covariant expansion in simultaneous eigenfunctions of both $\widehat{\delta}$ and δ_B:

$$\mathcal{S} = \sum_{k=0}^{\infty} \mathcal{S}_{(2k)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \mathcal{S}_{(2k,2\ell)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \int d^{d+1}x \mathcal{L}_{(2k,2\ell)}$$

where

$$\widehat{\delta}\mathcal{S}_{(2k,2\ell)} = (d+1-2k)\mathcal{S}_{(2k,2\ell)}, \quad \delta_B \mathcal{S}_{(2k,2\ell)} = (1-2\ell)\mathcal{S}_{(2k,2\ell)}, \quad 0 \leq \ell \leq k$$

- The operator $\widehat{\delta}$ counts derivatives

- The operator δ_B annihilates the projection operator $\sigma^i_j := \delta^i_j - Y^{-1}B^i B_j$ and counts derivatives contracted with B_i, which asymptotically become time derivatives since $B_i \sim B_{oi} \propto \mathbb{n}_i$

- Inserting the covariant expansion of $\mathcal{S}$ in simultaneous eigenfunctions of $\widehat{\delta}$ and δ_B in the Hamilton-Jacobi equation (Hamiltonian constraint) results in a system of recursive first order functional *linear* equations for the higher derivative terms:

$$\frac{1}{\alpha}\left(U_X-(d+1)\xi U+2\xi Y U_Y\right)\frac{\delta}{\delta\phi}\int\mathcal{L}_{(2k,2\ell)}+$$
$$\left((2Y+Z_\xi^{-1})U_Y+\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)\right)B_i\frac{\delta}{\delta B_i}\int\mathcal{L}_{(2k,2\ell)}-$$
$$\left(\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)(d+1-2k)+2YU_Y(1-2\ell)\right)\mathcal{L}_{(2k,2\ell)}=$$
$$e^{d\xi\phi}\mathcal{R}_{(2k,2\ell)}$$

- The inhomogeneous term $\mathcal{R}_{(2k,2\ell)}$ involves derivatives of lower order terms as well as the 2-derivative sources from the Hamiltonian constraint

Lifshitz boundary conditions

- The covariant expansion of $\mathcal{S}$ in simultaneous eigenfunctions of $\widehat{\delta}$ and δ_B, and hence the above recursion relations, is independent of the specific choice of boundary conditions

- In order to impose Lifshitz boundary conditions we must additionally expand $\mathcal{S}_{(2k,2\ell)}$ in $B_i - B_{oi}$ at each order of the covariant expansion as

$$\mathcal{L}_{(2k,2\ell)} = \mathcal{L}^{0}_{(2k,2\ell)}[\gamma(x),\phi(x)] + \int d^{d+1}x'(B_i(x')-B_{oi}(x'))\mathcal{L}^{1i}_{(2k,2\ell)}[\gamma(x),\phi(x);x'] + \mathcal{O}\left(B-B_o\right)^2$$

- Inserting this Taylor expansion in the above recursion relations eliminates the derivative with respect to B_i, resulting in first order linear functional differential equations in ϕ only. Such functional differential equations appear in the relativistic case as well, e.g. for non-conformal branes or Improved Holographic QCD, and they can be solved systematically [I.P. '11].

Solution of the recursion relations up to $\mathcal{O}(B - B_o)$

- The inhomogeneous solution of these linear functional differential equations takes the form

$$\mathcal{L}^0_{(2k,2\ell)} = e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} \mathcal{R}^0_{(2k,2\ell)},$$

$$\sigma^i_j \mathcal{L}^{1j}_{(2k,2\ell)} = Z_\xi^{\frac{1}{2}} e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} Z_\xi^{-\frac{1}{2}} \sigma^i_j \mathcal{R}^{1j}_{(2k,2\ell)},$$

$$B_{oj}(x)\mathcal{L}^{1j}_{(2k,2\ell)} = \Omega^{-1} e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} \Omega B_{oj} \widehat{\mathcal{R}}^{1j}_{(2k,2\ell)}$$

where $\mathcal{C}_{k,\ell} := d + 1 - 2k + (z-1)(1-2\ell)$,

$$\mathcal{K}(\phi) := \frac{\alpha}{e^{\xi\phi}\left(u_0' + \frac{Z'}{Z}u_1\right)} \sim -\frac{1}{\mu}, \quad e^{\mathcal{A}(\phi)} = Z_\xi^{-\frac{1}{2(\epsilon - z)}} \sim e^{\phi/\mu}$$

and the $\Omega(\phi)$ can be expressed in terms of u_0, u_1 and u_2.

