Non-Relativistic Holography, Newton–Cartan Geometry and Hydrodynamics

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Review of work done in various collaborations with:

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Introduction

- Non-AdS Holography:
  - Boundary not described by Lorentzian geometry
  - Examples: Lifshitz spaces (Newton–Cartan) & null Infinity of flat space (Carrollian)
  - Dual field theories are non-Lorentzian field theories
- non-Lorentzian field theories important for infrared effective descriptions of low energy physics (e.g. strange metals)
- New theories of gravity (e.g. Hořava–Lifshitz) but also novel 3D Chern–Simons theories based on non-Lorentzian symmetries

Terminology

- non-Lorentzian symmetry: any symmetry group, not Poincaré, that contains at least H (time translations), P_i (space translations) and J_{ij} (spatial rotations)
- Aristotelian (absolute rest) symmetries: H, P_i, J_{ij}
- May contain more symmetries like D (dilatations: $t \to \lambda^z t$ and $x^i \to \lambda x^i$) and boosts
- non-Lorentzian boosts:
  - G_i (Galilean boost: $t \to t$ and $x^i \to x^i + v^i t$)
  - C_i (Carrollian boost: $t \to t + \bar{v}^i x^i$ and $x^i \to x^i$)
- non-Lorentzian geometry: geometry obtained by gauging a non-Lorentzian symmetry

Important Algebras

- Galilei: H, P_i, G_i, J_{ij} $[H, G_i] = P_i$, $[P_i, G_j] = 0$
  $i = 1, \ldots, d$, contraction of Poincaré for $c \to \infty$
- Bargmann: Gal. + central N (mass), $[P_i, G_j] = N\delta_{ij}$
  subgroup of Poincaré in one dimension higher
  (commutant of null momentum), not a contraction
- Schrödinger: Barg. + dilatations (any z)
  enhancement for $z = 2$: $SL(2, \mathbb{R})$ subgroup: H, D, K
  (K =spec. conf.) subgroup of conf. group in one dimension higher
- Galilei conformal algebra: H, D ($z = 1$), K, P_i, G_i, K_i
  (K_i =accelerations), J_{ij}
  contraction of conformal algebra for $c \to \infty$

Important Algebras

- Schrödinger and Galilean conformal algebras contain infinite dimensional extensions in any dimension.
- Carroll: H, P_i, C_i, J_{ij}, $\qquad [P_i, C_j] = H\delta_{ij}$ (H central) contraction of Poincaré for $c \to 0$
- Lifshitz Carroll: Carroll+dilatations (any z)
- Lifshitz: H, P_i, J_{ij} + dilatations (any z)
- Galilei and Carroll are isomorphic in 1+1D.
- The finite/infinite dimensional Galilean conformal algebra in 1+1D is isomorphic to the Poincaré(2,1)/BMS$_3$ algebra.

Questions

- One of the main goals: Understand the landscape of non-Lorentzian field theories and derive the equations for their hydrodynamic limit
- What geometry do these non-Lorentzian field theories couple to?
- Which spaces have a non-Lorentzian boundary geometry?
- What theory of gravity arises when making non-Lorentzian geometries dynamical?
- Can we use this non-Lorentzian gravity as a bulk theory in holography?

Outline Talk

- Newton–Cartan Geometry
- Field theories on TNC backgrounds
- Lifshitz Scalar Models
- Carrollian Geometry and Field Theory
- Hydrodynamics
- Lifshitz Holography
- Chern–Simons Theories
- Summary/Outlook

Part I:

Newton–Cartan Geometry

Equivalence Principles

- Lorentzian geometry can be obtained by gauging the Poincaré algebra, replacing local translations by diffeomorphisms.
- Einstein's equivalence principle: locally a manifold is Minkowski space-time.
- Newton–Cartan geometry is a manifold such that locally space-time is flat in the sense of Galilei's principle of relativity and can be obtained by gauging the Bargmann algebra: H, P_a, G_a, J_{ab}, N.
- NC geometry (with torsion) is the natural geometric framework for HL gravity.

From Poincaré to GR

- Local Poincaré: P_a, M_{ab} (gauging), $a = 0, 1 \ldots, d$:

$$\mathcal{A}_\mu = P_a e^a_\mu + \frac{1}{2} M_{ab} \omega_\mu{}^{ab}$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu \,, \mathcal{A}_\nu] = P_a R_{\mu\nu}{}^a(P) + \frac{1}{2} M_{ab} R_{\mu\nu}{}^{ab}(M)$$

$$\delta \mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu \,, \Lambda] \,, \qquad \Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma \,, \qquad \Sigma = \frac{1}{2} M_{ab} \lambda^{ab}$$

$$\bar{\delta} \mathcal{A}_\mu = \delta \mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu \,, \Sigma]$$

- ∇_μ defined via VP : $\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho - \omega_\mu{}^a{}_b e^b_\nu = 0$
- Lorentz invariant $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. Affine $\Gamma^\rho_{\mu\nu}$: $\nabla_\mu g_{\nu\rho} = 0$.
- $R_{\mu\nu}{}^a(P) = 2\Gamma^\rho_{[\mu\nu]} =$ torsion
- $R_{\mu\nu}{}^{ab}(M) =$ Riemann curvature 2-form

Gauging Bargmann

- Gauging Bargmann [Andringa, Bergshoeff, Panda, de Roo, 2011]
  H, P_a, G_a, J_{ab}, N (a is a spatial index):

$$\mathcal{A}_\mu = H\tau_\mu + P_a e^a_\mu + G_a \Omega_\mu{}^a + \frac{1}{2} J_{ab} \Omega_\mu{}^{ab} + N m_\mu$$

$$\mathcal{F}_{\mu\nu} = H R_{\mu\nu}(H) + P_a R_{\mu\nu}{}^a(P) + G_a R_{\mu\nu}{}^a(G) + \frac{1}{2} J_{ab} R_{\mu\nu}{}^{ab}(J) + N R_{\mu\nu}(N)$$

$$\delta \mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu\,, \Lambda]\,, \quad \Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma\,, \quad \Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab} + N\sigma$$

$$\bar{\delta} \mathcal{A}_\mu = \delta \mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu\,, \Sigma]$$

