The M5-brane and Omega-Deformations of the Seiberg-Witten Effective Action

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Exact quantum fields and the structure of M-theory

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“surprisingly memorable”

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with D. Orlando and S. Reffert

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Introduction and Some Review

Introduction

M5-brane is a mysterious and powerful object:

- 6D (2,0) CFT

A major challenge to our understanding of QFT

No perturbative parameters but

- One can put it on manifolds: moduli become
- Can try deforming the theory: e.g. non-commutativity and Ω-deformation

PART I: An M5

An M5-brane in eleven dimensions has five dynamical scalars X^I and a self-dual 3-form h_{mnp} (and fermions)

The (bosonic) equations of [Howe,Sezgin][Howe,Sezgin,West] are (we've set $G_{IJKL} = 0$ for simplicity)

$$(m^2)^{mn}\nabla_m\nabla_n X^I = -\frac{2}{3}\hat{G}^I{}_{mnp}m_r{}^m h^{rnp} \qquad dH = -\frac{1}{4}\hat{G}$$

where

$$m_m{}^n = \delta_m{}^n - 2h_{mpq}h^{npq}$$

where the metric $\hat{g}_{mn}$ is the pull-back of the spacetime metric.

What is H_{mnp}? Well h_{mnp} is self-dual and

$$H_{mnp} = m_m{}^q m_n{}^r h_{qrp}$$

is complicated and satisfies a non-linear self-duality. (But for us $h = H$.)

Why? Consider flat space $dH = 0$ and reduce to D4-brane $F_{\mu\nu} = H_{\mu\nu 5}$.

- Bianchi $dF = 0$ comes from 5-component of $dH = 0$
- non 5-components are $H = \star_5 F + non - linear$
- $dH = 0$ gives non-linear equation which is just

$$d\left(\frac{\star F}{\sqrt{1-F^2}}\right) = 0 \qquad DBI!$$

PART II: The M5-brane, Seiberg-Witten and all that

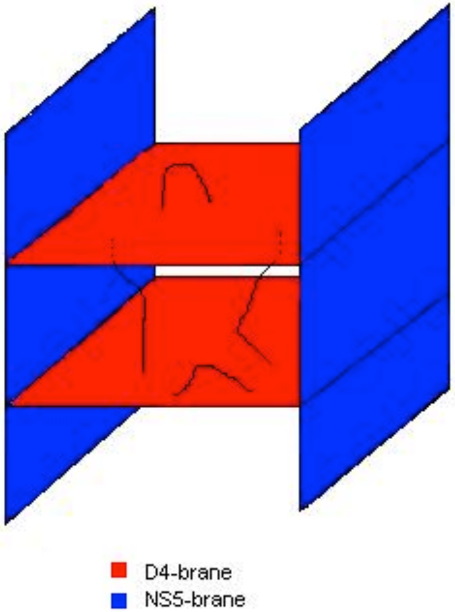
One of the most striking applications of the M5-brane was by [Witten]. Consider the following system of branes in type IIA string theory

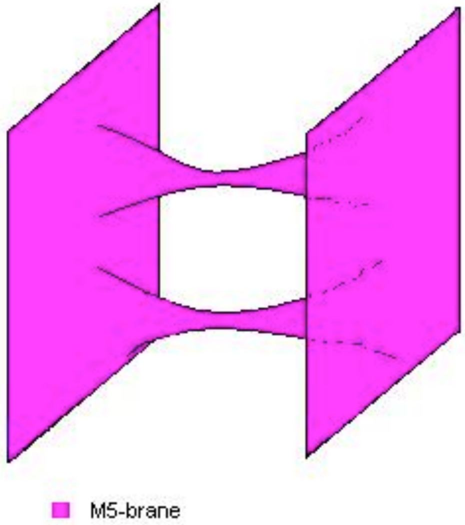
$D4$	0	1	2	3	6				
$NS5$	0	1	2	3		8	9		

is described by $D=4$, $N=2$ Yang-Mills at low energy.