- If $\mu = 0$ (e.g. for Einstein-Proca theory) the corresponding solutions can be expressed *algebraically* in terms of the source terms.

Structure of the HJ solution & the holographic dictionary

Structure of the HJ solution

- The solution of the HJ equation obtained via the above algorithm is of the form

$$\mathcal{S} = \sum_{k,\ell,m \,|\, \mathcal{C}_{k,\ell}+\theta-m\Delta_-\geq 0} \int \cdots \int (B-B_o)^m \mathcal{S}^m_{(2k,2\ell)} + \widehat{\mathcal{S}}_{ren} + \cdots$$

  where $\Delta_+ = d+z-\theta-\Delta_-$ (with Δ_- a complicated function of the parameters) is the dimension of the scalar operator dual to the **composite** mode

$$\psi := Y_o^{-1} B_o^j (B_j - B_{oj})$$

- $(B-B_o)^m \mathcal{S}^m_{(2k,2\ell)}$ has dilatation weight $\mathcal{C}_{k,\ell}+\theta-m\Delta_-$, and $\widehat{\mathcal{S}}_{ren}$ has weight 0.
- All terms $(B-B_o)^m \mathcal{S}^m_{(2k,2\ell)}$ for $\mathcal{C}_{k,\ell}+\theta-m\Delta_- \geq 0$ are recursively determined.
- For $\mathcal{C}_{k,\ell}+\theta-m\Delta_- < 0$ these terms are powerlike divergent in the UV, while terms with $\mathcal{C}_{k,\ell}+\theta-m\Delta_- = 0$ have a pole which via dimensional regularization leads to a logarithmic divergence.
- Such logarithmically divergent terms give rise to the conformal anomaly when $\mu = 0$, but they can be absorbed in the dilaton when $\mu \neq 0$.

- The local covariant counterterms that render the on-shell action finite and the variational problem with Lifshitz boundary conditions well posed are

$$\mathcal{S}_{ct} := -\sum_{k,\ell,m \mid \mathcal{C}_{k,\ell}+d\mu\xi-m\Delta_-\geq 0} \int \cdots \int (B-B_o)^m \mathcal{S}^m_{(2k,2\ell)}$$

- The renormalized part of the on-shell action is therefore given by the UV-finite term $\widehat{\mathcal{S}}_{ren}$, which corresponds to an independent contribution to the HJ solution and can be parameterized as

$$\widehat{\mathcal{S}}_{ren} = \int d^{d+1}x \left(\gamma_{ij}\widehat{\pi}^{ij} + B_i\widehat{\pi}^i + \phi\widehat{\pi}_\phi\right)$$

where $\widehat{\pi}^{ij}$, $\widehat{\pi}^i$ and $\widehat{\pi}_\phi$ are undetermined integration functions of the HJ equation.

Asymptotic expansions and symplectic variables

- The non-linear asymptotic solutions of the fields, with all modes parameterizing the symplectic space of such solutions, is obtained by inserting the asymptotic solution of the HJ equation in the first order flow equations.
- The sources modes correspond to integration constants of the flow equations, while the one-point functions are related to the integration constants of the HJ solution in $\widehat{\mathcal{S}}_{ren}$.
- In order to identify the modes that parameterize the space of asymptotic solutions it is necessary to decompose the induced fields on Σ_r as

$$\gamma_{ij}dx^i dx^j = -(n^2 - n_a n^a)dt^2 + 2n_a dt dx^a + \sigma_{ab}dx^a dx^b, \quad A_i dx^i = adt + A_a dx^a,$$

where the indices a, b run from 1 to d.

