Review of AC holographic conductivities

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Acknowledgments

- Beyond work by other people which I will refer to as I go along, this overview owes a lot to past and ongoing collaborations/discussions with Sera Cremonini, Richard Davison, Luca Delacrétaz, Aristomenis Donos, Simon Gentle, Sean Hartnoll and Elias Kiritsis.

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Disclaimer

Topics I find interesting and which are relevant to this review but won't talk about by lack of expertise and/or time

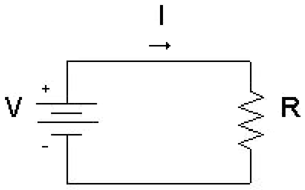
- Fermions
- Top-down constructions
- Anisotropic phases
- Inhomogeneous phases
- Insulators
- Intermediate scalings in the optical conductivity

What is the AC conductivity?

- Ohm's law

$$J = \sigma E$$

where
E is the source (applied electric field),
J the vev (electric current)
and σ the response coefficient:
conductivity.



- When E is frequency dependent: AC conductivity [My talk].

- When E is frequency independent: DC conductity [Aristos' talk].

- It is an important observable: metals, insulators, superfluids, charge density waves, information about the spectrum etc.

- 'Simple' to measure.

Modern reformulation of the conductivity

- The memory matrix formalism [Forster'75] is a useful tool when the Hamiltonian of a system can be decomposed as

$$H = H_0 + \epsilon \Delta H \,, \qquad \dot{A} = \epsilon i [A, \Delta H] \neq 0$$

where A is an operator overlapping with the electric current J.

- σ_{JJ} receives contributions from all operators with which J overlaps:

$$\sigma_{JJ} = \sum_{C,D} \chi_{JC} \left(\frac{1}{-i\omega\chi + M(\omega)} \right)_{CD} \chi_{DJ}$$

- Static susceptibility $\chi_{AB} = \delta O_A / \delta s_B$: overlap between 2 operators.

- Examples: momentum ($\chi_{JP} = \rho$ for rel hydro, $\chi_{JP} = 1$ for non-rel hydro), superfluid current ($\chi_{J_\phi J_\phi} = 1/\rho_s$ for superfluid hydro...)

Infinite vs finite DC conductivities

- M_{AB} is the memory matrix

$$M_{AB}(\omega) = \frac{1}{i\pi} \int_{-\infty}^{+\infty} d\omega' \frac{\mathrm{Im}\, G^R_{\dot{A}\dot{B}}(\omega')}{\omega'(\omega' - \omega)} \quad \Rightarrow M_{AA}(0) = \lim_{\omega \to 0} \frac{\mathrm{Im}\, G^R_{\dot{A}\dot{B}}(\omega)}{\omega}$$

- If all operators C, D are conserved, then $M = 0$ and the conductivity is a sum of δ-functions:

$$\sigma_{JJ} = \frac{i}{\omega} \sum_{C,D} \chi_{JC} (\chi^{-1})_{CD} \chi_{DJ} + \dots$$

- If there is a single long-lived operator A, then

$$\sigma_{JJ} = \frac{\chi^2_{JA}}{\chi_{AA}} \frac{1}{\Gamma - i\omega} + \dots , \qquad \Gamma = \frac{M_{AA}(0)}{\chi_{AA}} \ll k_B T$$

Gives rise to sharp Drude-like peak (width $O(\Gamma)$, height $O(1/\Gamma)$) in AC conductivity.

- Provides a transparent framework to analyze the various contributions to the conductivity.

Infinite vs finite DC conductivities

This suggests three ways for the DC conductivity to be finite

- $\chi_{AJ} = 0$: for instance, neutral CFTs ($\chi_{PJ} = \rho = 0$).
- $\dot{A} \neq 0$, $\chi_{AJ} \neq 0$: finite density and broken translation symmetry.
- Redefine the current such that $\chi_{A\tilde{J}} = 0$: incoherent currents.

All three possibilities will appear in this talk.

1. Zero density
2. Finite density, conserved U(1)
3. Finite density, conserved U(1), broken translations
4. Finite density, spontaneously broken U(1)
5. Finite density, spontaneously broken translation symmetry: CDWs

1. Zero density

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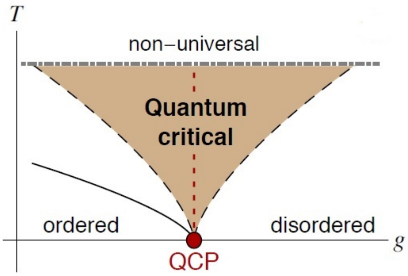
Quantum Critical Points



- A QCP separates an ordered from a disordered phase at zero temperature, when a coupling g is varied.
- It enjoys special scaling properties (e.g. conformal symmetry, Lifshitz invariance...).
- While it strictly speaking lies on the $T = 0$ axis, it governs the behaviour of the system in a quantum critical 'fan' at $T > 0$.

Correlation functions and OPE

- Correlation functions take simple forms, for instance

$$\sigma(\omega, T) = \frac{i}{\omega}\langle J(\omega, k=0)J(\omega, k=0)\rangle = T^{(d-2)/z}\Sigma(\omega/T)$$

$$\mathcal{O}(x)\,\mathcal{O}(0) \rightarrow \sum_n \frac{\mathcal{O}_n(0) + \text{desc.}}{x^{2\Delta-\Delta_n}}$$

- The high $\omega, k \gg T$ asymptotics are governed by the OPE of the JJ correlator, in particular by the few most relevant operators (stress-tensor, scalar...)

$$J(\omega,k)J(0,0) = |\omega^2 - k^2|^{2\Delta_J - D} \sum_{\mathcal{O}_n \; primary} \left(\frac{c_n(\omega,k)\mathcal{O}}{|\omega^2 - k^2|^{\Delta_n}} + \dots \right)$$

- The other regime $\omega, k \ll T$ is much harder to access as infinitely many operators contribute.