At strong coupling it lifts to a configuration in M-theory consisting only of M5-branes

$M5$	0	1	2	3	6			10
$M5$	0	1	2	3		8	9	


D4-brane
NS5-brane


M5-brane

But moreover this can be viewed as a single smooth M5-brane wrapped on a Riemann Surface Σ

- Σ is precisely the Seiberg-Witten curve!
- gives a systematic construction of Σ from braneology
- explains the appearance of Σ in quantum 4D gauge theory

From the point of view of the M5-brane this configuration appears as a 3-brane soliton. The effective low energy dynamics of this soliton is given by the Seiberg-Witten effective action [Howe, NL, West]

A single classical M5-brane can compute the exact low energy effective action of a 4D quantum gauge theory

- Perturbative part known since early 80's [Gates, Grisaru, Siegel],[Howe, Stelle, West]
- [Seiberg,Witten] gave this plus a prediction for all non-perturbative instanton terms
- The first few terms were then explicitly checked [Dorey, Khose, Mattis]

More recent work has vastly expanded this story Gaiotto, Gukov, Moore, Neitzke...everyone here...

PART II: Enter Ω

Based on previous work, in a seminal paper [Nekrasov] used the so-called Ω-deformation to compute all terms in the Seiberg-Witten prepotential exactly.

What is it? $D = 4$, $N = 2$ SYM is the dimensional reduction of $D = 5$, $N = 1$ SYM

The Ω-deformation is obtained by compactifying the Euclidean theory with a twist $\omega \in so(4)$

$$x^9 \to x^9 + 2\pi R \qquad x^\mu \to x^\mu + R\omega^\mu{}_\nu x^\nu$$

Typically one parameterizes

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \epsilon_1 & & \\ -\epsilon_1 & & & \\ & & 0 & \epsilon_2 \\ & & -\epsilon_1 & 0 \end{pmatrix}$$

This ‘localises the path integral of 4D N=2 SYM to point-like instantons a the origin.

- path integral reduces to zero-mode determinants.
- can be evaluated algebraically using group theory.

Many other applications (Integrable systems, topological strings, AGT)

The Ω-Deformation as a Flux Background

We can realise the Ω deformation as a Euclidean background in String theory [Hellerman,Orlando,Reffert]

Step 1) start with flat $\mathbb{R}^9 \times S^1 = (\mathbb{R}^2)^2 \times \mathbb{R}^5 \times S^1$ in polar coordinates (i=1,2)

$$(r_i, \theta_i),\ \theta_i \cong \theta_i + 2\pi R \qquad x^4, x^5, x^6, x^7, x^8 \qquad x^9$$

and metric

$$ds_{10}^2 = \sum_{i=1}^{3}(dr_i^2 + r_i^2 d\theta_i^2) + (dx^9)^2 + (dx^6)^2 + (dx^7)^2 + (dx^8)^2$$

Impose the orbifold

$$x^9 \cong x^9 + 2\pi R \qquad \theta_i \cong \theta_i + 2\pi\epsilon_i$$

This preserves supersymmetry if $\epsilon_1 = \pm\epsilon_2$.

More general possibilities exist but we will just consider this.

Step 2) introduce new coordinates that diagonalize the orbifold action

$$\phi_i = \theta_i - \frac{\epsilon_i}{R} x^9$$

$$x^9 \cong x^9 + 2\pi R \qquad \phi_i \cong \phi_i + 2\pi$$

But now the metric is off-diagonal.

$$\begin{aligned} ds^2_{10} &= \sum_{i=1}^{3} (dr_i^2 + r_i^2 d\phi_i^2 + \frac{\epsilon_i^2 r_i^2}{R^2}(dx^9)^2 + \frac{2\epsilon_i r_i^2}{R} d\phi_i dx^9) \\ & \quad + (dx^9)^2 + (dx^4)^2 + \ldots + (dx^8)^2 \end{aligned}$$

Step 3) T-dualize along x^9. This produces a non-flat background ($\mu, \nu = 0, 1, 2, 3$)

$$\begin{aligned} ds^2_{10} &= \left(\eta_{\mu\nu} - \frac{U_\mu U_\nu}{\Delta^2}\right) dx^\mu dx^\nu + (dx^4)^2 + \cdots + (dx^8)^2 \\ e^{-\phi} &= \Delta \\ B &= \frac{1}{\Delta^2} U \wedge dx^9 \end{aligned}$$

where

$$\Delta^2 = 1 + U_\mu U^\mu \qquad dU = \omega$$

Called the flux-trap since the dilaton has a maximum at the origin that localizes D-branes there.