- The sources appear as the leading modes in the expansions of n, n_a, σ_{ab}, ϕ and ψ, while the corresponding one-point functions are given by the combinations

$$\widehat{\mathcal{T}}^{ij} := -\frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\left(2\widehat{\pi}^{ij} + Y_o^{-1}B_o^i B_o^j B_{ok}\widehat{\pi}^k\right), \quad \widehat{\mathcal{E}}^i := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\sqrt{-Y_o}\varpi^i_j\widehat{\pi}^j$$

$$\widehat{\mathcal{O}}_\phi := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}\left(\widehat{\pi}_\phi + (\nu+\xi)B_{oi}\widehat{\pi}^i\right), \quad \widehat{\mathcal{O}}_\psi := \frac{e^{-d\xi\phi}}{\sqrt{-\gamma}}B_{oi}\widehat{\pi}^i$$

	one-point function	source
spatial stress tensor	$\widehat{\Pi}^i_j := \mathbb{o}^i_k \mathbb{o}_{jl} \mathcal{T}^{kl} \sim e^{-(d+z-\theta)r} \Pi^i_j(x)$	$\sigma_{(0)ab}$
momentum density	$\widehat{\mathcal{P}}^i := -\mathbb{o}^i_k \mathbb{n}_l \mathcal{T}^{kl} \sim e^{-(d+2-\theta)r} \mathcal{P}^i(x)$	$n_{(0)a}$
energy density	$\widehat{\mathcal{E}} := -\mathbb{n}_k \mathbb{n}_l \mathcal{T}^{kl} \sim e^{-(d+z-\theta)r} \mathcal{E}(x)$	$n_{(0)}$
energy flux	$\widehat{\mathcal{E}}^i \sim e^{-(d+2z-\theta)r} \mathcal{E}^i(x)$	0
dilaton	$\widehat{\mathcal{O}}_\phi \sim e^{-(d+z+d\mu\xi)r} \mathcal{O}_\phi(x)$	$\phi_{(0)}$
composite scalar	$\widehat{\mathcal{O}}_\psi \sim e^{-\Delta_+ r} \mathcal{O}_\psi(x)$	ψ_-

- This agrees with the energy-momentum complex discussed in [Ross '09]
- Note that there is no U(1) current operator!

- The momentum constraint of the radial Hamiltonian formalism leads to the diffeomorphism Ward identities

$$\mathbb{D}_j\widehat{\Pi}^i_i + \mathbb{q}_j\widehat{\Pi}^j_i + \mathbb{n}^j D_j\widehat{\mathcal{P}}_i + \mathbb{K}\widehat{\mathcal{P}}_i + \mathbb{K}^i_i\widehat{\mathcal{P}}_j + \mathbb{n}_i\mathbb{q}_j\widehat{\mathcal{P}}^j - \widehat{\mathcal{E}}\mathbb{q}_i + \widehat{\mathcal{O}}_\phi\mathbb{D}_i\phi + \widehat{\mathcal{O}}_\psi\mathbb{D}_i\psi = 0$$

$$\mathbb{n}^i D_i\widehat{\mathcal{E}} + \mathbb{K}\widehat{\mathcal{E}} - \mathbb{K}^i_j\widehat{\Pi}^j_i + \mathbb{D}_i\widehat{\mathcal{E}}^i + \widehat{\mathcal{O}}_\phi\mathbb{n}^i D_i\phi = 0$$

$$\mathbb{D}_i\widehat{\mathcal{P}}^i + 2\mathbb{q}_i\widehat{\mathcal{P}}^i = 0$$

where $\mathbb{D}_i$ is the covariant derivative w.r.t. σ_{ij}, $\mathbb{K}_{ij} = \mathbb{D}_i\mathbb{n}_j$ is the extrinsic curvature of the constant time slices, and $\mathbb{q}_i = \mathbb{n}^k D_k\mathbb{n}_i$.

- The transformation of the renormalized action under *local* anisotropic boundary Weyl transformations leads to the trace Ward identity

$$z\widehat{\mathcal{E}} + \widehat{\Pi}^i_i + \Delta_-\psi\widehat{\mathcal{O}}_\psi - \mu\widehat{\mathcal{O}}_\phi = 0, \quad \mu \neq 0,$$
$$z\widehat{\mathcal{E}} + \widehat{\Pi}^i_i + \Delta_-\psi\widehat{\mathcal{O}}_\psi = \mathcal{A}, \quad \mu = 0,$$

where $\mathcal{A}$ is the conformal anomaly, corresponding to all terms satisfying $\mathcal{C}_{k,\ell} + \theta - m\Delta_- = 0$.