Examples

- Main CM-motivated example: 2+1D O(N) CFTs.

$$S = \int d^3x \left[\frac{1}{2}(\partial\phi_a)^2 + \frac{v}{2N}\left(\phi_a^2 - N/g\right)^2 \right], \qquad a = 1 \ldots N$$

- $N = 1$: 2+1D Ising model
- $N = 2$: QCP in the same universality class as the Bose-Hubbard model with superfluid/insulator quantum phase transition.
- Strongly-coupled, no quasi particles at small N.
- Analyzed using $1/N$ expansion [Damle & Sachdev'97] or Quantum Monte Carlo simulations of lattice models [Witczak-Krempa et al'14].

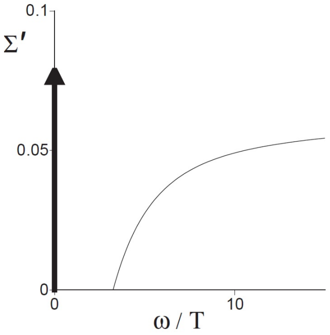
Large N analysis of the $O(N)$ model

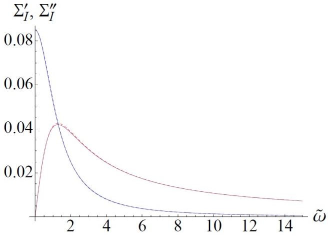
- At $N \to +\infty$, free theory: Wilson-Fisher fixed point.
- At $N \to +\infty$: infinitely lived quasiparticles $\Rightarrow$ δ-function in the conductivity (I).
- There is another, gapped contribution (II).
- At large N: strength of interactions $\sim 1/N$.

$$\sigma(\omega/T) = \frac{e^2}{\hbar} N\Sigma(N\omega/T)$$

- Σ_I is well approximated by a Drude form

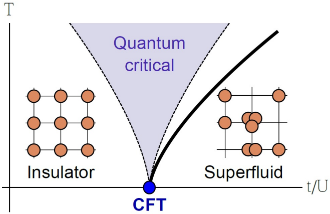
$$\Sigma_I(\omega) = \frac{\Sigma(0)}{1 - i\tau\omega}\,, \quad \tau \sim \frac{T}{N}$$

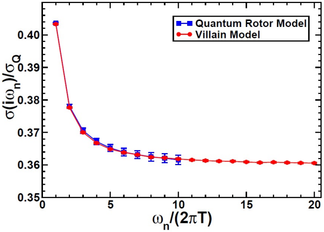




The $O(2)$ model

- The BHM displays a quantum phase transition between an insulating and a superfluid phase. t favours hopping, U favours on-site Coulomb repulsion.
- The QCP separating the two phases has an emergent conformal symmetry.
- The two lattice models are in the same universality class as the BHM (easier to simulate).
- They have the same (Euclidean) frequency dependence.
- Finite amount of points, no points at low frequency: analytic continuation?





Holographic approach

- Holography can be useful: can compute the OPE coefficients, model the relevant operators using bulk fields, etc.
- Complete knowledge of the spectrum (poles, zeroes) can be obtained from the QNMs of the corresponding black hole.
- No problem with analytic continuation: can be fitted to Quantum Monte Carlo data.
- Makes a reliable prediction for the low frequency regime.
- Ongoing research program by Sachdev, Myers, Witczak-Krempa et al [hep-th/0701036, 1010.0443, 1210.4166, 1302.0847, 1309.2941, 1312.3334, 1409.3841, 1501.03495, 1602.05599, 1608.02586]

Simplest holographic model at zero density

- Einstein-AdS (Schwarzschild black brane) + Maxwell action:

$$S = \int d^4x \sqrt{-g} \left[R + 6 - \frac{1}{4e^2} F^2 \right]$$

$$ds^2 = -r^2 \left(1 - \frac{r_0}{r}\right) dt^2 + \frac{dr^2}{r^2 \left(1 - \frac{r_0}{r}\right)} + r^2 \left(dx^2 + dy^2\right)$$

- Emergent bulk EM self-duality $F \leftrightarrow \star F$ in 3+1D spacetime:

$$\omega, k \neq 0 \qquad \langle J_x J_x \rangle \langle J_y J_y \rangle = -\frac{1}{e^4} \omega^2$$

$$k = 0 \quad \Rightarrow \quad \langle J_x J_x \rangle = \langle J_y J_y \rangle \quad \Rightarrow \quad \sigma(\omega) = \frac{i}{\omega} \langle JJ \rangle = \frac{1}{e^2}$$

- Exchanges zeroes and poles of $\perp$ and $\parallel$ correlators
- Akin to particle-vortex duality of the Bose-Hubbard model.
- No good model of real CM systems: conductivity is ω-independent.

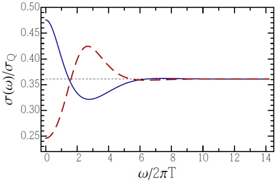
Option 1: higher-derivative couplings to the Maxwell field

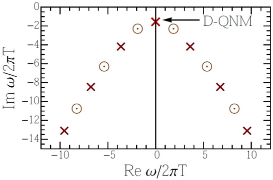
$$S_{EM} = \int d^4x\sqrt{-g}\left(-\frac{1}{4e^2}F^2 + \frac{\gamma}{e^2}C_{abcd}F^{ab}F^{cd}\right)$$

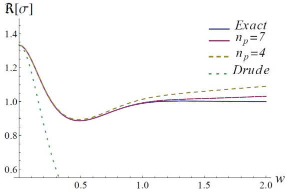
- $\gamma \neq 0$ formally breaks self-duality, but it can be restored at small γ by $\gamma \to -\gamma$
- There is a low-lying, purely imaginary QNM in the spectrum

$$\omega = -i\Gamma, \qquad \Gamma \sim T/\gamma^{0.66}$$

- $|\gamma| \leq 1/12$ by causality, so this pole cannot get arbitrarily close to the real axis.
- Turn on other terms: $\gamma_1 C^2 F^2$. γ_1 is unconstrained by causality (though see [Camahno et al'14]) and $\Gamma \sim T/\gamma_1^{0.83}$: Sharper Drude peak as $\gamma_1 \gg 1$. Effective field theory?



