Multicomponent Skyrmion lattices and their excitations

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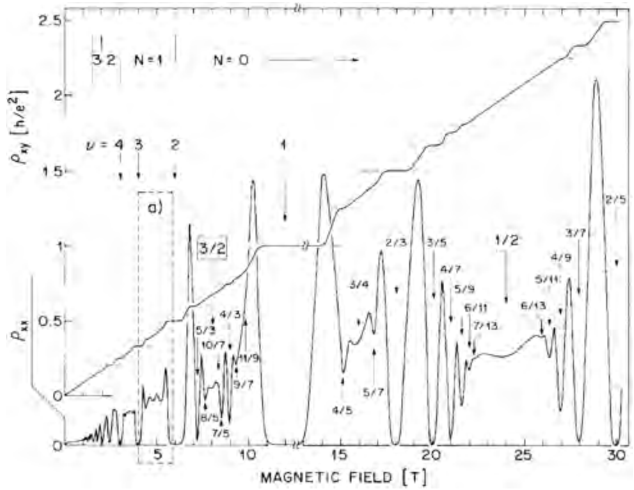
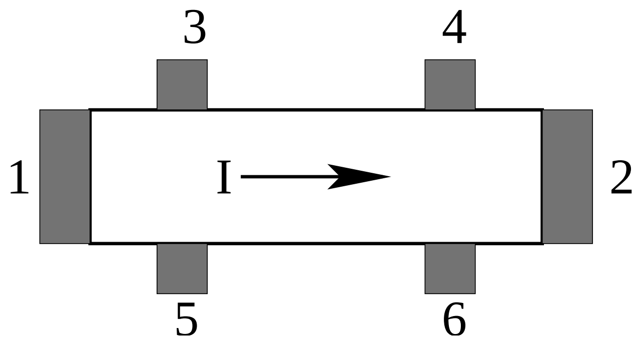
Quantum Hall effect (I)





$$R_{xx} = (V(3) - V(4))/I$$
$$R_{xy} = (V(3) - V(5))/I$$

Quantum nature of Hall resistance plateaus

Plateaus observed for (ν integer):

$$\rho_{xy} = \frac{B}{ne} = \frac{h}{\nu e^2}$$

$\longrightarrow$ Quantized electronic densities:

$$n = \nu \frac{eB}{h}$$

In terms of $\Phi_0 = \frac{h}{e}$: "Flux quantum"

$$N_{\text{electrons}} = \nu \frac{\text{Total magnetic flux}}{\Phi_0}$$

Energy spectrum for a single electron

$$H = \frac{1}{2m}(\mathbf{P} + e\mathbf{A})^2, \quad \mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} \ \text{ spatially uniform.}$$

Define gauge invariant $\boldsymbol{\Pi} = \mathbf{P} + e\mathbf{A} = m\mathbf{v}$

$\{p_i, r_j\} = \delta_{ij}, \quad i, j \in \{x, y\}, \quad \{\Pi_x, \Pi_y\} = eB$

$\rightarrow$ Harmonic oscillator spectrum: $E_n = \hbar\omega(n + 1/2)$, $\omega = eB/m$

Conserved quantities (also generators of magnetic translations)

$\mathbf{v} = \omega \hat{\boldsymbol{z}} \wedge (\mathbf{r} - \mathbf{R}), \quad \mathbf{R} = \mathbf{r} + \frac{\hat{\boldsymbol{z}} \wedge \boldsymbol{\Pi}}{eB}, \quad \{R_x, R_y\} = -\frac{1}{eB}, \quad \{R_i, \Pi_j\} = 0$