Outline

RG flows with Lifshitz or hvLf at the UV

- Non-relativistic RG flows with Lifshitz or hvLf at the UV can be obtained from the above action and take the form

$$ds^2 = dr^2 - e^{2f(r)}dt^2 + e^{2h(r)}\delta_{ab}dx^a dx^b, \quad A = a(r)dt, \quad \phi = \phi(r), \quad \omega = \omega(r)$$

- Such flows can be described in a first order formalism through the HJ ansatz

$$\mathcal{S}_{eff} = \frac{1}{\kappa^2} e^{f+dh} U(X,Y), \qquad X := \phi, \quad Y := -e^{-2f}a^2$$

- Inserting this in the HJ equation gives the 'superpotential' equation

$$\frac{1}{\alpha}\left(U_X - \xi(d+1)U + 2\xi Y U_Y\right)^2 - \frac{1}{d}(U - 2YU_Y)^2 - (U + 2YU_Y)(U - 2YU_Y)$$
$$+ 2Z_\xi^{-1} Y U_Y^2 = e^{2d\xi X}\left(W_\xi Y + V_\xi\right)$$

as well as the flow equations

$$\dot{f} = 2e^{-d\xi X}\left(YU_Y + \left(\frac{\alpha_\xi}{2d\alpha}U + \frac{\xi}{2\alpha}U_X - \frac{\alpha_\xi + d^2\xi^2}{d\alpha}YU_Y\right)\right)$$
$$\dot{h} = 2e^{-d\xi X}\left(\frac{\alpha_\xi}{2d\alpha}U + \frac{\xi}{2\alpha}U_X - \frac{\alpha_\xi + d^2\xi^2}{d\alpha}YU_Y\right)$$
$$\dot{a} = -e^{-d\xi X} Z_\xi^{-1}(X) U_Y a \quad \dot{\omega} = 0$$

- The ansatz

$$U(X,Y) = e^{d\xi X} w\left(Y Z_\xi(X)\right)$$

for some function $w(y)$ of $y \equiv Y Z_\xi(X)$ leads to the ODE

$$(\alpha_2 y + \alpha_1) y w'^2 + \beta y w w' + \gamma w^2 = \delta y + \varepsilon$$

where α_1, α_2, β, γ, δ and ε depend of the parameters of the model.

- The ansatz

$$U(X,Y) = \varepsilon_0 e^{d\xi X} \sqrt{\varepsilon_1 e^{2\xi X} u^2(X) + \varepsilon_2 v^2(X) Y}$$

where $\varepsilon_{0,1,2} = \pm 1$ are independent signs, leads to a set of equations for $u(X)$ and $v(X)$ that can be solved exactly.

RG flows with Lifshitz or hvLf at the IR

- Non-relativistic RG flows with AdS in the UV can be obtained from the action

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^{d+1}\mathbf{x}\sqrt{-g}\left(R[g] - (\partial\phi)^2 - Z(\phi)(\partial\chi)^2 - V(\phi,\chi) - \Sigma(\phi)F^2\right)$$

- Such flows take the form

$$ds_B^2 = dr^2 + e^{2A(r)}\left(-f(r)dt^2 + dx^2 + dy^2\right)$$
$$A_B = \alpha(r)dt + \frac{H}{2}(xdy - ydx)$$
$$\phi_B = \phi_B(r) \qquad \chi_B = \chi_B(r)$$

- They are in general **dyonic**, **finite density** and **finite temperature** solutions.

Superpotential and flow equations

- These RG flows can be described in terms of a 'superpotential' via the HJ ansatz

$$\mathcal{S} = -\frac{1}{\kappa^2}\int \mathrm{d}^d\mathbf{x}\left(e^{dA}f^{1/2}W(A,\phi,\chi) + 2Q\alpha\right)$$

- Inserting this in the HJ equation gives the superpotential equation

$$W_\phi^2 + Z^{-1}(\phi)W_\chi^2 - \frac{1}{d-1}\left(d+\partial_A\right)W^2 = V_{eff}(A,\phi,\chi)$$

$$V_{eff}(A,\phi,\chi) \equiv V(\phi,\chi) + 2\Sigma(\phi)e^{-4A}H^2 + 2\Sigma^{-1}(\phi)e^{-2(d-1)A}Q^2$$