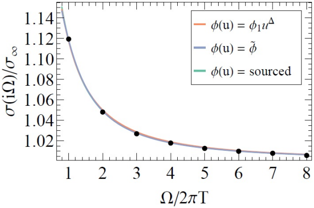


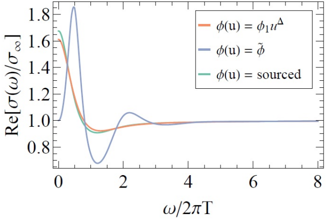


Option 2: Include the most relevant scalar operator

$$S_\phi = -\frac{1}{2}\int d^4x\sqrt{-g}\left[(\nabla\phi)^2 + m^2\phi^2 - 2\alpha_1\phi C^2 - \frac{1}{4e^2}(1+\alpha_2\phi)F^2\right]$$

- O(2) model: $\mathcal{O}_g$ mass operator, $\Delta_g \gtrsim 3/2$
- Work in the probe limit for the scalar, $\phi_{(0)} = 0$: QCP
- High freq expansion fixed by the OPE taking into account the vev for $\mathcal{O}_g$
- α_1, α_2 can be fixed by fitting to QMC data in Euclidean signature
- $\omega \ll T$: unconstrained by QMC data.
- The holographic conductivity can be plotted in Lorentzian signature: it is crucial to solve for ϕ consistently.

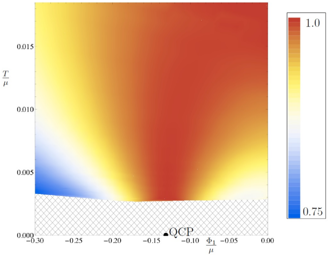




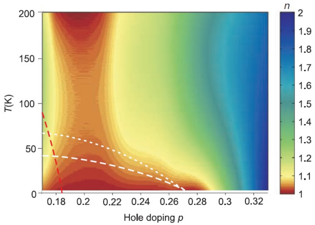
Open questions

- Interesting to consider other CFTs (2+1D Ising model) and other correlation functions (see [Witczak-Krempa'15]).
- Full, self-consistent backreaction of the scalar operator.
- Check match between DC conductivity predicted holographically and that obtained by other means.
- Detuning effects away from the QCP (see [Lucas et al'16]).

- Holographic quantum critical points vs phases? At finite density: $(T/\mu, \phi_{(0)}/\mu)$ [Hartnoll & Huijse'11, Adam & al'12, B.G. & Kiritsis'12].



[Adam et al'12]



[Cooper et al'09]

Intro

- A chemical potential is a relevant deformation of the UV fixed point.
- The UV asymptotics of the AC conductivity can be computed from a first-principle CFT analysis deformed by relevant operators.
- Here, we will examine what happens in the IR: low temperatures and/or low frequencies.
- Since translations are not broken: the effective theory governing transport is relativistic hydrodynamics.

Hydrodynamic approximation

- Hydrodynamics is an effective theory truncating the theory to the behaviour of a few collective variables: conserved charges and Goldstone bosons.
- It is valid at low frequencies and wavevectors compared to some UV scale (typically temperature).
- In real systems, $\tau_{ee} \ll \tau_{mr}$ for hydrodynamics to be a good approximation.
- The collective excitations around thermodynamic equilibrium are fluctuations of energy, charge and momentum density.
- They are sourced by fluctuations in temperature, chemical potential and fluid velocity.

Eoms and constitutive relations

- The equations of motion are

$$\nabla_\mu T^{\mu\nu} = 0\,, \qquad \nabla_\mu J^\mu = 0$$

- They are supplemented by constitutive relations:

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} - \eta\sigma^{\mu\nu}$$

$$J^\mu = \rho u^\mu - T\sigma_Q \partial^\mu \left(\frac{\mu}{T}\right)$$

- In conformal relativistic hydrodynamics, only two independent first-order transport coefficients: shear viscosity η, 'quantum critical' conductivity σ_Q.

Electric transport in relativistic hydrodynamics

- Linearize the eoms around thermodynamic equilibrium and solve for the fluctuations $(\delta\epsilon, \delta\rho, \pi)$ in terms of $(\delta T, \delta\mu, u)$.

- From the Ward identity for the current, get δj and read off the coefficient of $-\nabla\delta\mu$ which is the electric conductivity (Ohm's law):

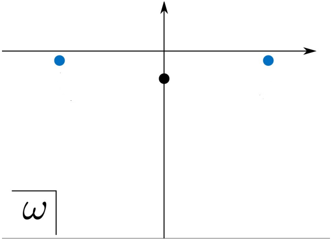
$$\sigma = \sigma_Q + \frac{\rho^2}{\epsilon + p}\frac{i}{\omega}$$

- The $\omega = 0$ pole originates from momentum conservation and finite density: a momentum flow also carries charge.

- Consistent with our earlier memory matrix formulation.

Incoherent charge transport

- There are two sound poles $\omega = \pm c_s k - iD_\pi k^2$ and a diffusion pole $\omega = -iD_{inc}k^2$.
- The diffusion pole comes from diffusion of the incoherent charge $\delta\rho_{inc} = s^2 T\delta(\rho/s)/(\epsilon + p)$ [Davison, B.G. & Hartnoll'15].



- The corresponding incoherent current $J_{inc} = J - \rho P/(\epsilon + p)$ verifies $\chi_{J_{inc}P} = 0$ and so has a finite conductivity

$$\sigma_{inc}(\omega) = \sigma_Q$$

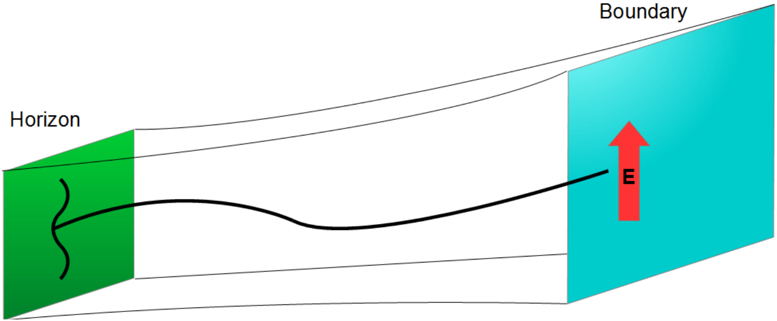
- This conductivity is totally incoherent at low frequency: no low-lying poles in the lower half plane (at $k = 0$).