- Vielbein postulates (introduction of $\Gamma^\rho_{\mu\nu}$):

$$\mathcal{D}_\mu \tau_\nu = \partial_\mu \tau_\nu - \Gamma^\rho_{\mu\nu} \tau_\rho = 0$$

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho - \Omega_\mu{}^a \tau_\nu - \Omega_\mu{}^a{}_b e^b_\nu = 0$$

Gauging Bargmann

- Transformations of τ_μ, e^a_μ and m_μ:

$$\bar\delta\tau_\mu = \mathcal{L}_\xi\tau_\mu\,, \quad \bar\delta e^a_\mu = \mathcal{L}_\xi e^a_\mu + \lambda^a\tau_\mu + \lambda^a{}_b e^b_\mu\,, \quad \bar\delta m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu\sigma + \lambda_a e^a_\mu$$

- Inverse vielbeins: v^μ and e^μ_a via:

$$v^\mu\tau_\mu = -1\,, \qquad v^\mu e^a_\mu = 0\,, \qquad e^\mu_a\tau_\mu = 0\,, \qquad e^\mu_a e^b_\mu = \delta^b_a$$

- Metric: $h^{\mu\nu} = \delta^{ab}e^\mu_a e^\nu_b$ and τ_μ
- $\Gamma^\rho_{\mu\nu}$ is affine and inert under G, J
- $\Omega_\mu{}^{ab} = \Omega_\mu{}^{[ab]}$ so that $\nabla_\mu h^{\nu\rho} = 0$. Also $\nabla_\mu\tau_\nu = 0$
- Torsion: $2\Gamma^\rho_{[\mu\nu]} = -v^\rho R_{\mu\nu}(H) + e^\rho_a R_{\mu\nu}{}^a(P)$
- Curvature: $[\nabla_\mu\,, \nabla_\nu]X_\sigma = R_{\mu\nu\sigma}{}^\rho X_\rho - 2\Gamma^\rho_{[\mu\nu]}\nabla_\rho X_\sigma$
- where $R_{\mu\nu\sigma}{}^\rho = e^\rho_a\tau_\sigma R_{\mu\nu}{}^a(G) - e_{\sigma a}e^\rho_b R_{\mu\nu}{}^{ab}(J)$

Affine Connection

- In GR one sets torsion to zero by hand because applying the Noether procedure to the gauging of Poincaré does not require torsion. Here a similar argument leads to [Festuccia, Hansen, JH, Obers, 2016]

$$\Gamma^{\rho}_{\mu\nu} = -\hat{v}^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}\bar{h}_{\nu\sigma} + \partial_{\nu}\bar{h}_{\mu\sigma} - \partial_{\sigma}\bar{h}_{\mu\nu}\right)$$

  where $\hat{v}^{\mu} = v^{\mu} - h^{\mu\nu}m_{\nu}$ and $\bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_{\mu}m_{\nu} - \tau_{\nu}m_{\mu}$ are G and J invariant.

- Torsion: $2\hat{\Gamma}^{\rho}_{[\mu\nu]} = -\hat{v}^{\rho}\left(\partial_{\mu}\tau_{\nu} - \partial_{\nu}\tau_{\mu}\right)$. Three cases :
  - No torsion: $\partial_{\mu}\tau_{\nu} - \partial_{\nu}\tau_{\mu} = 0$ (NC geometry)
  - Twistless torsion: $\partial_{\mu}\tau_{\nu} - \partial_{\nu}\tau_{\mu} = a_{\mu}\tau_{\nu} - a_{\nu}\tau_{\mu}$ (TTNC)
  - No constraint on τ_{μ} (TNC geometry)

ADM Decomposition

- Local G invariant vielbeins: τ_μ, $\hat{e}_\mu^a = e_\mu^a - \tau_\mu e^{\nu a} m_\nu$ and inverses: $\hat{v}^\mu$ and e_a^μ.
- Lorentzian metric: $g_{\mu\nu} = -\tau_\mu \tau_\nu + \hat{h}_{\mu\nu}$, $\hat{h}_{\mu\nu} = \delta_{ab} \hat{e}_\mu^a \hat{e}_\nu^b$
- $\hat{v}^\mu = g^{\mu\nu} \tau_\nu$ and $e_a^\mu = g^{\mu\nu} \hat{e}_{\nu a}$
- ADM: $ds^2 = -N^2 dt^2 + \gamma_{ij} \left(dx^i + N^i dt \right) \left(dx^j + N^j dt \right)$
- TTNC $\tau_\mu = \psi \partial_\mu \tau$ (τ is Khronon field of [Blas, Pujolas, Sibiryakov, 2010])
- Fix foliation $\tau = t$ this implies

$$\tau_t = N \,, \qquad \hat{h}_{ti} = \gamma_{ij} N^j \,, \qquad \hat{h}_{ij} = \gamma_{ij} \,, \qquad m_i = -N^{-1} \gamma_{ij} N^j$$

ADM Decomposition

- Since $\tau_t = N$ it follows that
  - NC: $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0$ is equivalent to $N = N(t)$: projectable HL gravity
  - TTNC: $N = N(t, x)$: non-projectable HL gravity, extra field (torsion) $a_i = N^{-1}\partial_i N$
- ADM decomposition becomes dynamical and is described by τ_μ (lapse), m_μ (shift) and $\hat{h}_{\mu\nu}$ (spatial metric on cst time slices).
- Actually $m_t = -\frac{1}{2N}\gamma_{ij}N^i N^j + N\tilde{\Phi}$ is an additional field (denoted by A in [Horava, Melby-Thompson, 2010]) where $\tilde{\Phi} = -v^\mu m_\mu + \frac{1}{2}h^{\mu\nu}m_\mu m_\nu$ is G, J invariant.