Heisenberg principle: $B \Delta R_x \Delta R_y \simeq \frac{h}{e} = \Phi_0$

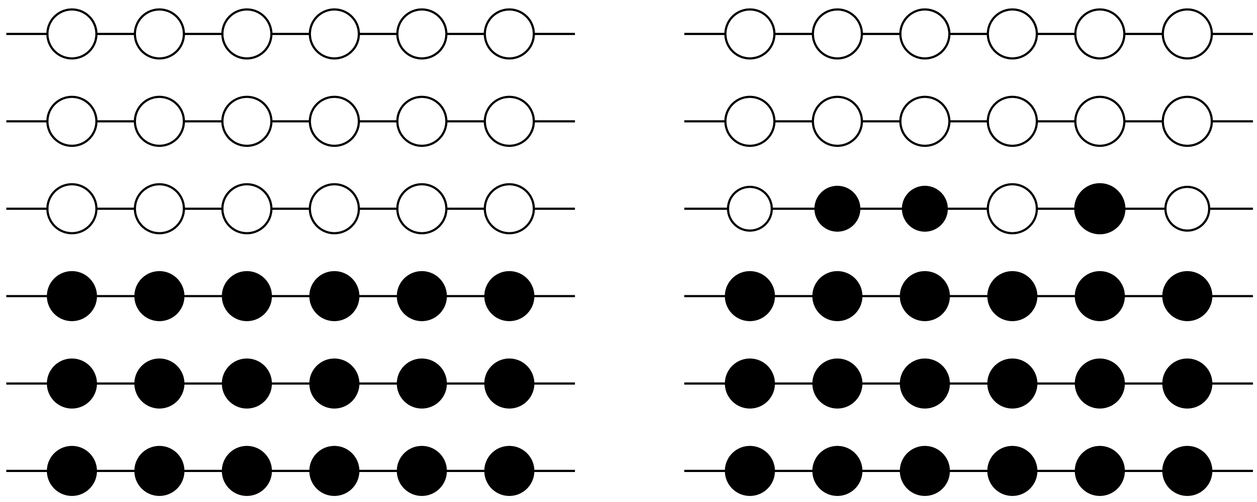
$\rightarrow$ Magnetic length $l = \sqrt{\frac{\hbar}{eB}}$

Landau levels are degenerate

Intuitively, each state occupies the same area as a flux quantum Φ_0, so that the number of states per Landau level =

$$\frac{\text{Total magnetic flux}}{\Phi_0}$$

ν is interpreted as the number of occupied Landau levels

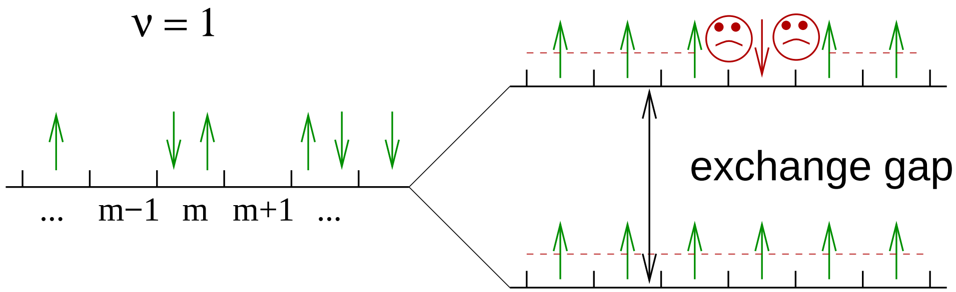


ν entier   3<ν<4

Ferromagnetism at $\nu = 1$

Coulomb repulsion favours anti-symmetric orbital wavefunction

$\longrightarrow$ spin wavefunction: symmetric (ferromagnet)



no interactions — with repulsive interactions

in QH systems : no kinetic−energy cost !
(LL ~ flat band)

A class of trial states near $\nu = 1$

Take antisymmetrized products of single particle states (Slater determinants or Hartree-Fock states): $|S_\psi\rangle = \bigwedge_{\alpha=1}^{N} |\Phi_\alpha\rangle$ where $\Phi_{\alpha,a}(r) = \chi_\alpha(r)\psi_a(r)$, $r = (x, y)$, $a \in \{1, ..., d\}$.
$\chi_\alpha(r) \longrightarrow$ electron position.
$\psi_a(r) \longrightarrow$ slowly varying spin background. ($\langle\psi(r)|\psi(r)\rangle = 1$).

In the $d = 2$ case, if σ_a denote Pauli matrices:
Associated classical spin field: $n_a(r) = \langle\psi(r)|\sigma_a|\psi(r)\rangle$
Topological charge: $N_{\text{top}} = \frac{1}{4\pi} \int d^{(2)}r \, (\partial_x \vec{n} \wedge \partial_y \vec{n}) \cdot \vec{n}$

Because of large magnetic field, we impose that orbital wave-functions $\Phi_{\alpha,a}(r)$ lie in the lowest Landau level.

Extra charges at $\nu = 1$ induce Skyrmion textures

Sondhi, Karlhede, Kivelson, Rezayi, PRB **47**, 16419, (1993)

$$\langle \Phi_\alpha | (P - eA)^2 | \Phi_\alpha \rangle = \langle \chi_\alpha | (P - eA_{\text{eff}})^2 + V_{\text{eff}} | \chi_\alpha \rangle$$

$$V_{\text{eff}} = \langle \nabla\psi | \nabla\psi \rangle - \langle \nabla\psi | \psi \rangle \langle \psi | \nabla\psi \rangle$$

$$A_{\text{eff}} = A - \Phi_0 \frac{1}{2\pi} \mathcal{A}$$

Berry connection: $\mathcal{A} = \frac{1}{i} \langle \psi | \nabla\psi \rangle$

Generalized topological charge: $\oint \mathcal{A}.d\mathbf{r} = 2\pi N_{\text{top}}$
(This coincides with the previous notion when $d = 2$).