Holographic computation of σ_{inc}

- Consider the family of holographic theories

$$S_{EMD} = \int dx^4 \sqrt{-g} \left[R - \frac{1}{2}\partial\phi^2 - \frac{1}{4}Z(\phi)F^2 - V(\phi) \right]$$

- To solve for the conductivity, we need to solve for the fluctuations δa_x, δg_{tx} of the bulk fields with ingoing boundary conditions at the horizon [Policastro, Son & Starinets'02]: 'shake the black hole'.



Holographic computation of σ_{inc}

- The system boils down to a single eom, which can be written as a total radial derivative:

$$\Pi_{inc}{}'(r) = 0\,, \qquad \Pi_{inc} = \frac{C^2 Z}{B}\left(\frac{D}{C}\right)'\delta a_x{}' - \frac{C Z^2 A_t^2}{B}\delta a_x$$

- At the boundary, it is proportional to J_{inc}. At the horizon, it is proportional to $E_{inc} \sim T\nabla(\mu/T)$.

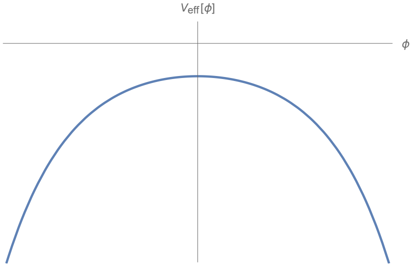
- We get (in $d+2$ dimensions) [Jain'10, Davison, B.G. & Hartnoll'15]

$$\sigma_Q = Z(\phi_h)\left(\frac{sT}{\epsilon + p}\right)^2 s^{(d-2)/d}$$

Effective IR holographic theories

- To evaluate the low-T scaling of σ_{inc}, we need to know the near-extremal geometry.
- Moreover we are interested in quantum critical phases.
- This motivates looking at the following class of solutions [Charmousis, B.G., Kim, Kiritsis & Meyer'10]

$$V_{IR} \sim e^{-\delta\phi}, \qquad Z_{IR} \sim e^{\gamma\phi}$$



$$ds_{IR}^2 = r^{\frac{2}{d}\theta}\left(-\frac{dt^2}{r^{2z}} + \frac{dr^2 + d\vec{x}_d^2}{r^2}\right), \quad \phi = \kappa(z,\theta)\log r\,, \quad A_t = Qr^{\zeta - z}$$

- **Three** critical exponents [B.G. & Kiritsis'12, Gath & al'12].
- When $\theta \neq 0$ or $\zeta \neq 0$, these solutions are only scale-covariant under 'Lifshitz' rescalings

$$t \to \lambda^z t\,, \quad (r, \vec{x}) \to \lambda(r, \vec{x})$$

$\Rightarrow$ Hyperscaling violation [B.G. & Kiritsis'11,'12, Huijse, Sachdev & Swingle'11, Gath & al'12, B.G.'13].

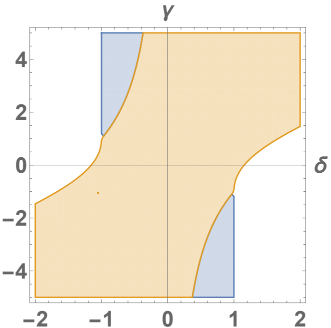
$$s \sim T^{(d-\theta)/z}$$

Classes of solutions

$$ds^2_{IR} = r^{\frac{2}{d}\theta}\left(-\frac{dt^2}{r^{2z}} + \frac{dr^2 + d\vec{x}_d^2}{r^2}\right), \quad \phi = \kappa(z,\theta)\log r\,, \quad A_t = Qr^{\zeta - z}$$

There are two classes of solutions

- $z \neq 1$, $\zeta = \theta - d$, $Q(z,\theta)$.
- $z = 1$, $\zeta \neq \theta - d$, Q is an independent scale in the IR solution.
- In both cases, Q is (proportional) to the UV charge density.



Low temperature scaling of σ_{inc}

- θ is related to an spatial effective dimensionality [B.G. & Kiritsis'11, Huijse, Sachdev & Swingle'11]

$$[s] = d - \theta \quad \Rightarrow \quad s \sim T^{(d-\theta)/z}$$

- ζ is related to an anomalous contribution to the dimension of the density in the IR [B.G.'13,'14, Karch'14]

$$[\rho] = d - \theta + \Phi\,, \qquad \Phi = \frac{1}{2}(\zeta + \theta - d)$$

- Recent proposal [Hartnoll & Karch] to explain DC transport data in the cuprates suggests $\theta = 0$, $\Phi = -2/3$, $z = 4/3$.

- Field theory proposals for Φ: 'multiband' models [Karch'15], 'unparticles' [Karch, Limtragool & Phillips'15].

Low temperature scaling of σ_{inc}

- ζ also governs the IR scaling of the electric conductivity for small T or ω: 'conduction' exponent [B.G.'13]. In particular,

$$z \neq 1: \quad \sigma_Q \sim T^{2+(d-2-\theta)/z} \qquad\qquad z = 1: \sigma_Q \sim T^{\zeta+2(d-\theta)}$$

- However, for $z = 1$, $\sigma(\omega, T)$ is not a function of ω/T:

$$\begin{array}{llll} z \neq 1: & Re[\sigma(\omega, T = 0)] & \sim & \omega^{2+(d-2-\theta)/z} + \#\delta(\omega) \\ z = 1: & Re[\sigma(\omega, T = 0)] & \sim & \omega^{-\zeta} + \#\delta(\omega) \end{array}$$

- The extra scale Q plays a role in making up the correct dimension

$$z = 1: Re[\sigma(\omega, T = 0)] \sim Q^4 \omega^{-\zeta} + \#\delta(\omega)\,, \qquad [Q] = (\zeta + d - \theta)/2$$

The horizon formula for σ_Q is crucial to get the low-T scaling.

Open directions

- How do we understand Φ holographically, given the U(1) current is conserved and its dimension should be protected?
- The last slides suggests that single-parameter analysis is not always enough to capture the low temperature behaviour of observables. It is also well-known that single parameter scaling is not enough to capture the behaviour of eg the cuprates.
- We can use explicit holographic computations to build consistent scaling theories.
- Probably also relevant for recent studies of violations of KSS bound with translation symmetry breaking [Davison & B.G.'14, Hartnoll, Ramirez & Santos'16, Alberte, Baggioli & Pujolas'16, Burikham & Poovuttikul'16].