Part II:
Field theories on TNC backgrounds

Fixed TNC Backgrounds

- The TNC geometry is described by τ_μ, e_μ^a (or v^μ, e_a^μ) and m_μ such that an action is invariant under local tangent space G, J, N transformations. To achieve J invariance use $h^{\mu\nu} = \delta^{ab} e_a^\mu e_b^\nu$. Sources are:

$$S = S[v^\mu, h^{\mu\nu}, m_\mu]$$

- Its variation is with $e = \det(\tau_\mu, e_\mu^a)$

$$\delta S = \int d^{d+1}x e \left(-T_\mu \delta v^\mu + \frac{1}{2} T_{\mu\nu} \delta h^{\mu\nu} + J^\mu \delta m_\mu \right)$$

- G invariance: $T_\mu e_a^\mu = J^\mu e_{\mu a}$ (mom. density=mass flux)
- N invariance: $\partial_\mu J^\mu = 0$ (mass conservation)

Energy-Momentum Tensor

- The EMT is $\mathcal{T}^{\mu}{}_{\nu} = -v^{\mu} T_{\nu} + h^{\mu\rho} T_{\rho\nu}$
  - $\mathcal{T}^{t}{}_{t}$ is the energy density
  - $\mathcal{T}^{i}{}_{t}$ is the energy flux
  - $\mathcal{T}^{t}{}_{i} = J^{i}$ is the momentum density
  - $\mathcal{T}^{j}{}_{i} = T^{j}{}_{i}$ is the stress
  - J^{t} is the mass density
- Diffeo and scale Ward identities on flat TNC space-time

$$\partial_\mu \mathcal{T}^{\mu}{}_{\nu} = 0 \ \ (\text{energy-momentum conservation}), \qquad z\mathcal{T}^{t}{}_{t} + \mathcal{T}^{i}{}_{i} = 0$$

- Making G inv. manifest: $S = S[\hat{v}^{\mu}, h^{\mu\nu}, \tilde{\Phi}]$ and $T^{\mu}{}_{\nu} = \mathcal{T}^{\mu}{}_{\nu} - J^{\mu} m_{\nu}$ but loose manifest N inv.

Example: Schrödinger Model

- Action for Schrödinger equation on a TNC background:

$$S = \int d^{d+1}x e \left[-i\phi^* \hat{v}^\mu \partial_\mu \phi + i\phi \hat{v}^\mu \partial_\mu \phi^* - h^{\mu\nu}\partial_\mu \phi \partial_\nu \phi^* - 2\phi\phi^* \Phi_N - V(\phi\phi^*)\right]$$

- On a flat NC background this becomes:

$$\begin{aligned} S &= \int d^{d+1}x \left[i\phi^* \left(\partial_t \phi + i\phi\partial_t M\right) - i\phi \left(\partial_t \phi^* - i\phi^* \partial_t M\right)\right. \\ & \left. -\delta^{ij}\left(\partial_i \phi + i\phi\partial_i M\right)\left(\partial_j \phi^* - i\phi^* \partial_j M\right) - V(\phi\phi^*)\right] \end{aligned}$$

- Wavefunction ψ defined as $\phi = e^{-iM}\psi$.
- Space-time symmetries for $M = \text{cst}$ is the Lifshitz subalgebra of Sch given by H, D, P_i and J_{ij}.
  Space-time symmetries for $M = x^i x^i/2t$ is the Lifshitz subalgebra of Sch given by K, D, G_i and J_{ij}.

Particle Number

- What about particle number?

$$S = \int d^{d+1}x e \left[-i\phi^* \hat{v}^\mu \partial_\mu \phi + i\phi \hat{v}^\mu \partial_\mu \phi^* - h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* - 2\phi\phi^* \Phi_N - V(\phi\phi^*) \right]$$

  can be rewritten as

$$S = \int d^{d+1}x e \left[-i\phi^* v^\mu D_\mu \phi + i\phi v^\mu D_\mu \phi^* - h^{\mu\nu} D_\mu \phi D_\nu \phi^* - V(\phi\phi^*) \right]$$

  where $D_\mu \phi = \partial_\mu \phi + i m_\mu \phi$. There is a local symmetry $\phi \to e^{-i\sigma}\phi$ and $m_\mu \to m_\mu + \partial_\mu \sigma$.

- Particle number corresponds to a global phase rotation of ϕ and is not a space-time symmetry of the TNC background.

Discussion

- Null reductions turn a Lorentzian space-time into a TNC geometry.
- Other examples of field theories on TNC backgrounds are null reductions of Maxwell, known as Galilean electrodynamics (massless) [Festuccia, Hansen, JH, Obers, 2016].
- Massless Galilean theories: momentum density is zero.
- By starting with a Galilean or Bargmann invariant field theory on flat space we can obtain TNC geometry by the Noether procedure [Festuccia, Hansen, JH, Obers, 2016].
- Typically first order in time derivatives.

Part III:

Lifshitz Scalar Models

Lifshitz Scalar Models

- Higher derivative single real scalar model

$$\mathcal{L} = \frac{1}{2}(\partial_t\theta)^2 - \frac{\lambda}{2}(\partial_i\partial_i)^n\theta(\partial_i\partial_i)^n\theta\,, \qquad z = 2n$$

- Shift symmetry: $\theta \to \theta + c$
  Scaling dimension: $[\theta] = (d - z)/2$
  Global symmetries: $\mathbb{R} \times$ Lif
  Hamiltonian is bounded from below
  Time reversal invariance
- For $z < d$ we can add a potential $V = V(\theta)$. For $z \geq d$ we keep the shift symmetry for otherwise we can add arbitrary high powers of θ.

Lifshitz Scalar Models

- Lifshitz is a subgroup of Schrödinger so the Schrödinger model is a Lifshitz scalar theory

$$\mathcal{L} = i\phi^\star\partial_t\phi - i\phi\partial_t\phi^\star - \partial_i\phi\partial_i\phi^\star - V_0\left(\phi\phi^\star\right)^{(d+2)/d}$$

$$\phi = \frac{1}{\sqrt{2}}\varphi e^{i\theta}\,, \qquad \mathcal{L} = -\varphi^2\left[\partial_t\theta + \frac{1}{2}\partial_i\theta\partial_i\theta\right] - \frac{1}{2}\partial_i\varphi\partial_i\varphi - V_0\varphi^{2(d+2)/d}$$

- Shift symmetry: $\theta \to \theta + c$

  Scaling dimension: $[\varphi] = d/2$, $[\theta] = 0$

  Global symmetries: Sch($z = 2$) $\supset U(1) \times$ Lif($z = 2$)

  Hamiltonian is bounded from below

  No time reversal invariance

Lifshitz Scalar Models

$$\mathcal{L} = -\varphi^2 \left[\partial_t\theta + \frac{1}{2}\partial_i\theta\partial_i\theta\right] - \frac{1}{2}\partial_i\varphi\partial_i\varphi - V_0\varphi^{2(d+2)/d}$$