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Sondhi, Karlhede, Kivelson, Rezayi, PRB **47**, 16419, (1993)

$$\langle \Phi_\alpha | (P - eA)^2 | \Phi_\alpha \rangle = \langle \chi_\alpha | (P - eA_{\rm eff})^2 + V_{\rm eff} | \chi_\alpha \rangle$$

Consequences:

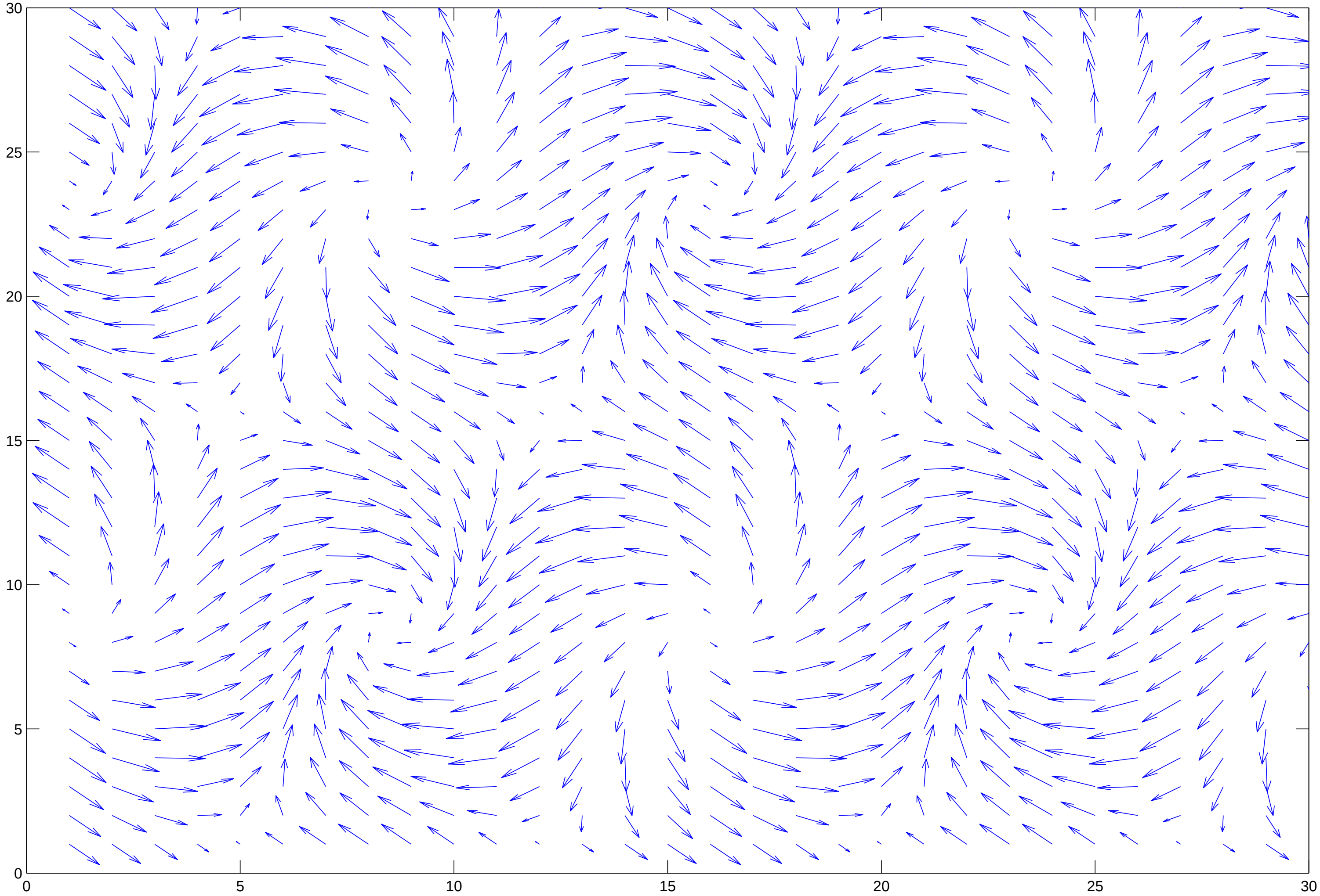
The charge orbitals $\chi_\alpha(r)$ lie in the lowest Landau level of $A_{\rm eff}$.
There are $N_{\rm eff} = \text{Effective flux}/\Phi_0$ states in this level.
Condition to minimize Coulomb energy:

