“Holographic DC conductivities and the hydrodynamic limit”

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Based on work in collaboration with:
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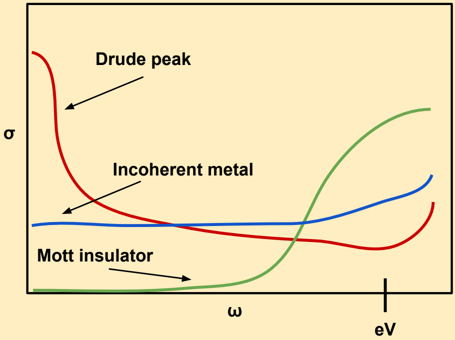
Outline

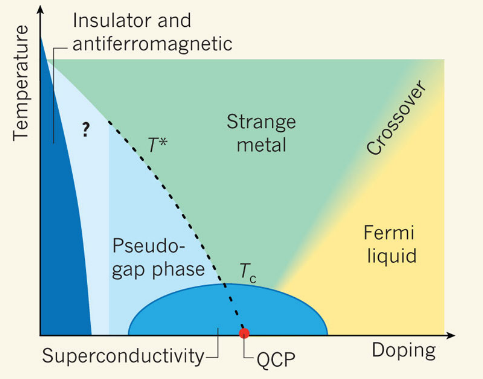
1. Motivation/Setup
2. Holography
3. Transport in magnetised materials
4. Universal (?) limits
   - Perturbative lattices
   - Hydrodynamic limit
5. Summary

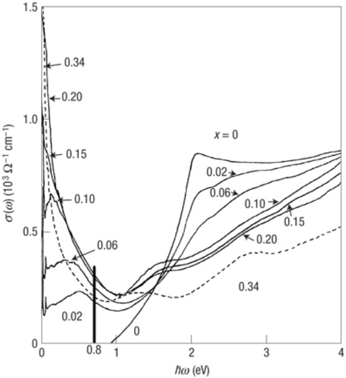
Charge transport in real materials



- Materials with charged d.o.f. can be
  - Coherent metals with a well defined Drude peak
  - Insulators
  - Incoherent conductors of electricity
- Interactions expected to become important in the incoherent phase $\rightarrow$ Possible description in AdS/CFT?

The Cuprates





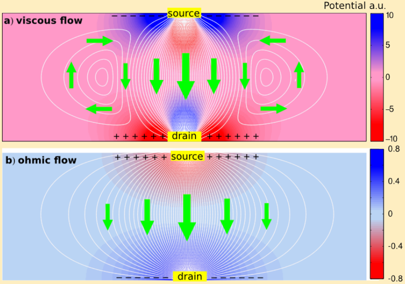
The Cuprates are real life example of :

- Incoherent transport
- Anomalous scaling of conductivity and Hall angle with T [Blake, AD]

$$\sigma_{DC}^{B=0} \propto T^{-1}, \quad \theta_H \propto T^{-2}$$

Electrons as a soup

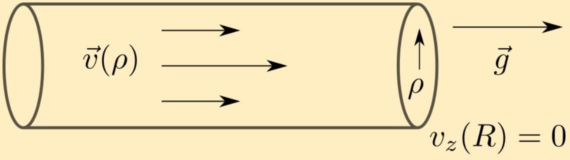
Recent evidence for viscous flows in strongly interacting electrons. [1508.00836],[1509.04165], [1509.05691]



- Hydrodynamics accurate in the high T, momentum (quasi-) conserving regime
- Holography famous for low viscosity/entropy. Can we model backflows in hydro regime?