- Galilean boosts (generator G_i):
  $x^i = x'^i - v^i t'\,, \qquad t = t'\,, \qquad \theta = \theta' - v^i x'^i + \frac{1}{2}v^i v^i t'$
  Invariant: $\partial_t\theta + \frac{1}{2}\partial_i\theta\partial_i\theta$
  $[P_i, G_j] = N\delta_{ij}$: mass N (shift θ)
- At an algebraic level we can break
  Sch $\to$ Lif by breaking N
- Or we can break
  Sch $\to U(1) \times$ Lif by breaking G_i

Lifshitz Scalar Models

$$\mathcal{L} = -\varphi^\alpha \partial_t \theta - \frac{1}{2}\varphi^2 \partial_i \theta \partial_i \theta - \frac{1}{2}\partial_i \varphi \partial_i \varphi - V_0 \varphi^{\frac{2(d+z)}{d+z-2}}\,, \qquad \alpha = \frac{2d}{d+z-2}$$

- Shift symmetry: $\theta \to \theta + c$

  Scaling dimensions: $[\varphi] = (d+z-2)/2, \quad [\theta] = 0$

  Global symmetries: $U(1) \times \text{Lif}(z)$ for $z \neq 2$

  General $z > 1$ scaling without higher spatial derivatives

  For $z \neq 2$ Galilean boosts are broken

  Hamiltonian is bounded from below

  No time reversal invariance

$$\mathcal{L} = (\phi\phi^\star)^{(\alpha-2)/2}\left[i\phi^\star\partial_t\phi - i\phi\partial_t\phi^\star\right] - \partial_i\phi\partial_i\phi^\star - V_0\left(\phi\phi^\star\right)^{\frac{d+z}{d+z-2}}$$

Discussion

- When breaking G_i we preserve a $U(1)$ but we break N. So mass current (defined as response to varying m_μ) is not conserved, instead there is another $U(1)$ current.
- Coupling to TNC geometry requires extra source χ whose response is the right hand side of $\partial_\mu J^\mu$.
- Other models: $\mathcal{L} = -\varphi^{\frac{2(d-z+2)}{z+d-2}}\left[\partial_t\theta + \frac{1}{2}\partial_i\theta\partial_i\theta\right] - \frac{1}{2}\partial_i\varphi\partial_i\varphi - V_0\varphi^{\frac{2(d+z)}{z+d-2}}$ Has Schrödinger invariance for general z.
- These Lifshitz models have first order time derivatives. How to understand

$$\mathcal{L} = \frac{1}{2}\left(\partial_t\theta\right)^2 - \frac{\lambda}{2}(\partial_i\partial_i)^n\theta(\partial_i\partial_i)^n\theta\,, \qquad z = 2n$$

Part IV:
Carrollian Geometry and Field Theory

Carroll Symmetries

- Ultra-relativistic limit of Poincaré
- Lorentz boosts: $L_i = ct\partial_i + \frac{1}{c}x^i\partial_t$. Send $c \to 0$ gives $C_i = x^i\partial_t$. Finite trafo: $t \to t + \bar{v}^i x^i$ and $x^i \to x^i$.
- Light cone collapses to a line
- $[P_i, C_j] = H\delta_{ij}$ no massive generalizations in $D > 2$
- Simple Carroll boost invariant model: $\mathcal{L} = \frac{1}{2}\left(\partial_t\phi\right)^2 - V(\varphi)$
- Time-reversal invariant

Carrollian Geometry

- Gauging Carroll: $A_\mu = H\tau_\mu + P_a e^a_\mu + C_a \Omega_\mu{}^a + \frac{1}{2} J_{ab} \Omega_\mu{}^{ab}$

$$\bar{\delta}\mathcal{A}_\mu = \delta\mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu \,, \Sigma] \,, \qquad \Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab}$$

- Square matrix (τ_μ, e^a_μ) has inverse $(v^\mu . e^\mu_a)$
- Metric: v^μ and $h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu$
- Like in TNC case we can Stückelberg the boost symmetry. Here this requires introducing M^μ so that $\hat{\tau}_\mu = \tau_\mu - h_{\mu\nu} M^\nu$ and $\hat{e}^\mu_a = e^\mu_a - M^\nu e_{\nu a} v^\mu$ are invariant.
- Examples: any null hypersurface of a Lorentzian metric is Carrollian including future/past null infinity of asymptotically flat space-times [JH, 2015].

Discussion

- Field theory on a fixed Carrollian geometry:

$$\delta S = \int d^{d+1}x e \left(-T_\mu \delta v^\mu + \frac{1}{2} T_{\mu\nu} \delta h^{\mu\nu} \right)$$

- Energy-momentum tensor: $T^\mu{}_\nu = -v^\mu T_\nu + h^{\mu\rho} T_{\rho\nu}$
- Carroll boost symmetry: $T_{\mu\nu} v^\mu e^\nu_a = 0$ (zero energy flux)
- We can now understand:

$$\mathcal{L} = \frac{1}{2} (\partial_t \theta)^2 - \frac{\lambda}{2} (\partial_i \partial_i)^n \theta (\partial_i \partial_i)^n \theta \,, \qquad z = 2n$$

  as a Lifshitz theory with broken Carrollian boosts.
- When coupling to Carrollian geometry the boost breaking is controlled by the coupling to M^μ.