Similar to the Melvin solution.

In the case of type IIA we can also do:

Step 4) Lift to M-theory

$$ds_{11}^2 = \Delta^{2/3}\left[\left(\eta_{\mu\nu} - \frac{U_\mu U_\nu}{\Delta^2}\right)dx^\mu dx^\nu + (dx^4)^2 + \cdots + (dx^8)^2 \right.$$
$$\left. + \frac{(dx^9)^2 + (dx^{10})^2}{\Delta^2}\right]$$
$$C = \frac{1}{\Delta^2} dx^9 \wedge dx^{10} \wedge U \, .$$

At first order the metric is flat and

$$G = dx^9 \wedge dx^{10} \wedge \omega$$

To see the effect on the D4/NS5-brane configuration consider the first order correction due to $B^{NSNS} = U \wedge dx^9$

$$\delta_\Omega S_{D4} = \frac{1}{g_4^2} \int d^5x U_\mu \partial_\nu X^9 F^{\mu\nu}$$

In the non-Abelian theory this becomes

$$\delta_\Omega S_{D4} = \frac{1}{g_4^2} \mathrm{tr} \int d^5x \, U_\mu D_\nu \mathbf{X}^9 \mathbf{F}^{\mu\nu} - i[\mathbf{X}^8, \mathbf{X}^9] U_\mu D^\mu \mathbf{X}^8$$

Here the second term arises following the discussion of Myers from imposing consistency with T–duality along x^8.

In particular the second term is $U_\mu \mathbf{D}_8 \mathbf{X}^9 \mathbf{F}^{\mu 8}$ with

- $\mathbf{D}_8 \mathbf{X}^9 = -i[\mathbf{X}^8, \mathbf{X}^9]$
- $\mathbf{F}^{\mu 8} = \mathbf{D}^\mu \mathbf{X}^8$

Going to higher order we find: ($\mathbf{\Phi} = \mathbf{X}^8 + i\mathbf{X}^9$)

$$S_{D4}^{\Omega} = -\frac{1}{g^2}\mathrm{Tr}\int d^4x \Big[\frac{1}{2}(\mathbf{D}_\mu\,\mathbf{\Phi} + i\frac{1}{2}\mathbf{F}_{\mu\lambda}U^\lambda)(\mathbf{D}^\mu\,\bar{\mathbf{\Phi}} - i\frac{1}{2}\mathbf{F}^{\mu\rho}U_\rho) + \frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} + \frac{1}{8}([\mathbf{\Phi},\bar{\mathbf{\Phi}}] + iU^\mu D_\mu(\mathbf{\Phi} - \bar{\mathbf{\Phi}}))^2\Big]$$

This agrees with the Ω-deformation of 4D $N = 2$ SYM given in [Nekrasov,Okounkov],[Ito,Nakajima,Saka]

The bottom line is that we can manufacture the Ω-deformation by putting branes in this background

The Alpha-Deformation

Given the M-theory flux solution corresponding to the Ω-deformation we can consider something else:

$$ds_{11}^2 = \Delta^{2/3}\left[\left(\eta_{\mu\nu} - \frac{U_\mu U_\nu}{\Delta^2}\right)dx^\mu dx^\nu + \ldots (dx^9)^2 + (dx^{10})^2 + \frac{(dx^6)^2 + (dx^8)^2}{\Delta^2}\right]$$

$$C = \frac{1}{\Delta^2} dx^6 \wedge dx^8 \wedge U\,.$$

i.e. $x^6 \leftrightarrow x^9$, $x^8 \leftrightarrow x^{10}$ - corresponding to a '9-11' flip.

The classic Witten M5-brane configuration in this background is still supersymmetric.