Electrons as a soup

- A Stoke's flow
  [1509.05691]



$$\cancelto{0}{\partial_t v_i} + \cancelto{0}{v^j \nabla_j v_i} + 2\,\nabla^j \left(\eta \nabla_{(j}\, v_{i)}\right) - \nabla_i p = -g, \quad \nabla_i v^i = 0$$

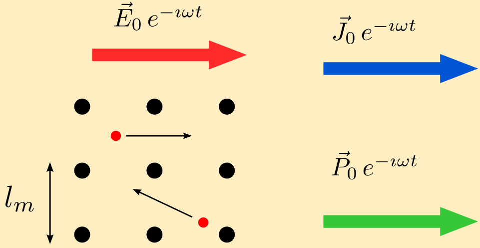
$$v_z = (g/4\eta)\,(R^2 - \rho^2)$$

$$\Rightarrow \sigma_{DC} = \frac{2}{g\,R^2} \int_0^R v_z\,\rho\,d\rho \approx R^2/\eta$$

- In this case σ_{DC} is finite because of no-slip boundary conditions
- Infinite systems need a lattice, viscosity will not help

Drude Model

- Lattice scattering (Drude physics)



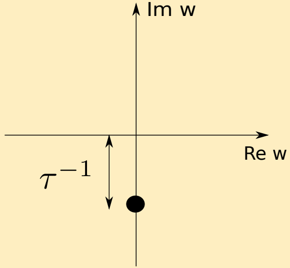
- Average momentum obeys

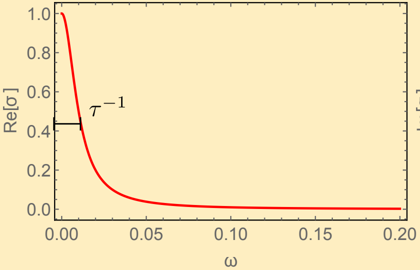
$$\langle \dot{p} \rangle = qE - \frac{1}{\tau} \langle p \rangle \Rightarrow \sigma = \frac{nq^2}{m} \frac{\tau}{1 - \imath\omega\tau} \Rightarrow \sigma_{DC} \approx \tau \approx l_m$$

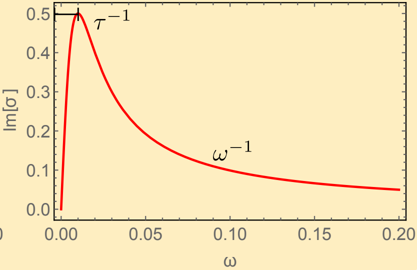
- Microscopically $\sigma = G_{JJ}(\omega)/(\imath\omega)$

Momentum Relaxation

Drude peaks are associated to perturbative breaking of translations. Give a mass to mode associated to boosts. [Hartnoll, Hofman]







Fourier/Ohm law

- We have electric currents J^i and a thermal current $Q^i = -{T^i}_t - \mu\, J^i$
- Transport coefficients are packaged in Ohm/Fourier law

$$\left(\begin{array}{c} J \\ Q \end{array}\right) = \left(\begin{array}{cc} \sigma & \alpha T \\ \bar{\alpha} T & \bar{\kappa} T \end{array}\right) \left(\begin{array}{c} E \\ -(\nabla T)/T \end{array}\right)$$

- With ∇T a temperature gradient

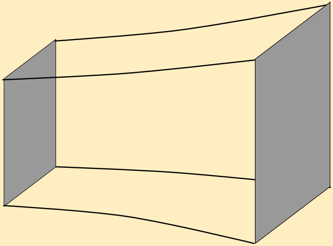
Setup

In $D = 4$ Einstein-Maxwell with AdS asymptotics:

$$
\begin{aligned}
\mathcal{L}_{EM} &= R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 12 \\
ds_4^2 &= -U(r)\, dt^2 + U(r)^{-1}\, dr^2 + r^2 \left(dx_1^2 + dx_2^2 \right) \\
A &= a(r)\, dt + B\, x\, dy
\end{aligned}
$$