Part V:
Hydrodynamics

Relativistic Perfect Fluids

- Energy-momentum tensor:

$$T^{\mu}{}_{\nu} = (\mathcal{E} + P)\, U^{\mu} U_{\nu} + P \delta^{\mu}_{\nu}$$

- geometry: Lorentzian with metric $g_{\mu\nu}$
- Velocity: $U^{\mu} U_{\mu} = -1$
- Landau frame: $T^{\mu}{}_{\nu} U^{\nu} = -\mathcal{E} U^{\mu}$
- Ward identities: $\nabla_{\mu} T^{\mu\nu} = 0$ and $T^{[\mu\nu]} = 0$
- Scale symmetry: $T^{\mu}{}_{\mu} = 0$: $\mathcal{E} = dP$
- Equation of state: $P = P(\mathcal{E})$. 1st law: $\delta\mathcal{E} = T\delta s$
- Fluid variables: T and U^{μ}

Galilean Perfect Fluids

$$T^{\mu}{}_{\nu} = (\mathcal{E} + P)\, u^{\mu}\tau_{\nu} + P\delta^{\mu}_{\nu} + u^{\mu}P_{\nu}\,, \qquad J^{\mu} = -\rho u^{\mu}$$

- Particle 4-mom. $P_{\nu} = \frac{1}{2}\rho h_{\kappa\lambda}u^{\kappa}u^{\lambda}\tau_{\nu} + \rho\,(h_{\nu\kappa}u^{\kappa} + m_{\nu})$
- geometry: TNC with metric τ_{μ} and $h^{\mu\nu}$
- Velocity: $u^{\mu}\tau_{\mu} = -1$
- Landau frame: $T^{\mu}{}_{\nu}u^{\nu} = -\mathcal{E}u^{\mu}$
- Ward identities (flat space): $\partial_{\mu}T^{\mu\nu} = 0$, $\partial_{\mu}J^{\mu} = 0$, $T^{t}{}_{i} = J^{i}$ and $T^{i}{}_{j} = T^{j}{}_{i}$
- Scale symmetry: $zT^{t}{}_{t} + T^{i}{}_{i} = 0$: $z\mathcal{E} = dP$
- Equation of state: $P = P(\mathcal{E})$. 1st law: $\delta\mathcal{E} = T\delta s$
- Fluid variables: T, ρ and u^{μ}

Speed of Sound of Galilean perfect fluid

- Fluctuate around a constant background (perform G-boost: $\partial_{t'} = \partial_t + V_0^i \partial_i$ and $\partial'_i = \partial_i$)

$$\partial_{t'}^2 \delta\mathcal{E} - \frac{\mathcal{E}_0 + P_0}{\rho_0} \partial'_i \partial'_i \delta P = 0$$

- Assume equation of state $2\mathcal{E} = dP$

$$\partial_{t'}^2 \delta\mathcal{E} - \frac{2}{d} \frac{\mathcal{E}_0 + P_0}{\rho_0} \partial'_i \partial'_i \delta\mathcal{E} = 0$$

- 1st law: $\delta\mathcal{E} = T\delta s = \frac{\mathcal{E}+P}{\rho}\delta\rho + T\rho\delta\left(\frac{s}{\rho}\right)$
- Speed of sound $c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_{s/\rho} = \frac{2}{d}\frac{\mathcal{E}_0 + P_0}{\rho_0}$

Carrollian Perfect Fluids

$$T^{\mu}{}_{\nu} = (\mathcal{E} + P)\, v^{\mu} \bar{u}_{\nu} + P \delta^{\mu}_{\nu}$$

- geometry: Carrollian with metric v^{μ} and $h_{\mu\nu}$
- Velocity: $v^{\mu} \bar{u}_{\mu} = -1$
- Landau frame: $T^{\mu}{}_{\nu} \bar{u}_{\mu} = -\mathcal{E} \bar{u}_{\nu}$
- Ward identities (flat space): $\partial_{\mu} T^{\mu\nu} = 0$, $T^{i}{}_{t} = 0$ and $T^{i}{}_{j} = T^{j}{}_{i}$
- Scale symmetry: $z T^{t}{}_{t} + T^{i}{}_{i} = 0$: $z\mathcal{E} = dP$
- Equation of state: $P = P(\mathcal{E})$. 1st law: $\delta\mathcal{E} = T\delta s$
- Fluid variables: T and $\bar{u}_{\mu}$
- Follows from $c \to 0$ limit of rel. PF

Null Reduction

- Minkowski in null coordinates: $ds^2 = 2dtdu + dx^i dx^i$
- Null reduction of EMT: $t^{\mu}{}_{u} = T^{\mu}$, $t^{u}{}_{u} = T^{t}{}_{t}$, $t^{\mu}{}_{\nu} = T^{\mu}{}_{\nu}$
- Perfect fluid: $t^{A}{}_{B} = (E+P)U^A U_B + P\delta^A_B$, $U^2 = -1$
- Reduction of fluid:

$$U_u^2 = \frac{\rho}{E+P}\,, \qquad U_t = -\frac{1}{2}U_u\left(V^i V^i + U_u^{-2}\right)$$

$$U_i = U_u V^i\,, \qquad E = 2\mathcal{E} + P\,,$$

- Lower-dimensional EMT and mass current:

$$T^{\mu}{}_{\nu} = \left(\mathcal{E} + P + \frac{1}{2}\rho V^i V^i\right) u^{\mu}\tau_{\nu} + P\delta^{\mu}_{\nu} + \rho u^{\mu} h_{\nu\rho} u^{\rho}\,, \quad T^{\mu} = -\rho u^{\mu}$$

$$\tau_{\mu} u^{\mu} = -1\,, \quad u^i = -V^i\,, \qquad \tau_{\mu} = \delta^t_{\mu}\,, \quad h_{t\mu} = 0\,, \quad h_{ij} = \delta_{ij}$$

Lifshitz Fluids from Null Reduction

- Add a scalar with shift symmetry and consider the Ward identities: $\partial_A t^A{}_B = -O_\psi \partial_B \psi$ and $t^A{}_A = 0$
- Twisted null reduction: $\psi = -u$
- Lower-dimensional Ward identities: $\partial_\mu T^\mu{}_\nu = 0$, $\partial_\mu T^\mu = O_\psi$ (mass no longer conserved), $2T^t{}_t + T^i{}_i = 0$
- Fluid equations imply a conserved entropy current if we take

$$\mathcal{E} + P - \frac{1}{2}\rho V^2 = Ts\,, \qquad \delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho$$

- Velocity V^i becomes a chemical potential
- Lifshitz PF from Schrödinger PF by breaking N

Discussion

- In the example from twisted null reduction we can take e.g. $O_\psi = c\rho\partial_i V^i$. This leads to a conserved current $\partial_t \rho^{\frac{1}{1-c}} + \partial_i\left(\rho^{\frac{1}{1-c}} V^i\right) = 0$. In this case speed of sound the same as in Galilean case.
- work in progress:
  - Systematically classify transport coefficients of Lifshitz fluids by breaking Gal, Lor or Car boosts by adding terms to the boost Ward identities that break these symmetries.
  - Most general hydro assuming only Lifshitz symmetries.
  - Hydro from Finite temperature Lifshitz field theories.