$$N_{\rm electrons} = N_{\rm eff}$$

Finally:

$$N_{\rm electrons} = N(\nu = 1) - N_{\rm top}$$

Picture of a Skyrmion crystal

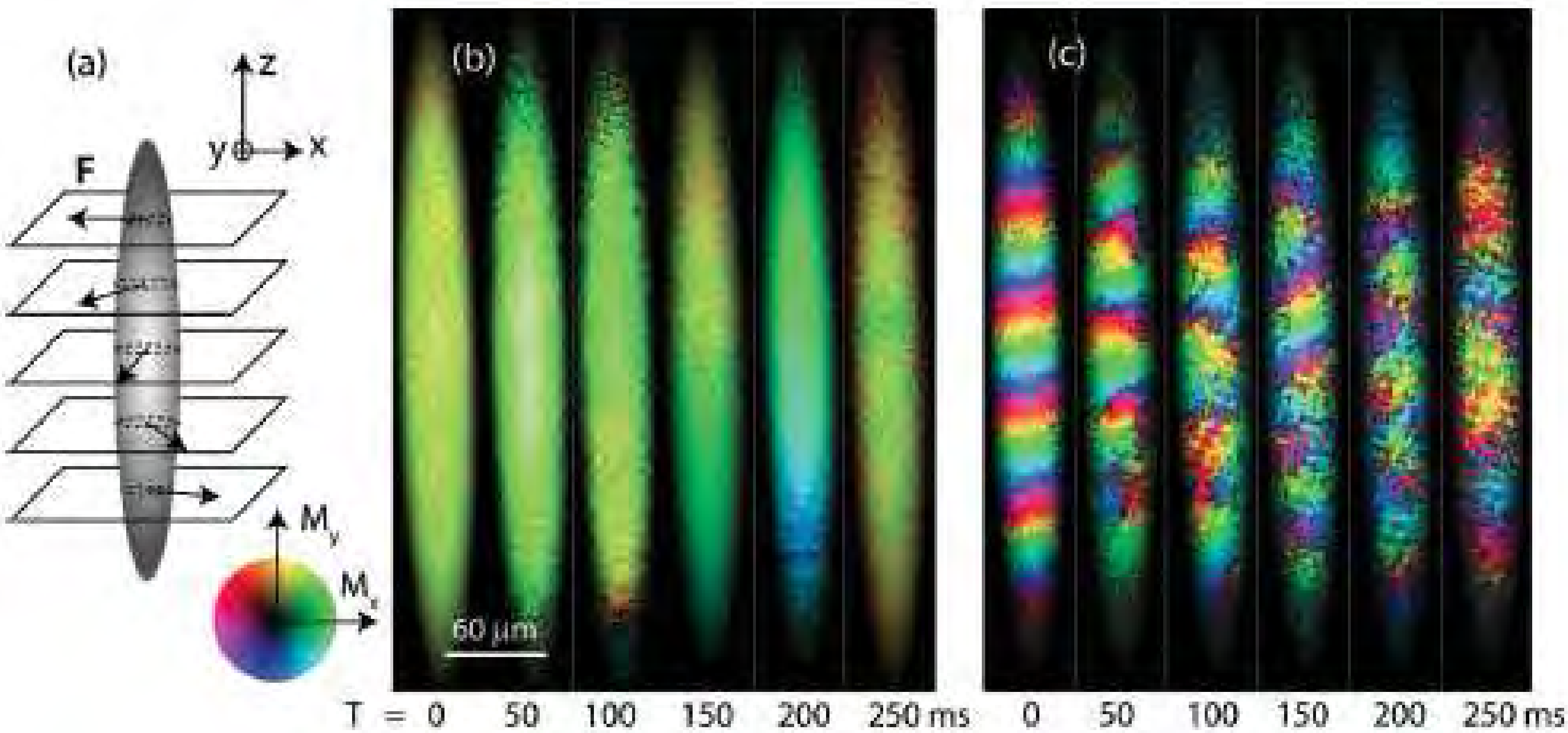


Skyrmion crystals in electronic systems

Theoretical prediction: Brey, Fertig, Côté and MacDonald, PRL **75**, 2562 (1995)
Specific heat peak: Bayot et al. PRL **76**, 4584 (1996) and PRL **79**, 1718 (1997)
Increase in NMR relaxation: Gervais et al. PRL **94**, 196803 (2005)
Raman spectroscopy: Gallais et al, PRL **100**, 086806 (2008)
Microwave spectroscopy: Han Zhu et al. PRL **104**, 226801 (2010)