Again we can consider the type IIA D4/NS5 brane configuration. To first order $C^{RR} = U \wedge dx^6 \wedge dx^8$

$$\delta_A S_{D4} = \frac{i}{2g_4^2} \int d^5x \, \varepsilon^{\mu\nu\lambda\rho} U_\mu F_{\nu\lambda} \partial_\rho X^8$$

To check compatibility with T-duality along x^9 we note that

$$C_3^{RR} \to C_4^{RR} = C_3^{RR} \wedge dx^9$$

This time there is no extra term a la Myers and:

$$\delta_A S_{D4} = \frac{i}{g_4^2} \mathrm{tr} \int d^5x \, \mathbf{D}_\mu \mathbf{X}^8 U_\nu \star \mathbf{F}^{\mu\nu}.$$

At first order and for abelian fields this essentially agrees with the Ω-deformation after an integration by parts.

At higher order we find the deformed action

$$S_{D4}^{A} = -\frac{1}{g^2}\mathrm{Tr}\int d^4x\left[\frac{1}{2\Delta^2}\left(\mathbf{D}_\mu\mathbf{X}^8 + iU^\lambda\star\mathbf{F}_{\mu\lambda}\right)\left(\mathbf{D}^\mu\mathbf{X}^8 + iU_\rho\star\mathbf{F}^{\mu\rho}\right)\right.$$
$$+\frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} + \frac{1}{2}\mathbf{D}_\mu\mathbf{X}^9\mathbf{D}^\mu\mathbf{X}^9$$
$$\left.+\frac{1}{2\Delta^2}(U^\mu\mathbf{D}_\mu\mathbf{X}^8)^2 + \frac{1}{2}(U^\mu\mathbf{D}_\mu\mathbf{X}^9)^2 - \frac{1}{2\Delta^2}([\mathbf{X}^8,\mathbf{X}^9])^2\right] ,$$

'S-dual' to the Ω-deformation (roughly $\mathbf{F}\to\star\mathbf{F}$)

More concretely, in the Abelian case, S_{D4}^{A} can be obtained from S_{D4}^{Ω} by dualizing $F\to{}^*F=\star F$.

Deforming the M5-brane

So we can revisit the M5-brane, wrapped on a Riemann Surface, in this spacetime to obtain the Seiberg-Witten effective action in the presence of the Ω-deformation.

In fact we will just consider the first order corrections in ϵ_1, ϵ_2 and low energy fluctuations, to quadratic order in ∂_μ

Switch to complex coordinates

$$
\begin{aligned}
ds_{11}^2 &= dx_\mu dx^\mu + dzd\bar{z} + dsd\bar{s} + (dx^4)^2 + (dx^5)^2 + (dx^7)^2 \\
C_3 &= \frac{1}{2}(z+\bar{z})(ds+d\bar{s}) \wedge \omega - \frac{1}{2}(s+\bar{s})(dz+d\bar{z}) \wedge \omega
\end{aligned}
$$

where $s = x^6 + ix^{10}, z = x^8 + ix^9$

To this order the M5-brane is still embedded by a holomorphic function $s(z)$

- can take the same, undeformed Seiberg-Witten curve Σ
- depends on moduli u_i as well as other non-dynamical parameters such as hypermultiplet masses.
- we will just consider the case of a single modulus u corresponding to gauge group $SU(2)$
  - Σ is defined by $t^2 - 2(z^2 - u)t + 1 = 0$, $t = e^{-s}$
  - Σ has a single holomorphic differential $\lambda = (\partial s/\partial u)dz$

M5-brane dynamics come from letting $u \to u(x^\mu)$ and turning on fluctuations of h_3

First, since we are Euclidean, we take $i * h_3 = h_3$ and, to the order we are working, we can simply take $h_3 = H_3$

The equation of motion for h_3 is

$$dh_3 = -\frac{1}{4} d\hat{C}_3$$

To solve this we write

$$h_3 = -\frac{1}{4}(\hat{C}_3 + \star\hat{C}_3 + \Phi)$$

where $\Phi = i \star \Phi$ takes the form

$$\begin{aligned} \Phi &= \frac{\kappa}{2}\mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge dz + \frac{\bar{\kappa}}{2}\bar{\mathcal{F}}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge d\bar{z} \\ &\quad + \frac{1}{1+|\partial s|^2}\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\left(\partial^\tau s\bar{\partial}\bar{s}\,\kappa\mathcal{F}_{\sigma\tau} - \partial^\tau \bar{s}\partial s\,\bar{\kappa}\mathcal{F}_{\sigma\tau}\right) dx^\mu \wedge dx^\nu \wedge dx^\rho \end{aligned}$$

with

$$\mathcal{F} = -\star_4 \mathcal{F} \qquad \bar{\mathcal{F}} = \star_4 \bar{\mathcal{F}}$$