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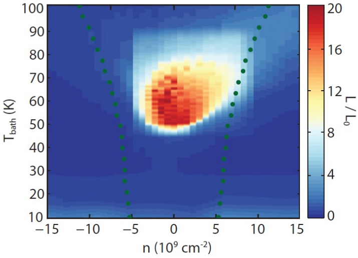
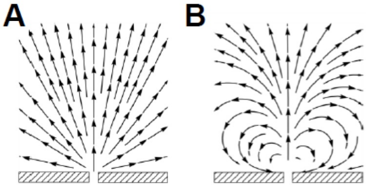
Quasi-hydrodynamic regimes in real metals

- Most realistic metals: $\tau_{mr} \ll \tau_{ee}$. Momentum is not a long-lived quantity: not part of the long wavelength description.
- However: graphene [Crossno et al'16, Bandurin et al'16], delafossite [Moll et al'16]. Violations of the Wiedemann-Franz law, viscous vorticity flows, viscous contributions to the resistivity.
- $\Rightarrow$ hydrodynamics with almost conserved momentum.



[Crossno et al'16]



[Bandurin et al'16]

A long-lived operator means a sharp peak

- If P is conserved, $\dot{P} = [H, P] = 0$:

$$\sigma_{JJ} = \frac{\chi_{JP}^2 i}{\chi_{PP}\omega}$$

- If P is long-lived, $\dot{P} = [H, P] = O(\epsilon)$, the $\omega = 0$ pole is relaxed:

$$\sigma_{JJ} = \frac{\chi_{JP}^2}{\chi_{PP}} \frac{1}{\Gamma - i\omega} + \dots , \qquad \Gamma = \frac{M_{PP}}{\chi_{PP}} = O(\epsilon)^2$$

- M_{PP} is a certain memory matrix element computed from knowledge of $[H, P]$.

- If P is almost conserved, $\Gamma \ll T \Rightarrow$ sharp peak.

- If P is more efficiently relaxed (lattice, impurities, etc) Γ increases and the dynamics is no longer governed by the Drude pole.

Hydrodynamics with almost-conserved momentum

- First pass [Hartnoll & al'07]: turn on a momentum relaxation term in the Ward identity

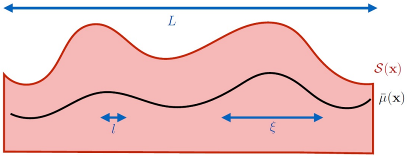
$$\nabla_\mu T^{\mu\nu} = -\Gamma \delta^{\nu i} T^{0i}\,, \qquad \nabla_\mu J^\mu = 0$$

$$\Rightarrow \sigma = \sigma_Q + \frac{\rho^2}{\epsilon + p} \frac{1}{\Gamma - i\omega}$$

Valid if $\Gamma \ll T$ but not under very good theoretical control.

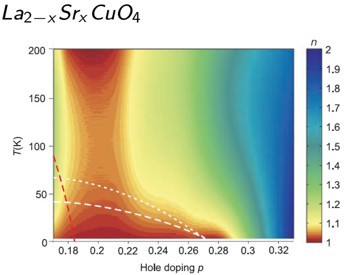
- More elaborate computation [Lucas'15, Lucas et al'16]: hydro around inhomogeneous background. No modification of Ward identity necessary. See also [Davison, Schalm & Zaanen'13]

$$\Gamma \sim \frac{1}{\sigma_Q} + \frac{\eta}{\xi^2}$$

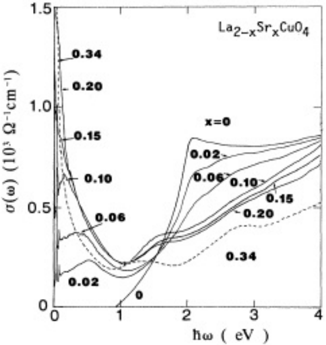


[Lucas'15]

Unconventional electric transport in LSCO



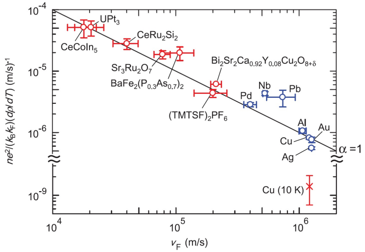
[COOPER ET AL, SCIENCE'09]



[UCHIDA ET AL, PRB'91]

- **Sharp peak** at low frequencies for high doping: quasiparticles, resistivity quadratic in T, Fermi liquid.
- At intermediate doping, the peak decreases and broadens out: strong coupling regime, resistivity linear in T, no quasiparticles.
- Need effective theory of transport without quasiparticles.

Universality of T-linearity



[Bruin et al'16]

- Many dirty metals, both conventional and unconventional, display a resistivity linear in T, [Bruin et al'16]. Planckian timescale: $\tau_P \sim \hbar/k_B T$, [Damle & Sachdev'97, Zaanen'04].

- These results can be reformulated in term of a bound saturated by the charge diffusion constant [Hartnoll'14]:

$$D \gtrsim \hbar v^2/k_B T$$

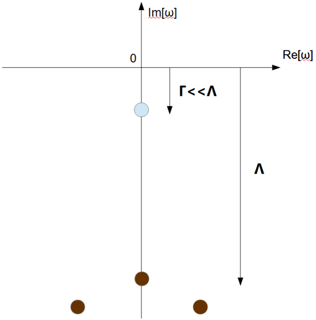
- What is v in a strongly-coupled system? Recent proposal [Blake'16]: butterfly velocity.