Recent observation (neutron scattering) on the chiral itinerant magnet MnSi: Mühlbauer et al, Science **323**, 915 (2009)

Textures in spinor condensates



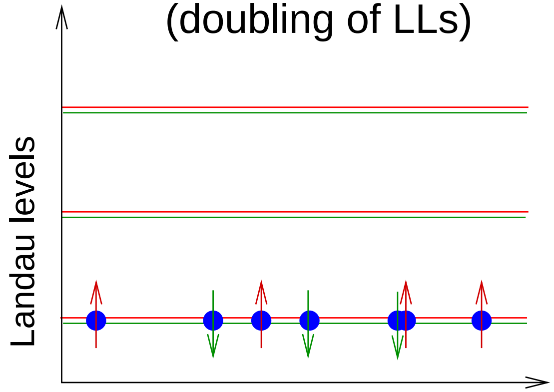
M. Vengalattore et al. PRL **100**, 170403 (2008)

"Helical spin textures in a $^{87}\mathrm{Rb}\ F = 1$ *spinor Bose-Einstein condensate are found to decay spontaneously toward a spatially modulated structure of spin domains. The formation of this modulated phase is ascribed to magnetic dipolar interactions that energetically favor the short-wavelength domains over the long-wavelength spin helix."*

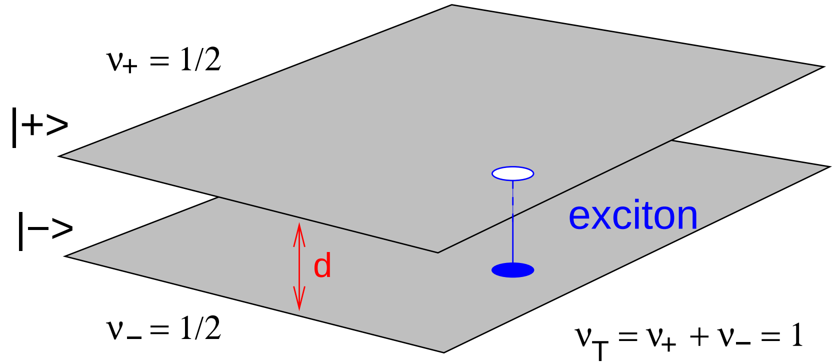
Multi-Component Systems (Internal Degrees of Freedom)

Ⓐ physical spin: SU(2)

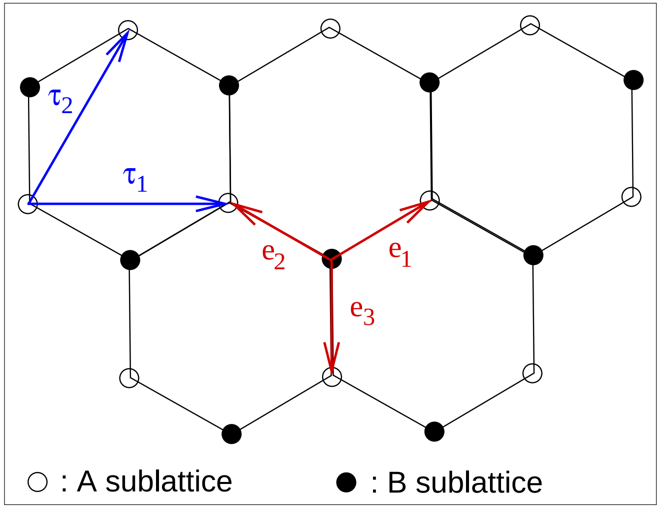
(doubling of LLs)



Ⓑ bilayer: SU(2) isospin



Ⓒ graphene (2D graphite)



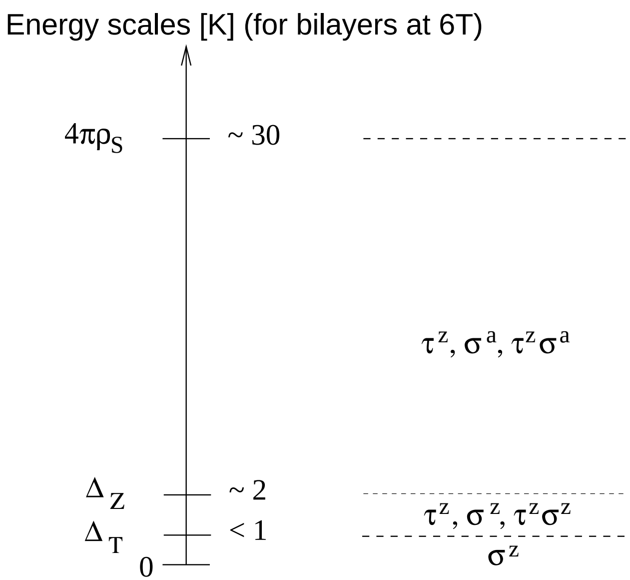
two−fold valley
degeneracy
⟶ SU(2) isospin