This explains our odd choice for C:

$$C_3 = \frac{1}{2}\,(z+\bar{z})\,(ds+d\bar{s})\wedge\omega - \frac{1}{2}\,(s+\bar{s})\,(dz+d\bar{z})\wedge\omega$$

Because in this case

$$d(\hat{C}_3 + i\star\hat{C}_3) = -\frac{1}{4}\hat{G}_4$$

if $\partial_\mu s = 0$.

Furthermore it is supersymmetric:

$$h_3\cdot\Gamma\epsilon \sim \omega\cdot\Gamma\epsilon = 0$$

Other choices seem very problematic and do not lead to manageble integrals.

Note that

$$G = d(z + \bar{z}) \wedge d(s + \bar{s}) \wedge \omega$$

so this is an Alpha-deformation.

The corresponding Ω-deformation would have

$$G = d(z - \bar{z}) \wedge d(s - \bar{s}) \wedge \omega$$

To handle this we could take

$$C_3 = \frac{1}{2}\,(z - \bar{z})\,(ds - d\bar{s}) \wedge \omega - \frac{1}{2}\,(s - \bar{s})\,(dz - d\bar{z}) \wedge \omega$$

But this has an explicit dependence on $x^{10} = (s - \bar{s})/2i$.

- Not a good gauge choice for type IIA
- Nevertheless just computing we find the same answer.

The equation of motion for for h_3 now takes the form

$$0 = d\Phi + \star\hat{C}_3 = \star_4(E_{\mu z}dz + \bar{E}_{\mu\bar{z}}d\bar{z})$$

with

$$\begin{aligned} E_{\mu z} \quad &= \quad \partial_\mu(\kappa\mathcal{F}_{\mu\nu} - \hat{C}_{\mu\nu z}) + \partial\left[\frac{\bar{\partial}\bar{s}\partial_\nu s}{1+|\partial s^2|}(\kappa\mathcal{F}_{\mu\nu} - \hat{C}_{\mu\nu z})\right] \\ &\qquad -\partial\left[\frac{\partial s\partial_\nu\bar{s}}{1+|\partial s|^2}(\bar{\kappa}\bar{\mathcal{F}}_{\mu\nu} - \hat{C}_{\mu\nu\bar{z}})\right] \end{aligned}$$

To reduce this equation we evaluate

$$0 = \int_\Sigma E_{\mu z}dz \wedge \bar{\lambda} \qquad 0 = \int_\Sigma \bar{E}_{\mu\bar{z}}d\bar{z} \wedge \lambda$$

Doing this one encounters the following integrals

$$I_0 = \int_\Sigma \lambda \wedge \bar{\lambda} = \frac{da}{du} (\tau - \bar{\tau}) \frac{d\bar{a}}{d\bar{u}}$$

$$K = \int_\Sigma d \left[\frac{\lambda_z \bar{\partial} \bar{s}}{1 + |\partial s|^2} \right] \wedge \lambda = - \left(\frac{da}{du} \right)^2 \frac{d\tau}{du}$$

$$J = \int_\Sigma d \left[\frac{\partial_\mu \bar{s} \partial s}{1 + |\partial s|^2} \bar{z} \bar{\partial} \bar{s} \right] \wedge \bar{\lambda} = 0$$

The first is easily evaluated using the Riemann bi-linear identity

The second is tricky but was evaluated years ago in [Lambert, West] by cutting holes of size δ in Σ around the branch points e_i, of $t(z)$, doing the line integrals and taking $\delta \to 0$:

$$K = \sum_i \oint \left[\frac{\lambda_z \bar{\partial} \bar{s}}{1 + |\partial s|^2} \right] \lambda = -\pi i \sum_i \frac{1}{e_i} \frac{1}{\prod_{j \neq i} (e_i - e_j)}$$