Momentum relaxation vs diffusion [Hartnoll et al'13, Hartnoll'14]

Slow momentum relaxation $\Gamma \ll \Lambda$

Transport dominated by momentum relaxation with rate Γ

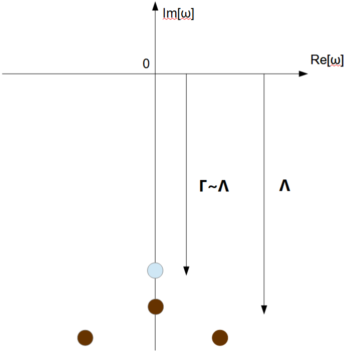
Purely imaginary pole $\omega \sim -\imath\Gamma$ close to the real axis $\Rightarrow$ sharp peak



Fast momentum relaxation $\Gamma \gtrsim \Lambda$

Transport dominated by energy/charge diffusion

No long-lived low energy collective excitation $\Rightarrow$ broad peak $\sim$ constant optical conductivity



Holographic models of momentum relaxation

- Holography ideally suited to transport computations.
- Allows non-perturbative computations. No need for hydrodynamic approximation etc. Does not assume quasiparticles: non-Boltzman equation treatment.
- Need holographic models relaxing momentum. By now there are many: lattices, disorder, Bianchi VII, massive gravity, axions, Q-lattices. [Amoretti, Andrade, Arean, Blake, Baggioli, Davison, Donos, Gauntlett, Ge, Gentle, B.G., Grozdanov, Hartnoll, Horowitz, K-Y Kim, Kiritsis, Krikun, Ling, Lucas, Musso, Pando Zayas, Phillips, Pujolas, Rangamani, Rozali, Sachdev, Salazar Landea, Santos, Schalm, Tong, Vegh, Withers, Zaanen...]

The axion model

$$S = \int dx^4 \sqrt{-g} \left[R - \frac{1}{2}\partial\phi^2 - \frac{1}{4}Z(\phi)F^2 - V(\phi) - \frac{1}{2}Y(\phi)\partial\vec{\psi}^2 \right]$$

- Turn on a scalar source linear in the boundary spatial coordinates: $\psi_i = mx^i$. For a massless bulk scalar, this is actually a solution for the whole bulk [Bardoux et al'12, Andrade & Withers'13].

- The Ward identity is modified

$$\nabla_\mu \langle T^{\mu\nu} \rangle = \partial_\nu \psi_i^0 \langle O_{\psi_i} \rangle$$

- Projecting along $\nu = x^i$ shows momentum is relaxed.

Horizon formulæ for holographic DC conductivities

- The DC conductivity can be computed in terms of horizon data, exact in m [Aristos' talk tomorrow]:

$$\sigma_{DC} = s^{(d-2)/d} Z(\phi_h) + \frac{\rho^2}{s m^2 Y(\phi_h)}$$

- For slow momentum relaxation $m \gg T$, the second term dominates. If $m = 0$, it diverges: related to the weight of $\omega = 0$ pole?

- For fast momentum relaxation $m \ll T$, the first term dominates.

- Similar formulæ in all holographic momentum relaxation models [Aristos' talk tomorrow].

- Scaling solutions exist at low temperatures, parameterized by (θ, z, ζ) and whether the axion deformation is marginal or irrelevant in the IR [B.G.'14, Donos & Gauntlett'14].

- Both metallic and insulating behaviours can be found.

Interpretation of formula

$$\sigma_{DC} = s^{(d-2)/d} Z(\phi_h) + \frac{\rho^2}{s m^2 Y(\phi_h)}$$

Compare with the formula from quasi-hydro [Hartnoll & al '07]

$$\sigma = \sigma_Q + \frac{\rho^2}{\epsilon + p} \frac{1}{\Gamma}$$

$$\Rightarrow \qquad \sigma_Q = s^{(d-2)/d} Z(\phi_h)\,, \qquad \Gamma = s m^2 Y(\phi_h)/(\epsilon + p)$$

But we calculated before [Jain'10, Davison, B.G. & Hartnoll'15]

$$\sigma_Q = s^{(d-2)/d} Z(\phi_h) \left(\frac{sT}{\epsilon + p} \right)^2$$

Slow momentum relaxation AC conductivity

- Set the dilaton to zero: analytical black brane [Bardoux et al'12, Andrade & Withers'13].

- The gauge-invariant perturbations decouple ($\psi_\pm$): two orthogonal currents $J_\pm$ [Davison & B.G.'15].

- $J_+ = J - \rho P/(\epsilon + p)$, $J_- = P$ for $m \ll T$!

- Can compute the AC conductivity in a small $\omega, m \ll T$ expansion.

$$\sigma_+ = \sigma_Q + O(\omega, \Gamma)\,, \qquad \sigma_- = \frac{\frac{\rho^2}{\epsilon+p} + \Gamma(1-\sigma_Q + \lambda\mu^2)}{\Gamma - i\omega} + O(\omega, \Gamma)\,,$$

where

$$\Gamma = \frac{sm^2}{4\pi(\epsilon+p)}(1 + \lambda m^2 + O(m^4, \mu^6))$$

- Quasi hydrodynamics [Hartnoll & al'07] does not capture all corrections at $O(\Gamma^0)$. See also fluid/gravity calculation by [Blake'15].

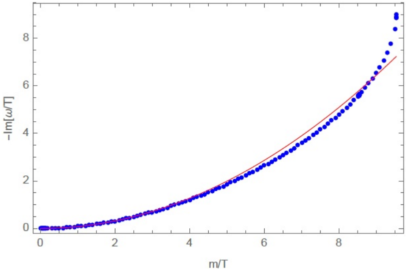
Quantitative comparison for the decay rate (1)

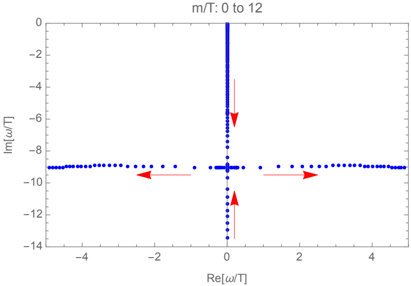
We have obtained the following expression for the decay rate

$$\Gamma = \frac{sm^2}{4\pi(\epsilon + p)}(1 + \lambda m^2 + O(m^4, \mu^6))$$

How good is our formula? We can compare it to the exact location of the pole determined numerically.

At zero density [Davison & B.G.'14], the agreement is excellent up to $m/T \simeq 4$: crossover to the incoherent regime. This happens some time before the pole collision with another purely imaginary pole.