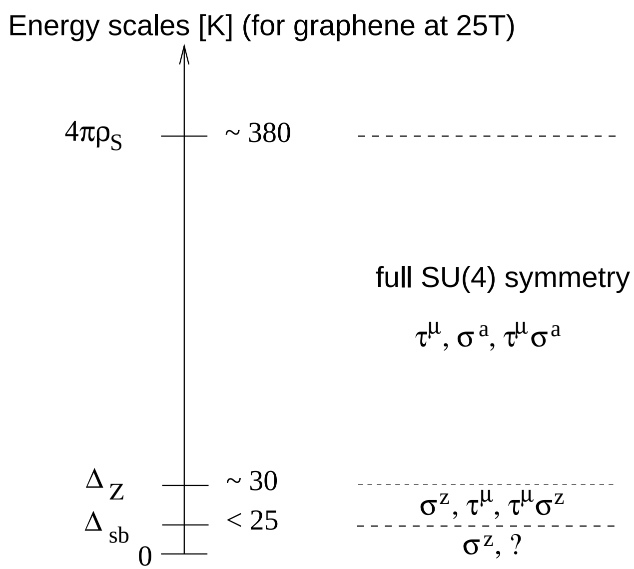
spin + isospin : SU(4)

Realistic anisotropies

Hamiltonian can approximately have high $SU(4)$ symmetry

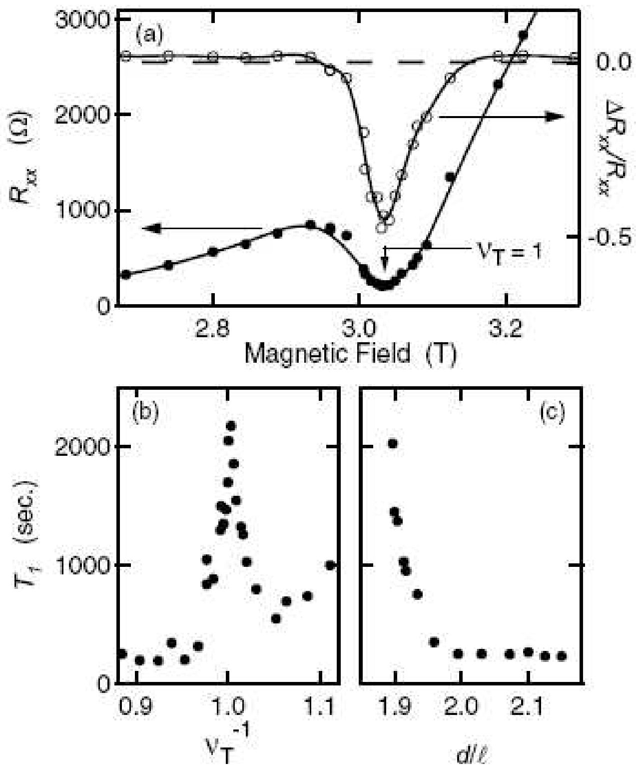
- Zeeman anisotropy: $SU(2) \to U(1)$
- Graphene: valley weakly split, $O(a/l_B)$
- Bilayers: charging energy: $SU(2) \to U(1)$; neglect tunnelling





NMR experiments in quantum Hall bilayers (I)

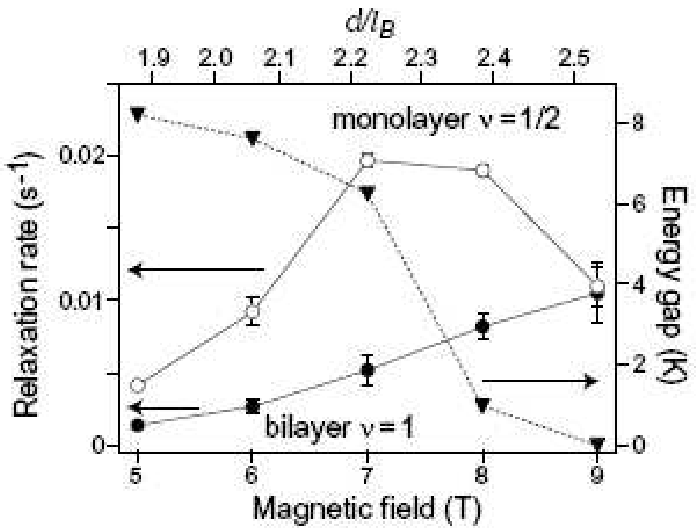
Heat or NMR pulse → increases effective electron Zeeman energy
Nuclear spin relaxation is detected resistively