$r = r_+$

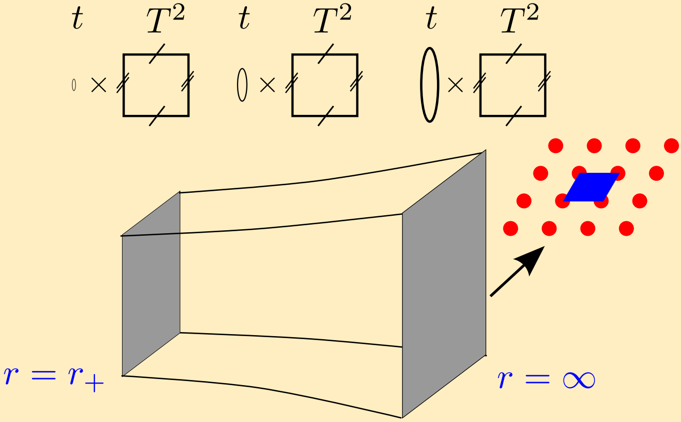
$r = \infty$



$a = \mu - q\, r^{-1} + \cdots$

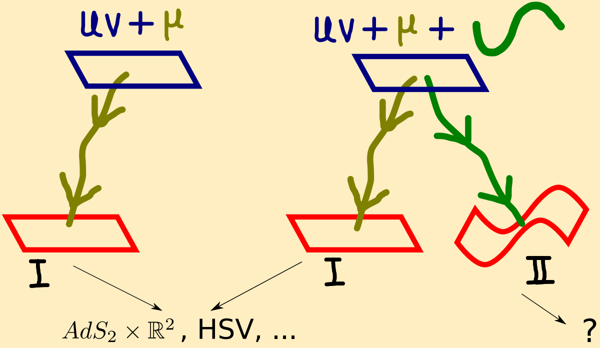
Background black hole has temperature T , energy E, pressure P, entropy s and charge q.

Setup



- Introduce periodic lattice (deformation) on the boundary
- Focus on simple black hole topologies
- More general statements
  [AD, Gauntlett, Griffin, Melgar]

RG/Holographic picture

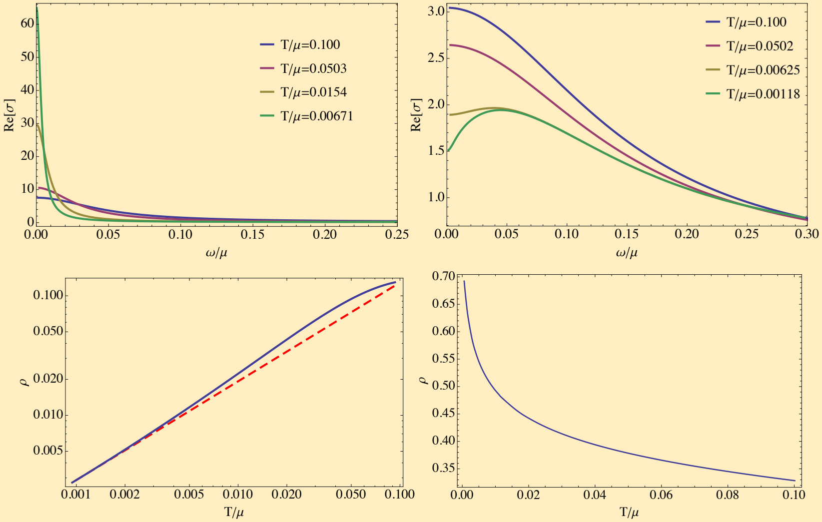


I Charge dominated RG flows, translations restored in IR $\rightarrow$ Coherent transport

II Lattice dominated RG flows, translations broken in IR $\rightarrow$ Incoherent transport / Insulators

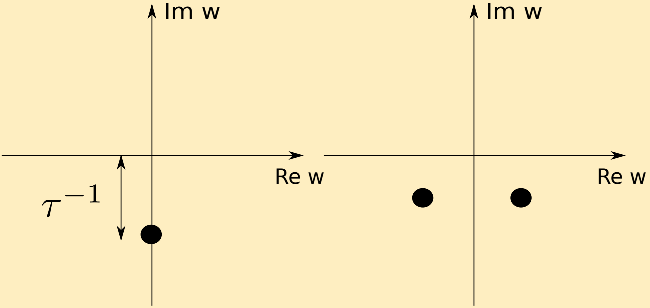
[AD, Hartnoll] [AD, Gauntlett]

Conductivity from Q-lattices [AD, Gauntlett]



- Can model Metal - Insulator transitions
- Similar story for inhomogeneous lattices [Rangamani, Rozali, Smyth]

QNM Point of view



- QNM on the axis $\rightarrow$ coherent transport
- QNM off the axis $\rightarrow$ incoherent transport
- Recent check and also transition T [Davison, Gouteraux]