Part VI:

Lifshitz Holography

Lifshitz Holography

- Types of models:
  - Horava–Lifshitz gravity in the bulk [Griffin, Horava, Melby-Thompson, 2012], [Janiszewski, Karch, 2012] which is dynamical TTNC geometry in the bulk [JH, Obers, 2015].
  - Bulk GR with massive vectors or a massless vector with a dilaton: [Baggio, de Boer, Holsheimer, 2011], [Ross, 2011], [Griffin, Horava, Melby-Thompson, 2012], [Chemissany, Geissbuhler, JH, Rollier, 2012], [Chemissany, Papadimitriou, 2014].
- Here focus is on the massive vector model.

Massive Vector Model

- For a bulk theory of the form

$$S=\int d^4x\sqrt{-g}\left(R-\frac{1}{4}Z(\Phi)F^2-\frac{1}{2}(\partial\Phi)^2-\frac{1}{2}W(\Phi)B^2-V(\Phi)\right)$$

Lifshitz solutions are given by

$$ds^2=-\frac{dt^2}{r^{2z}}+\frac{1}{r^2}\left(dr^2+dx^2+dy^2\right)\,,\quad B=\frac{2(z-1)}{zZ_0}\frac{dt}{r^z}\,,\quad \Phi=0$$

provided the functions Z, W and V obey

$$V_0 \;=\; -\left(z^2+z+4\right)\,,\quad W_0=2zZ_0\,,\quad V_1=(z-1)\left(z\frac{Z_1}{Z_0}+2\frac{W_1}{W_0}\right)$$

where we denote: $V(\Phi)=V_0+V_1\Phi+\dots$

Massive Vector Model

- For a bulk theory of the form

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}W(\Phi)B^2 - V(\Phi)\right)$$

$$ds^2 = \frac{dr^2}{R(\Phi)r^2} - E^0E^0 + \delta_{ab}E^aE^b\,, \qquad B_M = A_M - \partial_M\Xi$$

the AlLif boundary conditions are [Ross, 2011], [Christensen, JH, Obers, Rollier, 2013], [JH, Kiritsis, Obers, 2014] $(1 < z \leq 2)$:

$$\begin{aligned} E^0_\mu &= r^{-z}\tau_\mu + \ldots + r^{z-2}\left(m_\mu - \partial_\mu\chi\right) + \ldots & E^a_\mu &= r^{-1}e^a_\mu + \ldots \\ A_\mu &= \alpha(\Phi)E^0_\mu + \ldots & A_r &= (z-2)r^{z-3}\chi + \ldots \\ \Xi &= r^{z-2}\chi + \ldots & \Phi &= r^\Delta\phi + \ldots \end{aligned}$$

Transformations of the sources

- The local bulk symmetries are: local Lorentz transf., gauge transf. acting on A_M and Ξ and diffeos preserving the metric gauge (PBH transf.).
- The way these symmetries act on the sources τ_μ, e^a_μ, m_μ, χ is the same as the action of the Bargmann algebra plus local dilatations, i.e. the Sch algebra.
- There is thus a Schrödinger Lie algebra valued connection given by ($\tilde{m}_\mu = m_\mu + (z-2)\chi b_\mu$):

$$A_\mu = H\tau_\mu + P_a e^a_\mu + N\tilde{m}_\mu + \frac{1}{2}J_{ab}\omega_\mu{}^{ab} + G_a\omega_\mu{}^a + Db_\mu$$

whose transformations reproduce those of the sources [Bergshoeff, JH, Rosseel, 2014]. Bdry geom.=TTNC geom.

Boundary Geometry

- In the metric formalism the sources are:

source	$\bar{h}_{\mu\nu}$	τ_μ	$\tilde{\Phi}$	ϕ
scaling dimension	-2	$-z$	$2(z-1)$	Δ

- Schematically they appear in the asymptotic exp. as

$$ds^2 = \frac{dr^2}{Rr^2} + h_{\mu\nu}dx^\mu dx^\nu \qquad B = B_r dr + B_\mu dx^\mu$$

$$\begin{aligned} h_{\mu\nu} &= -r^{-2z}\tau_\mu\tau_\nu + \ldots + r^{-2}\left(\bar{h}_{\mu\nu} + \tilde{\Phi}\tau_\mu\tau_\nu\right) + \ldots \\ B_\mu &= r^{-z}\tau_\mu + \ldots + r^{2-z}\tilde{\Phi}\tau_\mu + \ldots \\ \Phi &= r^\Delta\phi + \ldots + r^{2(z-1)}\tilde{\Phi} + \ldots \end{aligned}$$

Leading Order Solutions

- i). $1 < z < 2$ and $\Delta > 0$

$$R_{(0)} = \frac{1}{2z}\frac{W_0}{Z_0}, \qquad \alpha^2_{(0)} = \frac{2(z-1)}{z}\frac{1}{Z_0}$$

$$V_0 = -\frac{1}{2z}(z^2+z+4)\frac{W_0}{Z_0}, \qquad V_1 = (z-1)\left[\frac{1}{2}\frac{Z_1}{Z_0} + \frac{1}{z}\frac{W_1}{W_0}\right]\frac{W_0}{Z_0}$$

We can set $Z_0 = 1$. R_0 is the Lifshitz radius.

- ii). $1 < z < 2$ and $\Delta = 0$

- iii). $z = 2$ and $\Delta > 0$

$$R_{(0)} = (Z(\phi))^{-\frac{z-1}{z+1}}, \quad \alpha^2_{(0)} = \frac{2(z-1)}{z}(Z(\phi))^{-1}$$

$$W(\Phi) = 2z(Z(\phi))^{\frac{2}{z+1}} + \ldots, \quad V(\Phi) = -(z^2+z+4)(Z(\phi))^{-\frac{z-1}{z+1}} + \ldots$$

- iv). $z = 2$ and $\Delta = 0$

Vevs and Ward identities

- Assuming holographic renormalizability the variation of the on-shell action takes the form:

$$\delta S^{\text{os}}_{\text{ren}} = \int d^3xe\left[-T_\mu\delta v^\mu + \frac{1}{2}T_{\mu\nu}\delta h^{\mu\nu} + J^\mu\delta m_\mu + \langle O_\chi\rangle\delta\chi + \langle O_\phi\rangle\delta\phi - \mathcal{A}\frac{\delta r}{r}\right]$$

- The vevs and sources can be used to define the G, J, N invariants:

$$T^\mu{}_\nu = -v^\mu T_\nu + e^\mu_a T^a_\nu - J^\mu\left(m_\nu - \partial_\nu\chi\right) \qquad \text{bdry EM tensor}$$

$$J^\mu = -J^0 v^\mu + J^a e^\mu_a \qquad \text{mass current}$$

- Vielbein components of $T^\mu{}_\nu$ provide the energy density, energy flux, momentum density and stress.