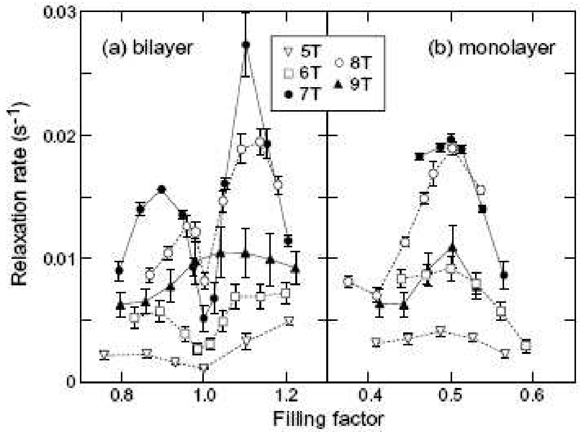


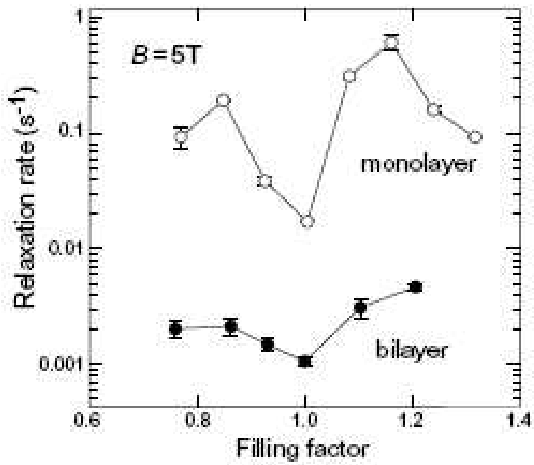
Spielman et al., Phys. Rev. Lett. **94**, 076803, (2005)

NMR experiments in quantum Hall bilayers (II)