J is evaluated in a similar way

The result is

$$(\tau-\bar{\tau})\left(\partial_\mu \mathcal{F}_{\mu\nu}+\partial_\mu a\,\omega_{\mu\nu}\right)+\partial_\mu\tau\mathcal{F}_{\mu\nu}-\partial_\mu\bar{\tau}\bar{\mathcal{F}}_{\mu\nu}=0$$
$$(\tau-\bar{\tau})\left(\partial_\mu \bar{\mathcal{F}}_{\mu\nu}+\partial_\mu \bar{a}\,\omega_{\mu\nu}\right)+\partial_\mu\tau\mathcal{F}_{\mu\nu}-\partial_\mu\bar{\tau}\bar{\mathcal{F}}_{\mu\nu}=0$$

Taking the difference we find

$$\partial_\mu(\mathcal{F}_{\mu\nu}-\bar{\mathcal{F}}_{\mu\nu})=-\partial_\mu\,(a-\bar{a})\,\omega_{\mu\nu}$$

which is solved by writing

$$\begin{cases}\mathcal{F}=(1-*)\,F-(a-\bar{a})\,\omega^- \\ \bar{\mathcal{F}}=(1+*)\,F+(a-\bar{a})\,\omega^+\end{cases}$$

where $F=dA$ satisfies the standard Bianchi identity

Next we look at the scalar equations

$$
\begin{aligned}
E &= \partial_\mu \partial_\mu s - \partial \left[\frac{\partial_\rho s \partial_\rho s \bar{\partial} \bar{s}}{1 + |\partial s|^2} \right] - \frac{16 \partial^2 s}{\left(1 + |\partial s|^2\right)^2} h_{\mu\nu\bar{z}} h_{\mu\nu\bar{z}} \\
&\quad - 2\omega^-_{\mu\nu} \mathcal{F}_{\mu\nu} \left(\frac{da}{du} \right)^{-1} \lambda_z + 2\omega^+_{\mu\nu} \bar{\mathcal{F}}_{\mu\nu} \left(\frac{d\bar{a}}{d\bar{u}} \right)^{-1} \bar{\lambda}_{\bar{z}} = 0 \\
\bar{E} &= \partial_\mu \partial_\mu \bar{s} - \bar{\partial} \left[\frac{\partial_\rho \bar{s} \partial_\rho \bar{s} \partial s}{1 + |\partial s|^2} \right] - \frac{16 \bar{\partial}^2 \bar{s}}{\left(1 + |\partial s|^2\right)^2} h_{\mu\nu z} h_{\mu\nu z} \\
&\quad - 2\omega^-_{\mu\nu} \mathcal{F}_{\mu\nu} \left(\frac{da}{du} \right)^{-1} \lambda_z + 2\omega^+_{\mu\nu} \bar{\mathcal{F}}_{\mu\nu} \left(\frac{d\bar{a}}{d\bar{u}} \right)^{-1} \bar{\lambda}_{\bar{z}} = 0
\end{aligned}
$$

and reduce by evaluating

$$
0 = \int_\Sigma E dz \wedge \bar{\lambda} \qquad 0 = \int_\Sigma \bar{E} d\bar{z} \wedge \lambda
$$

The details of the calculation are similar

$$
\begin{aligned}
&(\tau-\bar{\tau})\,\partial_\mu\partial_\mu a+\partial_\mu a\partial_\mu\tau+\frac{d\bar{\tau}}{d\bar{a}}\bar{\mathcal{F}}_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}\\
&-2\,(\tau-\bar{\tau})\,\omega_{\mu\nu}\mathcal{F}_{\mu\nu}+2\,(L_1-L_2)\left(\frac{d\bar{a}}{d\bar{u}}\right)^2\omega_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}=0\\
&(\tau-\bar{\tau})\,\partial_\mu\partial_\mu\bar{a}-\partial_\mu\bar{a}\partial_\mu\bar{\tau}-\frac{d\tau}{da}\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\\
&-2\,(\tau-\bar{\tau})\,\omega_{\mu\nu}\bar{\mathcal{F}}_{\mu\nu}+2\left(\bar{L}_1-\bar{L}_2\right)\left(\frac{da}{du}\right)^2\omega_{\mu\nu}\mathcal{F}_{\mu\nu}=0
\end{aligned}
$$