Vevs and Ward identities

- The Ward identities are (ignoring the dilaton ϕ):

$$\begin{aligned}
0 &= -\hat{e}^a_\mu J^\mu + \tau_\mu e^{\nu a} T^\mu{}_\nu && \text{boosts} \\
0 &= \hat{e}^a_\mu e^{\nu b} T^\mu{}_\nu - (a \leftrightarrow b) && \text{rotations} \\
\mathcal{A} &= -z\hat{v}^\nu \tau_\mu T^\mu{}_\nu + \hat{e}^a_\mu e^\nu_a T^\mu{}_\nu + 2(z-1)\tilde{\Phi}\tau_\mu J^\mu && \text{dilatations} \\
\langle O_\chi \rangle &= e^{-1}\partial_\mu \left(e J^\mu\right) && \text{gauge trafo} \\
0 &= \nabla_\mu T^\mu{}_\nu + 2\Gamma^\rho_{[\mu\rho]} T^\mu{}_\nu - 2\Gamma^\mu_{[\nu\rho]} T^\rho{}_\mu + \tau_\mu J^\mu \partial_\nu \tilde{\Phi} && \text{diffeos}
\end{aligned}$$

- We used Galilean boost invariant vielbeins τ_μ, $\hat{v}^\mu$, e^μ_a, $\hat{e}^a_\mu$ and density $e = \det\left(\tau_\mu, e^a_\mu\right)$.
- ∇_μ denotes that affine TNC connection for which $\nabla_\mu \tau_\nu = 0$, $\nabla_\mu h^{\nu\rho} = 0$ and $\nabla_\mu \hat{v}^\nu = 0$.

Moving Lifshitz Black Branes [JH, Obers, Sanchioni, 2016]

- SS reduction of $\mathcal{L}_{(5)} = \sqrt{-\gamma}\left(R + 12 - \frac{1}{2}(\partial\psi)^2\right) + \mathcal{L}_{\text{ct}}$

$$\mathcal{L}_{(4)} = \sqrt{-g}\left(R - \frac{1}{4}e^{3\Phi}F^2 - 2B^2 - \frac{3}{2}(\partial\Phi)^2 - 2e^{-3\Phi} + 12e^{-\Phi}\right) + \mathcal{L}_{\text{ct}}$$

- Admits $z = 2$ and $\Delta = 0$ Lifshitz solutions
- Near boundary ($r = 0$) exp.: $ds_4^2 = e^{\Phi}\frac{dr^2}{r^2} + h_{\mu\nu}dx^\mu dx^\nu$

$$\begin{aligned} \Phi &= -\frac{1}{8}r^2\rho - r^4\left(\frac{1}{6}T^t{}_t + \frac{1}{64}\rho^2\right) + \mathcal{O}(r^6) \\ B_t &= r^{-2} + \frac{1}{4}\rho + r^2\left(\frac{1}{12}T^t{}_t + \frac{1}{16}\rho^2\right) + \mathcal{O}(r^4) \\ B_i &= -\frac{1}{4}r^2 T^t{}_i + \mathcal{O}(r^4) \end{aligned}$$

likewise for $h_{\mu\nu}$

Moving Lifshitz Black Branes

- Ansatz for full solution (all functions depend on r only):

$$ds_4^2 = -r^{-4}F_1 dt^2 + \frac{dr^2}{r^2 F_2} + \frac{F_3}{r^2}dx^2 + \frac{F_4}{r^2}(dy + N^y dt)^2$$

$$B = r^{-2}G dt + A_y (dy + N^y dt) \,, \qquad \Phi = \Phi(r)$$

- Effective action for this ansatz has two scale symmetries:

$$1). \quad [G] = 1\,, \quad [F_1] = 2\,, \quad [F_{3,4}] = 1\,, \quad [A_y] = -1/2\,, \quad [N^y] = 3/2$$

$$2). \quad [F_3] = 2\,, \quad [F_4] = -2\,, \quad [A_y] = -1\,, \quad [N^y] = 1$$

Associated Noether charges Q_1 and Q_2 are 1st int. of motion along r (similar to [Bertoldi, Burrington, Peet, 2009]).

Thermodynamics

- Near horizon regularity: G, F_1, F_2 first order zeros at $r = r_h$, the rest are nonzero
- Horizon generator $X = \partial_t - N^y(r_h)\partial_y$ gives temperature and chemical potential ($N^y(r_h) = -V$)
- Conserved Noether charges $Q_1 - \frac{3}{2}Q_2 = Ts$ at the horizon and $\mathcal{E} + P - \frac{1}{2}\rho V^2$ at the boundary
- 1st law from on-shell action, grand potential as a function of temperature T and chemical potential V

$$\mathcal{E} + P - \frac{1}{2}\rho V^2 = Ts\,, \qquad \delta\mathcal{E} = T\delta s + \frac{1}{2}V^2\delta\rho$$

- Other holographic setups: EMD [Kiritsis, Matsuo, 2015] and compare with approach by [Hoyos, Kim, Oz, 2013].