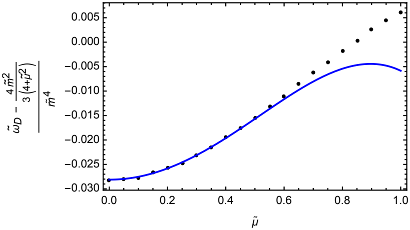


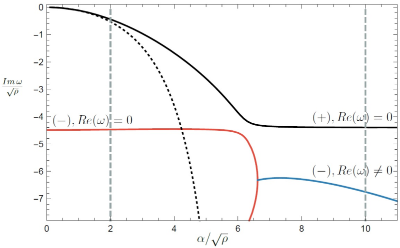


Quantitative comparison for the decay rate (2)

The agreement is also very good at non-zero density [Davison & B.G.'15].

There is no longer a collision involving the Drude pole when entering the incoherent regime [Withers'16].





Fast momentum relaxation

- The two decoupled currents become $J_+ = J$, $J_- = Q$. This means the matrix of thermoelectric conductivities diagonalizes:

$$\begin{pmatrix} J \\ Q \end{pmatrix} = \begin{pmatrix} \sigma & O(1/m) \\ O(1/m) & \bar{\kappa} T \end{pmatrix} \begin{pmatrix} E \\ -\nabla T/T \end{pmatrix}.$$

  So the heat and electric currents decouple at leading order in $1/m$.

- The diffusion constants are also diagonal (heat, charge) and are given by Einstein relations $D = \sigma/\chi$.

- General or specific to this holographic model?

- Important question which could lead to progress in understanding the bound of [Hartnoll'14] applied to incoherent black holes [Blake'16].

1. Zero density
2. Finite density, conserved U(1)
3. Finite density, conserved U(1), broken translations
4. Finite density, spontaneously broken U(1)
5. Finite density, spontaneously broken translation symmetry: CDWs

Superfluidity

- In a superfluid, a U(1) symmetry is spontaneously broken: a complex order parameter condenses.

- Its phase is a Goldstone boson and by gauge invariance

$$\dot{\phi} = -\mu$$

Taking a spatial derivative

$$\dot{u}_\phi = -\frac{1}{m}\nabla\mu\,, \qquad u_\phi = \frac{1}{m}\nabla\phi$$

A phase gradient sources an electric field! The (conserved) superfluid current couples to the electric current.

- The electric conductivity contains a normal and a superfluid delta function

$$\sigma(\omega) = \sigma_0 + \frac{\rho_n^2}{(\epsilon + p - \mu\rho_s)}\frac{i}{\omega} + \frac{\rho_s}{\mu}\frac{i}{\omega}$$

Holographic superfluids with momentum relaxation

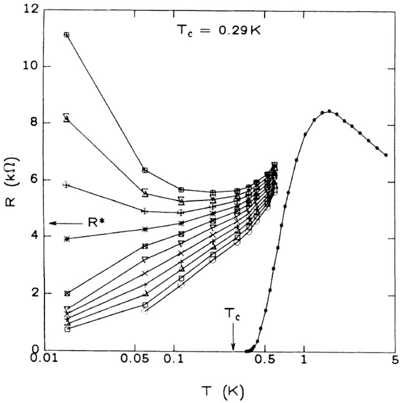
Naive expectation

$$\sigma(\omega) = \tilde{\sigma}_0 + \frac{K_n}{\Gamma - i\omega} + \frac{iK_s}{\omega}$$

- This is indeed what is seen with holographic lattices [Horowitz and Santos'13], Q-lattices [Ling et al'14] or within the axion model [Andrade & Gentle'14], [Kim, Kim & Park'14].
- Many open questions: superfluid hydrodynamics with slowly relaxing momentum? Fate of superfluid/normal densities, as well as horizon/bulk charge? Collective excitations when momentum is strongly relaxed? Quantum critical superfluid phases? etc.
- What about the superfluid delta function? Can this be relaxed? Relevant? Yes! [Davison, Delacrétaz, B.G. & Hartnoll'16]

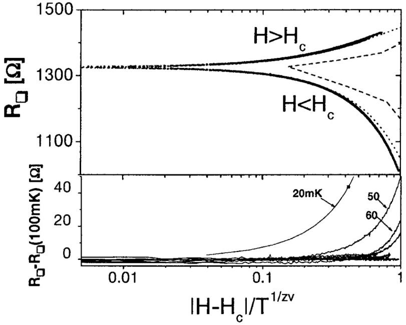
Superfluid/insulator transitions in thin superfluid films

Superfluidity is destroyed in 2D superfluid films upon turning up a large enough magnetic film.



[Hebard & Paalanen'90]

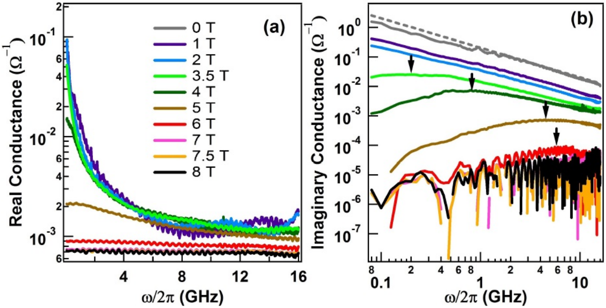
For weakly-disordered films, a $T = 0$ intermediate metallic phase appears.



[Mason & Kapitulnik'99]

AC measurements: peaks in the conductivity

Sharp Drude-like peaks appear in the real part of the conductivity.
The superfluid $1/\omega$ pole in the imaginary part is resolved.

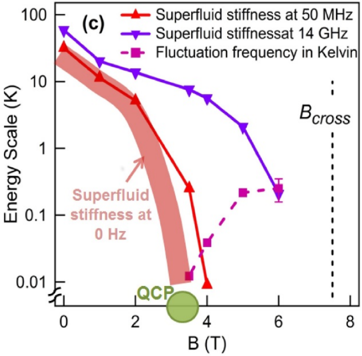


[Liu-Pan-Wen-Kim-Sambandamurthy-Armitage'13]

The width Ω of the Drude-like peak depends on the magnetic field B and vanishes when superfluidity is restored: suggests quantum origin.