Setup

- Deform by chemical potential μ_0 and magnetic field B
- Hold at finite temperature T
- Introduce periodic sources that can relax momentum:
  - Local chemical potential $\nabla\mu$
  - Local temperature ∇T
  - Magnetic impurities
  - Local stress + rotation
- Probe with external electric field $\nabla\delta\mu = E$ and thermal gradient $-\nabla\delta T/T = \zeta$ to extract conductivities

Currents At Equilibrium

- For homogeneous systems we have T_0, μ_0, $\vec{B}_0$, ...
- First consider hydrodynamic limit
- Weakly break translations $\mu_0 \to \mu_0 + \delta\mu(x)$, $\vec{B}_0 \to \vec{B}_0 + \delta\vec{B}(x)$, $T \to T + \delta\mu(x)$
- In hydrodynamic limit, magnetization becomes local

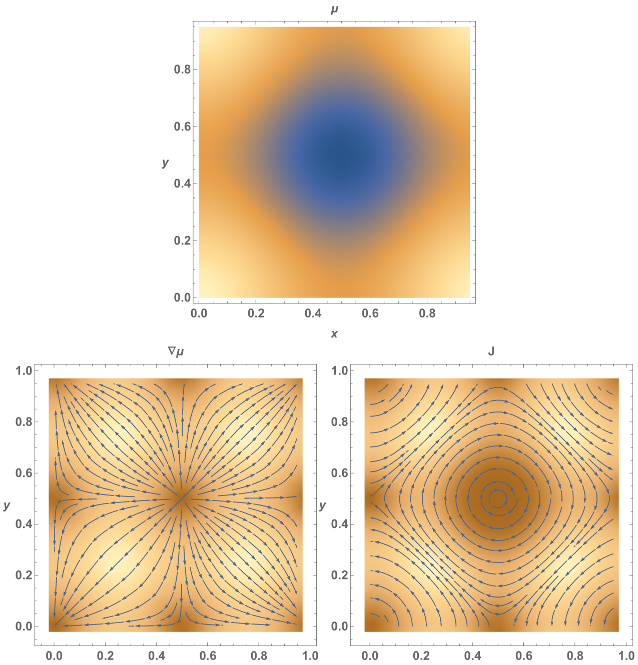
$$\delta\vec{M} = \partial_\mu \vec{M}_0 \, \delta\mu(x) + \partial_T \vec{M}_0 \, \delta T(x) + \cdots$$

$\Rightarrow$ Presence of local magnetization currents

$$\vec{J} = \vec{\nabla} \times \delta\vec{M}$$

Similar for heat currents

Currents At Equilibrium



Currents At Equilibrium

In the non-hydrodynamic regime $k = \partial_t$ is a symmetry

$$\mathcal{L}_k * J = 0 \Rightarrow i_k(d * J) + d(i_k * J) = 0$$
$$d(i_k * J) = 0$$

Assuming $\mathcal{R}_t \times M_{D-1}$ topology

$$i_k * J = d *_{D-1} M + \omega$$

with ω harmonic. Currents relax

$$\int_{C_{D-2}} i_k * J = 0 \Rightarrow \omega = 0 \Rightarrow J^i = \partial_j(\sqrt{g_{D-1}} M^{ij})$$

Similarly for the heat current

$$Q^i = -T^i{}_\mu k^\mu - k^\mu A_\mu J^i = \partial_j(\sqrt{g_{D-1}} M_T^{ij})$$

DC conductivities from BH horizons

- Bulk theory is Einstein-Maxwell
- Consider E/M charged, static black branes

$$ds^2 = -UG\,(dt+\chi)^2 + \frac{F}{U}\,dr^2 + ds^2(\Sigma_d)$$

$$A = a_t\,(dt+\chi) + a_i\,dx^i$$

$$ds^2(\Sigma_d) = g_{ij}(r,x)dx^i dx^j$$

- Asymptotically, $r \to \infty$

$$U \to r^2, \qquad F \to 1$$

$$a_t(r,x) \to \mu(x), \qquad a_i(r,x) \to a_i(x)$$

$$G \to \bar{G}(x), \qquad g_{ij}(r,x) \to r^2 \bar{g}_{ij}(x), \quad \chi_i(r,x) \to \bar{\chi}_i(x)$$