where L_1 and L_2 are the integrals

$$
\begin{aligned}
L_1&=-\int_\Sigma\partial\left(\frac{\partial s}{1+|\partial s|^2}\right)(\bar{s}+\bar{s}-z\bar{\partial}\bar{s}-\bar{z}\bar{\partial}\bar{s})\lambda_{\bar{z}}dz\wedge\bar{\lambda},\\
L_2&=\int_\Sigma\bar{\lambda}_{\bar{z}}dz\wedge\bar{\lambda}
\end{aligned}
$$

Some analysis, complimented by a numerical check, shows that $L_1=L_2$.

Putting everything together we arrive at the following Lagrangian that gives these equations of motion

$$i\mathcal{L} = -\left(\tau - \bar{\tau}\right)\Big[\frac{1}{2}\partial_\mu a \partial^\mu \bar{a} + F_{\mu\nu}F^{\mu\nu} + \left(a - \bar{a}\right) \star \omega_{\mu\nu}F^{\mu\nu} - 2\partial_\mu \left(a + \bar{a}\right) \star F^{\mu\nu *}U_\nu\Big] + \left(\tau + \bar{\tau}\right)\left[F_{\mu\nu} \star F^{\mu\nu} + \left(a - \bar{a}\right)\omega_{\mu\nu}F^{\mu\nu} + 2\partial_\mu \left(a - \bar{a}\right) \star F^{\mu\nu *}U_\nu\right]$$

where $\omega = dU$ and $\star\omega = d^* U$.

This is low-energy effective action of the M5-brane on Σ in a flux background and hence can be identified with the Seiberg-Witten solution of 4D N=2 Yang-Mills theory with an A- or Ω-deformation.

It is natural to complete the squares and generalize to arbitrary gauge group:

$$
\begin{aligned}
i\mathcal{L} = -\left(\tau_{ij}-\bar{\tau}_{ij}\right)\Big[\frac{1}{2}&\left(\partial_\mu a^i + 2\left(\frac{\bar{\tau}}{\tau-\bar{\tau}}\right)_{ik} \star F^{k}_{\mu\nu}{}^{*}U^{\nu}\right) \\
&\quad\times\left(\partial^\mu \bar{a}^j - 2\left(\frac{\tau}{\tau-\bar{\tau}}\right)_{jl} \star F^{l\,\mu\lambda *}U_\lambda\right) \\
&+\left(F^i_{\mu\nu} + \frac{1}{2}\left(a^i-\bar{a}^i\right)\star\omega_{\mu\nu}\right)\left(F^{j\,\mu\nu}+\frac{1}{2}\left(a^j-\bar{a}^j\right)\star\omega^{\mu\nu}\right)\Big] \\
+\left(\tau_{ij}+\bar{\tau}_{ij}\right)&\left(F^i_{\mu\nu}+\frac{1}{2}\left(a^i-\bar{a}^i\right)\star\omega_{\mu\nu}\right)\left(\star F^{j\mu\nu}+\frac{1}{2}\left(a^j-\bar{a}^j\right)\omega^{\mu\nu}\right)
\end{aligned}
$$

Similar structure to the Ω-deformation of the original theory

Conclusions

Analysed the Ω-deformation as a flux background in M-theory.

Considered an S-dual A-deformation and evaluated the M5-brane in this background.

Hope to shed light on the Ω-deformation and the M5-brane using a 11D supergravity realisation

Currently looking at

- Second order corrections: looks as if the SW curve is essentially unaltered (although the complex structure gets deformed by Δ).
- More general SW curves (e.g. $SU(N_c)$ with N_f flavours) - conceptually no change but the non-holomorphic integrals are more involved.
- When can you lift a D4/NS5-brane system to a smooth M5? - tells us about non-Coulomb branch vacua of the (2,0) theory.

Other applications: non-Lagrangian theories, AGT...?