Part VII:
Chern–Simons theories

Metrics on non-semisimple algebras

- The Galilei algebra and most of its extensions do not admit a symmetric non-degenerate bilinear form ("trace"). There are a couple of exceptions.
- One can combine the 2D Galilean conformal algebra (H, D, K, S, Y, Z) with the 3D $z = 2$ Schrödinger algebra (H, D, K, N, J, P_a, G_a) such that the $SL(2,\mathbb{R})$ subalgebras (H, D, K) are the same and

$$[G_a\,, G_b] = S\epsilon_{ab}\,, \quad [P_a\,, P_b] = Z\epsilon_{ab}\,, \quad [P_a\,, G_b] = N\delta_{ab} - Y\epsilon_{ab}\,.$$

Trace: $$B(H,S) = -B(J,N) = c_1\,, \quad B(P_a, G_b) = c_1\epsilon_{ab}\,,$$
$$B(G_a, G_b) = c_2\delta_{ab}\,, \quad B(J,S) = c_2\,, \quad B(J,J) = c_3$$

Chern–Simons Theories

- $A = H\tau + P_a e^a + G_a \Omega^a + J\Omega + Nm + Db + Kf + S\eta + Y\alpha + Z\beta$

$$\mathcal{L}_{\text{CS}} = \text{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

- Admits $z = 2$ Lifshitz solutions with a ‘metric’ in the sense of HL gravity:

$$\text{“metric”}\ ds^2 = -\tau^2 + (e^1)^2 + (e^2)^2 : \qquad \tau = \frac{dt}{r^2}\,,\ e^1 = \frac{dr}{r}\,,\ e^2 = \frac{dx}{r}\,,$$

$$\text{“matter”} : \qquad b = -\frac{dr}{r}\,,\ \beta = -\frac{dx}{r}$$

- Algebra admits infinite dimensional extensions with 2 central charges.

Chern–Simons Theories

- What are the boundary conditions such that the asymptotic symmetry algebra is BMS_3 combined with Schrödinger–Virasoro (and $U(1)$ current algebra for J)?
- There must exist a novel 2D class of field theories with these infinite symmetries.
- What are thermal states in the bulk? Black holes?
- Similar to lower-spin gravity [Hofman, Rollier, 2014].
- In metric formulation this leads to new versions of HL gravity called Schrödinger CS gravity.
- Novel type of holographic duality with a non-Einsteinian bulk gravity theory.

Outlook

- New types of geometries by gauging non-relativistic symmetries.
- non-AdS holography and infrared effective field theories requires new classes of field theories.
- novel types of holographic dualities with non-Einsteinian bulk theories.
- What about Kerr/CFT, Schrödinger holography, etc.?
- Many ways to realize Lifshitz invariance in field theory. New models with general z scaling without higher spatial derivatives.
- Most general Lifshitz hydodynamics.

Part VIII:
Relation with Hořava–Lifshitz gravity

Effective Actions

- Extrinsic curvature: $\nabla_\mu \hat{v}^\rho = -h^{\rho\sigma} K_{\mu\sigma}$ where $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\hat{v}}\hat{h}_{\mu\nu}$
- Integration measure $e = \det\left(\tau_\mu, e^a_\mu\right)$ is G, J invariant.
- Add terms (built out of tangent space invariants) to the action that are relevant or marginal (up to dilatation weight $d+z$)

invariant	τ_μ	$\hat{h}_{\mu\nu}$	$\hat{v}^\mu$	$h^{\mu\nu}$	e	$\tilde{\Phi}$	χ
dil. weight	$-z$	-2	z	2	$-(z+d)$	$2(z-1)$	$z-2$

- We work in 2+1 dimensions with $1 < z \leq 2$. Weight of each term is determined by number of $h^{\mu\nu}$ and $\hat{v}^\mu$.

Effective Actions

- In 2+1 dimensions there is one curvature invariant: $\mathcal{R} = h^{\mu\nu} R_{\rho\mu\nu}{}^{\rho}$ (2) which is the Ricci curvature of γ_{ij}.
- Do not allow terms that break time reversal invariance.
- Two kinetic terms (the HL λ parameter):

$$c_1 \nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu + c_2 \nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu = C \left(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - \lambda \left(h^{\mu\nu} K_{\mu\nu} \right)^2 \right)$$

- The potential term matches [Blas, Pujolas, Sibiryakov, 2010], [Zhu, Shu, Wu, Wang, 2010]

$$\begin{aligned} \mathcal{V} \ &= c_3 h^{\mu\nu} a_\mu a_\nu + c_4 \mathcal{R} + \delta_{z,2} \left[c_5 \left(h^{\mu\nu} a_\mu a_\nu \right)^2 + c_6 h^{\mu\rho} a_\mu a_\rho \nabla_\nu \left(h^{\nu\sigma} a_\sigma \right) \right. \\ &\left. + c_7 \nabla_\nu \left(h^{\mu\rho} a_\rho \right) \nabla_\mu \left(h^{\nu\sigma} a_\sigma \right) + c_8 \mathcal{R}^2 + c_9 \mathcal{R} \nabla_\mu \left(h^{\mu\nu} a_\nu \right) + c_{10} \mathcal{R} h^{\mu\nu} a_\mu a_\nu \right] \end{aligned}$$

Local $U(1)$

- TTNC identity (note $\lambda = 1$):

$$\delta_N \left(\nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu - \nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu\right) = -\mathcal{R}\hat{v}^\mu \partial_\mu \sigma + \text{torsion terms}\,,$$

- The additional field $\tilde{\Phi}$ transforms as: $\delta_N \tilde{\Phi} = -\hat{v}^\mu \partial_\mu \sigma$.

$$S = \int d^3 x e \left[C\left(h^{\mu\rho}h^{\nu\sigma}K_{\mu\nu}K_{\rho\sigma} - \left(h^{\mu\nu}K_{\mu\nu}\right)^2 - \tilde{\Phi}\mathcal{R}\right) - \mathcal{V}\right]$$

  Is the $U(1)$ invariant HMT action for projectable HL gravity in 3D [Horava, Melby-Thompson, 2010].

- The non-projectable HMT action can only be made $U(1)$ invariant by adding a Stückelberg scalar [HMT]. By replacing m_μ by $m_\mu - \partial_\mu \chi$ we reproduce precisely all the terms of [Zhu, Shu, Wu, Wang, 2010].

Outlook

- What if we drop condition that the actions is time-reversal invariant? More $U(1)$ invariant possibilities?
- Black holes: in 4D up to second order in spatial derivatives (Einstein–Aether theories) there are universal horizons [Barausse, Jacobson, Sotiriou, 2011] but with higher spatial derivatives that does not seem to be the case [Kiritsis, Kofinas, 2009].
- Fluid/gravity correspondence [Davison, Grozdanov, Janiszewski, Kaminski, 2016].
- What are good probes? String propagation on TNC backgrounds ....