- Local μ, B, T, mag impurities, surface forces

DC conductivities from BH horizons

DC sources at infinity

$$\delta ds^2 = -2\,\delta\phi\,g_{tt}\,dt^2, \quad \delta A_t = -\delta\phi\,\mu$$
$$F_{ti} = -E_i$$

In the DC limit

$$\partial_t \phi = 0, \qquad \zeta_i = -\partial_i \ln T = \partial_i \delta\phi$$
$$\partial_t E_i = 0, \qquad \partial_{[j} E_{i]} = 0$$

Convenient to redefine $t \to (1 + \delta\phi)t$ mapping the sources to

$$\delta ds^2 = -2\,g_{tt}\,t\,\zeta_i\,dt dx^i, \quad \delta A_i = t\zeta_i\,\mu\,dx^i$$
$$\delta A = -t\,E_i dx^i$$

DC conductivities from BH horizons

For the perturbation write

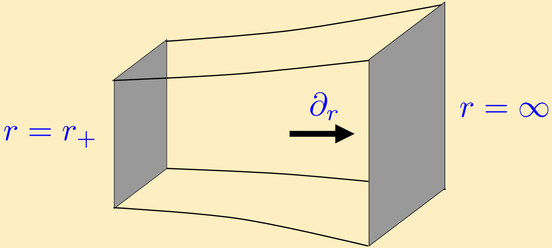
$$\delta(ds^2) = \delta g_{\mu\nu}(r,x)dx^\mu dx^\nu - 2tGU\zeta_i dx^i\,(dt+\chi),$$
$$\delta A = \delta a_\mu(r,x)dx^\mu - tE_i dx^i + ta_t\zeta_i dx^i$$

- $E(x^i)$ and $\zeta(x^i)$ are closed forms
- ζ is boundary temperature gradient
- E is boundary electric field
- Count functions:
  - $g_{\mu\nu} \to \frac{1}{2}\,(d+2)\,(d+3) - (d+2)$ functions
  - $A_\mu \to (d+2) - 1$ functions

Radial Hamiltonian

- Imagine radial foliation by hypersurfaces e.g. normal to ∂_r
- Radial evolution Hamiltonian is sum of constraints

$$H_{\partial_r} = \int N\,\mathcal{H} + N_\mu\,\mathcal{H}^\mu + D\,\mathcal{G} + \mathrm{b.t.}$$

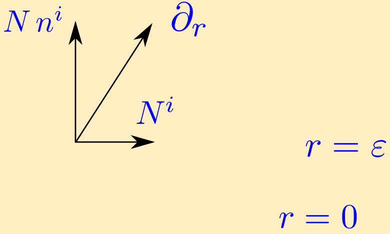


- At infinity they yield Ward identities

$$\nabla_\mu \langle T^{\mu\nu} \rangle = F^{\mu\nu}\ \langle J_\nu \rangle\,, \qquad \nabla_\mu \langle J^\mu \rangle = 0, \qquad \langle T^\mu{}_\mu \rangle = \mathrm{anom}$$

- Meaningful but not closed system without hydro

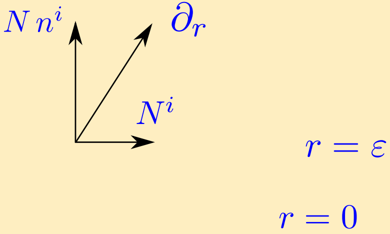
DC conductivities from BH horizons



- Projection of metric $h_{\mu\nu}$ and gauge field b_μ on $r = \varepsilon$ surface
- Conjugate momentum densities $\pi^{\mu\nu}$ and π^μ with respect to ∂_r
- "Evolution" equations

$$\dot{h}_{\mu\nu} = \frac{\delta H_{\partial_r}}{\delta \pi^{\mu\nu}}, \quad \dot{\pi}^{\mu\nu} = -\frac{\delta H_{\partial_r}}{\delta h_{\mu\nu}}$$
$$\dot{b}_\mu = \frac{\delta H_{\partial_r}}{\delta \pi^\mu}, \quad \dot{\pi}^\mu = -\frac{\delta H_{\partial_r}}{\delta b_\mu}$$