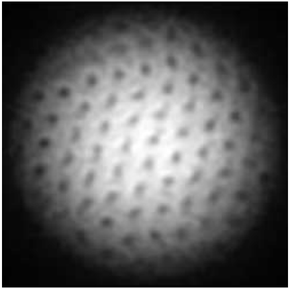
Quantum critical point at the superfluid/metal transition?

[Liu-Pan-Wen-Kim-Sambandamurthy-Armitage'13]



Vortices in two dimensions

- At finite temperature, vortices can proliferate due to thermal fluctuations and destroy quasi long range order (BKT transition).
- At a vortex, the amplitude of the order parameter vanishes.



[CREDIT: ANDRE SCHIROTZEK (MIT)]

- This requires that the circulation of the superfluid velocity is quantized

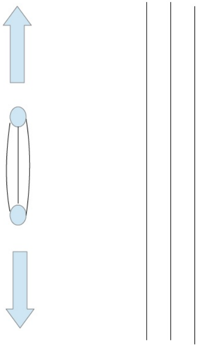
$$\oint_{\text{vortex}} u_\phi = \frac{2\pi n}{m}$$

- At a vortex location, the superfluid velocity is no longer a pure gradient

$$u_\phi = \frac{1}{m}\left(\nabla\phi + \epsilon \times \nabla\psi\right)$$

Vortices in two dimensions

- Vortices are nucleated in pairs of vortices anti-vortices: pinned vortices do not relax the supercurrent.
- Mobile vortices will relax the supercurrent, $\partial_t u_\phi \neq 0$, by (un)winding the phase.



- As vortex cores are not superconducting, expect that mobile vortices produce dissipation and regulate the conductivity

$$\sigma = \sigma_0 + \frac{\rho_s}{m^2} \frac{1}{-i\omega + \Omega}$$

Classically [Bardeen & Stephen'65]

$$\Omega \sim \frac{n_f}{\sigma_n}$$

Decay rate from the memory matrix formalism

- In our case, the slow operators are J, J_ϕ and J_Q (not momentum).

- Let us consider a slowly-decaying supercurrent:

$$H = H_0 + \varepsilon \Delta H , \qquad \varepsilon \ll 1$$

$$\partial_t J_\phi = \varepsilon i \left[\Delta H, J_\phi \right] , \qquad J_\phi = \frac{1}{m} \int_{T^2 \backslash \{\text{vortex cores}\}} d^2 x \nabla \phi$$

- The conductivity is Drude-like

$$\sigma = \sigma_0 + \frac{\rho_s}{m^2} \frac{1}{-i\omega + \Omega}$$

- Importantly, we also get a formula for Ω

$$\Omega = \rho_s M_{J_\phi J_\phi} = \varepsilon^2 \rho_s \lim_{\omega \to 0} \frac{\mathrm{Im}\, G^R_{i[\Delta H, J_\phi] i[\Delta H, J_\phi]}}{\omega} \ll k_B T$$

Onsite Coulomb interaction

- A natural starting point is the commutation relation between the charge density and the phase, which are canonical variables

$$[\phi(x), \rho(y)] = i\delta(x-y)$$

- This relation encodes a Heisenberg uncertainty relation:

$$\Delta\phi\Delta\rho \gtrsim \hbar$$

- Coulomb interactions penalize charge density fluctuations [Efetov'80, Doniach'81... Emery-Kivelson'95]. Consequently phase fluctuations are enhanced.
- A simple choice is an on-site density-density interaction [Doniach'84, Sachdev-Starykh'00]

$$\Delta H = \frac{1}{2\chi_{\rho\rho}} \int d^2x\, \rho(x)^2$$

- Since $\int_{T^2} d^2x \nabla\rho(x) = 0$, we obtain

$$\partial_t J_\phi = \frac{1}{m\chi_{\rho\rho}} \int_{T^2\setminus\{\text{vortex cores}\}} \nabla\rho(x) = -\frac{1}{m\chi_{\rho\rho}} \int_{\{\text{vortex cores}\}} \nabla\rho(x)$$

Decay rate due to onsite Coulomb interaction

- The calculation of the decay rate

$$\Omega = \varepsilon^2 \frac{\rho_s}{m^2 \chi_{\rho\rho}} \lim_{\omega \to 0} \frac{\mathrm{Im}\, G^R_{\rho\rho}}{\omega}$$

  now involves knowledge of the density-density retarded Green's function **of the normal fluid inside the vortex cores**, for which we assume the form (neglecting thermoelectric effects)

$$G^R_{\rho\rho} = \frac{k^2 D \chi_{\rho\rho}}{-i\omega + Dk^2}\,, \qquad D = \frac{\sigma_n}{\chi_{\rho\rho}}$$

- We can compute the decay rate due to this interaction as

$$\Omega = \frac{\rho_s}{m^2} \frac{n_f \pi r_v^2}{2\sigma_n}$$

  where r_v is the typical radius of a vortex core, n_f the vortex density and σ_n the conductivity of the normal state.

- This is exactly [Bardeen-Stephen'65], in a fully quantum treatment.

Open directions

- In the last few slides: transparent, fully quantum formalism for quantum fluctuating superconductivity. Important for experiments.
- The computation of the phase relaxation rate involves knowledge of the phase relaxation mechanisms: vortices are one possibility, other interactions exist (Chern-Simons).
- Holographic superfluids have been a popular topic of study in the past few years. The presence of a superfluid delta function is well established.
- Holographic dirty superfluids exist in the literature, as well as vortex solutions.
- Including phase fluctuations means taking into account $1/N$ corrections. Also need mobile vortices. Sounds like a nice numerical holography challenge, and would give another calculable model of phase relaxation.

Charge density waves

- Many Condensed Matter systems exhibit a phase with spontaneous breaking of translation symmetry, accompanied by formation of periodic charge order. This is true in particular in high T_c superconductors, where this phase competes with the superconducting phase.
- The AC conductivity has very characteristic features: δ-function if no disorder, pinned collective mode in the presence of disorder, sliding density wave and nonlinear conductivity...
- These phases have been constructed holographically [Donos, Gauntlett, Jarvinen, Jokela, Ling, Lippert, Withers, Zhang...].
- Their holographic AC conductivity has not been much studied.