- The radial profile of the fields is determined by the first order flow equations

$$\dot{A} = -\frac{1}{d-1}W, \qquad \frac{\dot{f}}{f} = -\frac{2}{(d-1)}W_A, \qquad \dot{\phi} = W_\phi, \qquad \dot{\chi} = Z^{-1}W_\chi$$
$$\dot{\alpha} = -\Sigma^{-1}e^{-(d-2)A}f^{1/2}Q$$

- A class of exact solutions is obtained by the separable ansatz

$$W(A,\phi) = W_o(\phi)\sqrt{1+q^2e^{-4A}}$$

where $W_o(\phi)$ is arbitrary and the parameter q is defined through

$$H^2\Sigma_0 + Q^2\Sigma_0^{-1} = q^2L^{-2}$$

- The scalar potentials $V(\phi)$ and $\Sigma(\phi)$ are determined in terms of $W_o(\phi)$ as

$$V(\phi) = W_o'^2 - \frac{3}{2}W_o^2$$
$$H^2\Sigma(\phi) + Q^2\Sigma^{-1}(\phi) = \frac{q^2}{2}\left(W_o'^2 + \frac{1}{2}W_o^2\right)$$

- Depending on the choice of $W_o(\phi)$, these solutions flow to a hyperscaling violating Lifshitz background in the IR, with

$$\theta = z+1, \quad 1 < z < 3$$

Embedding in $U(1)^4$ gauged supergravity

- Two purely magnetic cases (i.e. $Q = 0$, $H \neq 0$) can be embedded in the $U(1)^4$ gauge supergravity in 4D.

- $z = 2$, $\theta = 3$:

$$W_o(\phi) = -\frac{2}{L}\cosh(\phi/\sqrt{2})$$

$$V(\phi) = -\frac{2}{L^2}\left(2 + \cosh(\sqrt{2}\phi)\right), \quad \Sigma(\phi) = \Sigma_0 \cosh(\sqrt{2}\phi)$$

- $z = 3/2$, $\theta = 5/2$:

$$W_o(\phi) = -\frac{1}{2L}\left(3e^{-\phi/\sqrt{6}} + e^{\sqrt{\frac{3}{2}}\phi}\right)$$

$$V(\phi) = -\frac{6}{L^2}\cosh\left(\sqrt{\frac{2}{3}}\phi\right), \quad \Sigma(\phi) = \frac{1}{4}\Sigma_0\left(e^{\sqrt{6}\phi} + 3e^{-\sqrt{\frac{2}{3}}\phi}\right)$$

Outline

Summary

- The space of sources and local operators in QFT admits a symplectic structure, which arises naturally in the context of the Local Renormalization Group.
- A holographic dictionary amounts to identifying the symplectic space of bulk asymptotic solutions with the space of sources and operators in the dual QFT.
- Given a bulk theory, the symplectic space of asymptotic solutions can be constructed algorithmically using general techniques, which apply equally well to non-relativistic backgrounds.
- The same techniques allow one to describe any solution of the second order equations in terms of first order flow equations, which provides an efficient tool for a number of different approximations relevant in holography and cosmology.

Open questions

- Non-relativistic theories are not as well understood as their relativistic siblings, both on the field theory and on the holographic sides.
- On the field theory side, it would be instructive to know more about, e.g., **correlation functions** at strong coupling (e.g. Lifshitz symmetry is not sufficient to fix the two- and three-point functions), **quantum anomalies**, **hydrodynamics**, or to what extent **Newton-Cartan** geometry is general or necessary in the context of Lifshitz theories.
- On the bulk side, it would be interesting to see the effect of **higher derivatives** or **non-relativistic gravity** (e.g. Horava gravity) on the holographic dictionary, as well as to develop **real-time holography** and address the **initial value problem**.
- It would also be interesting to better understand the **UV completions** of such theories, e.g. through **string embeddings**.
- On a more technical level, it would be useful to clarify the connection between **null reductions** and **second class constraints**, as well as understanding the match or mismatch between Newton-Cartan geometry and the holographic dictionary obtained from bulk relativistic gravity in the metric formulation.
- Finally, as for the entire AdS/CMT program, it would be highly desirable to compare the holographic results to **experimental data**.