DC conductivities from BH horizons



- And constraints

$$\mathcal{H}_\nu = D_\mu t^\mu{}_\nu - \frac{1}{2} f_{\nu\rho} j^\rho = 0$$
$$\mathcal{G} = D_\mu j^\mu = 0$$

- With $t^{\mu\nu} = (-h)^{-1/2}\, \pi^{\mu\nu}$ and $j^\mu = (-h)^{-1/2}\, \pi^\mu$.
- Continuity equations on the surface

DC conductivities from BH horizons

Examine constraints close to the horizon

- Impose infalling conditions
- Define

$$v_i \equiv -\delta g_{it}^{(0)}, \qquad w \equiv \delta a_t^{(0)},$$

$$p \equiv -4\pi T \frac{\delta g_{rt}^{(0)}}{G^{(0)}} - \delta g_{it}^{(0)} g_{(0)}^{ij} \nabla_j \ln G^{(0)}$$

DC conductivities from BH horizons

Constraints on the horizon give

$$\mathcal{H}^t \Rightarrow \quad \nabla_i v^i = 0$$

$$\mathcal{G} \Rightarrow \quad \nabla^2 w + \nabla_i (F^{(0)i}{}_k v^k) + v^i \nabla_i a_t^{(0)} = -\nabla_i E^i$$

$$\begin{aligned} \mathcal{H}^j \Rightarrow \quad & 2\,\nabla^i \nabla_{(i} v_{j)} + a_t^{(0)} \nabla_j w - \nabla_j\, p \\ & + 4\pi T\, d\chi^{(0)}_{ji}\, v^i + F^{(0)}_{ji}\, (\nabla^i w + a_t^{(0)} v^i + F^{i(0)}{}_k\, v^k) \\ & = -4\pi T\, \zeta_j - a_t^{(0)} E_j - F^{(0)}_{ji} E^i \end{aligned}$$

- Solve for a Stokes flow on the curved black hole horizon
- Closed system of equations in d dimensions
- Nowhere made hydro assumptions!
- Related work

  [Damour][Thorne, Price][Eling, Oz][Bredberg, Keeler, Lysov, Strominger]

DC conductivities from BH horizons

Electric Current

Define

$$J^i = \sqrt{-g} F^{ir}$$

- At $r \to \infty$ gives field theory current densities J^i_∞
- Anywhere in the bulk

$$\partial_r J^i = \partial_j \left(\sqrt{-g} F^{ji}\right) + \sqrt{-g}\, F^{ij}\, \zeta_j$$
$$\partial_i J^i = J^i \zeta_i$$

DC conductivities from BH horizons

Heat Current

Let $k = \partial_t$ and define

$$G^{\mu\nu} = -2\,\nabla^{[\mu}k^{\nu]} - k^{[\mu}F^{\nu]\sigma}A_\sigma - \tfrac{1}{2}\,(\phi - \theta)\,F^{\mu\nu}$$

and

$$Q^i = \sqrt{-g}G^{ir}$$

- At $r \to \infty$ gives field theory heat current densities

$$Q^i_\infty = -\left\langle \delta T^i{}_t \right\rangle - \mu\,\left\langle \delta J^i \right\rangle$$

- Anywhere in the bulk

$$\partial_r Q^i = \partial_j\left(\sqrt{-g}G^{ji}\right) + 2\sqrt{-g}G^{ij}\zeta_j + \sqrt{-g}ZF^{ij}E_j$$
$$\partial_i Q^i = 2Q^i\zeta_j + J^iE_i$$

DC Conductivities from BH horizons

For the background ($E_i = \zeta_i = 0$) we have

$$J_\infty^{(B)i} = \partial_j M^{(B)ij}, \quad Q_\infty^{(B)i} = \partial_j M_T^{(B)ij}$$

with the magnetizations

$$M^{ij}(x) = -\int_{r_+}^{\infty} dr \sqrt{-g}\, F^{ij}, \quad M_T^{ij}(x) = -\int_{r_+}^{\infty} dr \sqrt{-g}\, G^{ij}$$

satisfying

$$\partial_i J_\infty^{(B)i} = 0, \quad \partial_i Q_\infty^{(B)i} = 0$$

and giving no net fluxes!

DC Conductivities from BH horizons

Back to perturbations we write...

$$J^i_\infty = J^i_{(0)} + \partial_j M^{ij} - M^{(B)ij}\zeta_j$$
$$Q^i_\infty = Q^i_{(0)} + \partial_j M^{ij}_T - M^{(B)ij}E_j - 2\,M^{(B)ij}_T\zeta_j$$

The "transport components" of the currents are then

$$\mathcal{J}^i_\infty = J^i_{(0)}, \quad \mathcal{Q}^i_\infty = Q^i_{(0)}$$

Important point is

$$\partial_i \mathcal{J}^i_\infty = 0, \quad \partial_i \mathcal{Q}^i_\infty = 0$$

- Meaningful to examine fluxes through $d-1$ cycles!
- The currents on the horizon are the transport currents!

DC conductivities from BH horizons

- Solutions for v^i, w and p are uniquely fixed by sources E and ζ
- Then

$$J^i_{(0)} = \frac{s}{4\pi}\left(\partial^i w + E^i + F^{(0)i}{}_j v^j\right) + \rho\, v^i$$

$$Q^i_{(0)} = T s\, v^i, \quad s = 4\pi\sqrt{g_{(0)}}, \quad \rho = \sqrt{g_{(0)}}\, a_t^{(0)}$$

- To find field theory currents $\bar{\mathcal{J}}^i_\infty$ and $\bar{\mathcal{Q}}^i_\infty$ in e.g. $d=2$

$$\bar{\mathcal{J}}^1_\infty = \int dx^2\, \mathcal{J}^1_\infty, \quad \bar{\mathcal{J}}^2_\infty = \int dx^1\, \mathcal{J}^2_\infty$$

- Conductivities determined by BH horizon data!

Hydro temptation

- Meaningful quantities are

$$Q = \mathrm{vol}_d^{-1} \int \sqrt{g_{(0)}}\, a_t^{(0)}, \quad S = \mathrm{vol}_d^{-1} \int 4\pi \sqrt{g_{(0)}}$$

- Can also write as

$$\begin{aligned}
&\nabla_i v^i = 0 \\
&\nabla^2 \delta\mu + v^i \nabla_i \rho + \nabla_i (F^{(0)i}{}_k v^k) = -\nabla_i E^i \\
&2\,\nabla^i \nabla_{(i} v_{j)} + d\chi^{(0)}_{ji}\, Q^i_{(0)} + F^{(0)}_{ji}\, J^i_{(0)} \\
&\qquad = T\, s\, (\zeta_j + T^{-1} \nabla_j \delta T) + \rho\, (E_j + \nabla_j \delta\mu)
\end{aligned}$$

- Tempting to see it as first order hydro
- It does give an exact answer!

DC conductivities from BH horizons

Can show (strict) positivity of transport coefficients:

$$0 < \int d^d x \sqrt{h^{(0)}} \Big(2\nabla^{(i} v^{j)} \nabla_{(i} v_{j)} + |E_i + \nabla_i w + F^{(0)}_{ij} v^j|^2 \Big)$$

$$= \int d^d x (J^i_{(0)} E_i + Q^i_{(0)} \zeta_i)$$

$$= \begin{pmatrix} \bar{E}_i & \bar{\zeta}_i \end{pmatrix} \begin{pmatrix} \sigma^{ij} & \alpha^{ij}\, T \\ \bar{\alpha}^{ij}\, T & \bar{\kappa}^{ij}\, T \end{pmatrix} \begin{pmatrix} \bar{E}_i \\ \bar{\zeta}_i \end{pmatrix}$$

In the absence of Killing vectors

$$\mathcal{L}_v g^{(0)}_{ij} = 2\,\nabla_{(i} v_{j)} = 0, \quad \mathcal{L}_v a^{(0)}_t = 0$$

- The eigenvalues are positive definite... No insulators at finite T with regular BH horizons.
- In some cases can come up with specific numbers for the bound.

  [Grozdanov, Lucas, Sachdev, Schalm]

Onsager relations

We can easily find the time reversed background bh horizons by simply

$$\chi_i^0 \to -\chi_i^{(0)}, \quad F_{ij}^{(0)} \to -F_{ij}^{(0)}$$

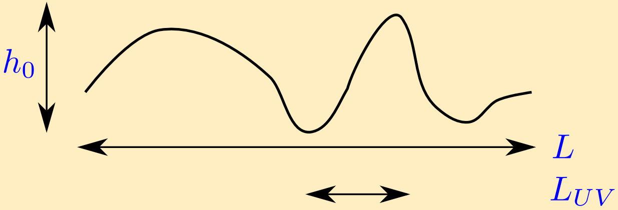
- The transport coefficients of the new geometry are simply related to the original ones through Onsager relations

$$\begin{pmatrix} \tilde{\sigma} & \tilde{\alpha} \\ \tilde{\bar{\alpha}} & \tilde{\bar{\kappa}} \end{pmatrix} = \begin{pmatrix} \sigma & \alpha \\ \bar{\alpha} & \bar{\kappa} \end{pmatrix}^T$$

- If the background is symmetric under time reversal then these reduce to a relation among the transport coefficients
- Non-obvious after subtracting magnetisation currents in the UV theory. Proof relatively easy!

Interesting limits

$$S = S_{CFT} + \int h(x)\,\mathcal{O}(x)$$



Limits to consider for the background black holes:

- Perturbative lattice $h_0/T^{d-\Delta} << 1$
- Hydrodynamic limit $L_{UV}\,T >> 1$
- ! For marginal operators hydro doesn't imply perturbative

Perturbative lattices

- Consider perturbative, periodic lattices about flat black brane

$$g_{(0)ij} = g\,\delta_{ij} + \lambda\,h^{(1)}_{ij} + \lambda^2\,h^{(2)}_{ij} + \cdots$$
$$a^{(0)}_t = a + \lambda\,a_{(1)} + \lambda^2\,a_{(2)} + \cdots$$

- The bh horizon is a small λ expansion about flat space
- Leading order is homogeneous system, e.g. AdS-RN horizon

Perturbative lattices

- Solve Navier-Stokes perturbatively in λ

$$v^i = \frac{1}{\lambda^2}\, v^i_{(0)} + \frac{1}{\lambda}\, v^i_{(1)} + v^i_{(2)} + \cdots\,, \qquad w = \frac{1}{\lambda}\, w_{(1)} + w_{(2)} + \cdots\,,$$
$$p = \frac{1}{\lambda}\, p_{(1)} + p_{(2)} + \cdots\,.$$

- The leading fluid velocity term dominates the expressions for the currents

$$v^i_{(0)} = (M^{-1})^{ij}\, \left(4\pi T \zeta_j + a E_j\right)$$

- At leading order the currents are homogeneous

$$\bar{J}^i \approx \lambda^{-2}\, \rho v^i_{(0)}\,, \qquad \bar{Q}^i \approx \lambda^{-2}\, T s v^i_{(0)}\,,$$

Perturbative lattices

- The conductivities are

$$\bar{\kappa} = M^{-1}\frac{4\pi s T}{\lambda^2}\,, \qquad \alpha = \bar{\alpha} = M^{-1}\frac{4\pi\rho}{\lambda^2}\,, \qquad \sigma = \lambda^{-2}\,M^{-1}\frac{4\pi\rho^2}{\lambda^2\,s}$$

- M depends on the UV details of the lattice
- They drop out of the Lorenz ratio

$$L = \frac{\bar{\kappa}}{\sigma T} = \frac{s^2}{\rho^2}$$

- Wiedemann - Franz law conjectures $L = \frac{\pi^2 k_B^2}{3\,e^2}$
- More general treatment using memory matrix formalism
  [Mahajan, Barkeshli, Hartnoll]

Summary

- Argued that DC transport currents are fixed by horizon "hydro"
- Some general results on perturbative lattices
- Heat current flows in hydro limit
- Inhomogeneous ground states?
- Holographic insulators with gap?