Current-pump and resistive detection



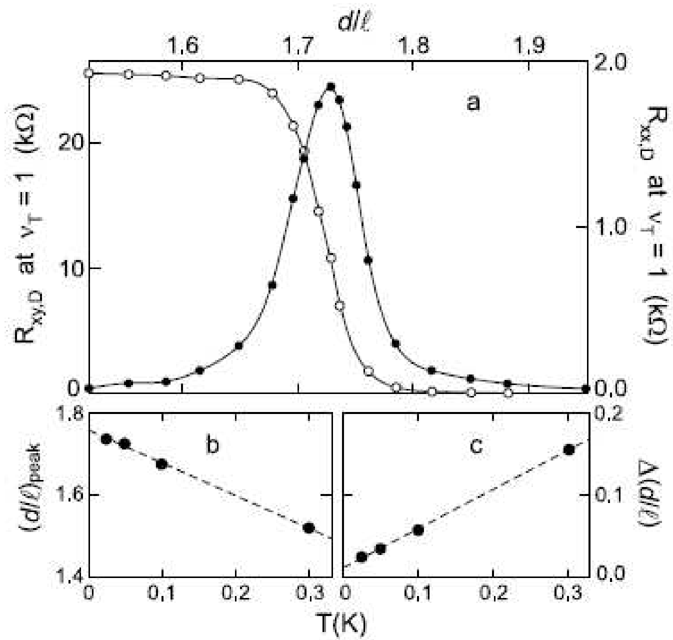




Kumada at al., Phys. Rev. Lett. **94**, 096802

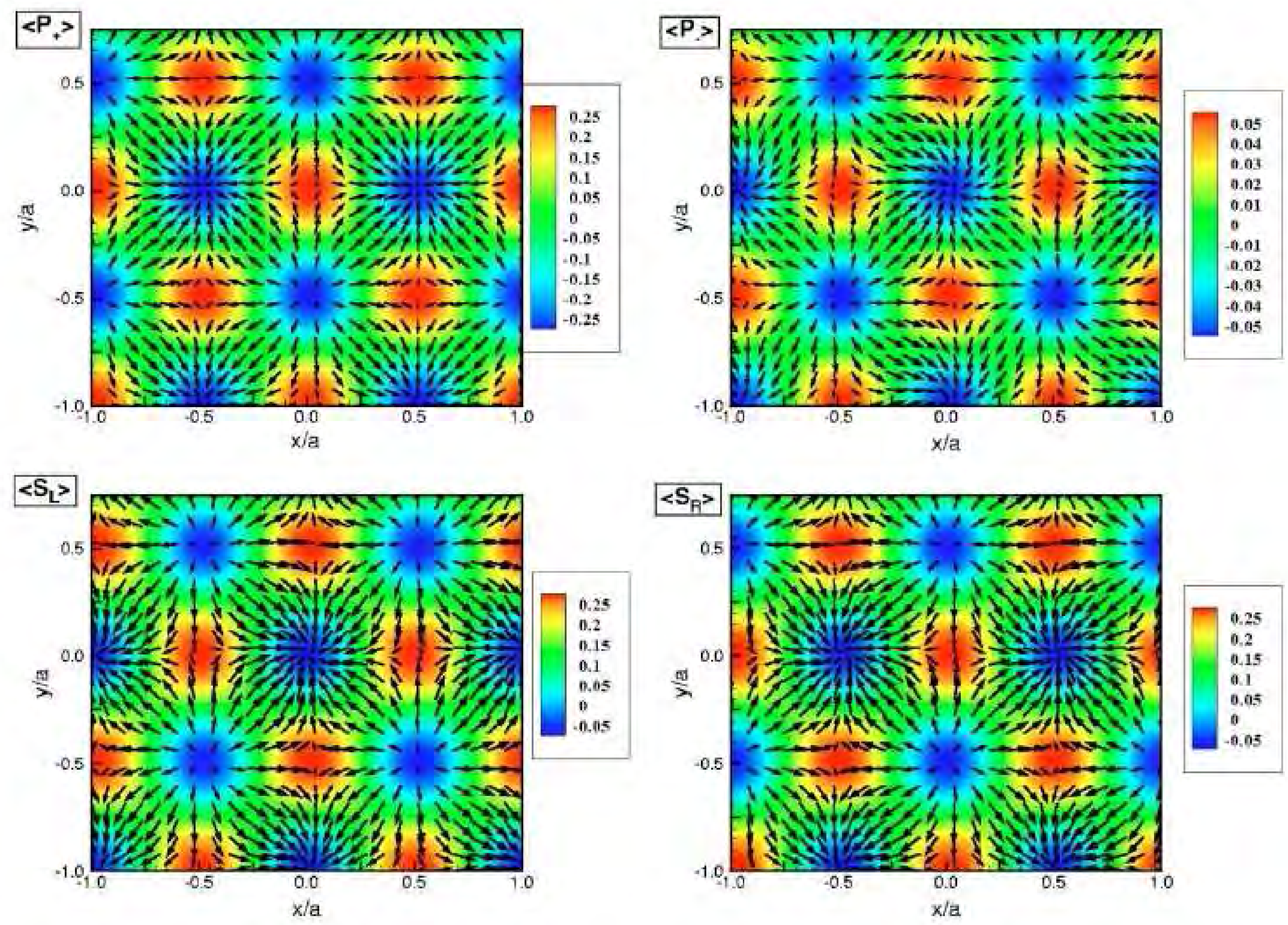
Phase coexistence scenario

- Theoretical suggestion of first order transition (Schliemann, Girvin, MacDonald, 2001)
- Explanation of the longitudinal Coulomb drag peak (Stern, Halperin, 2002)



Kellog et al. Phys. Rev. Lett. **90**, 246801, (2003)

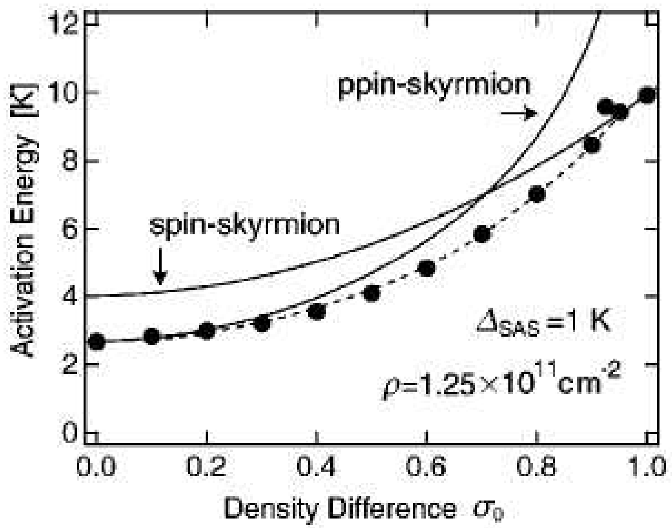
The case for entangled textures (I)



Bourassa et al, Phys. Rev. B **74**, 195320 (2006)

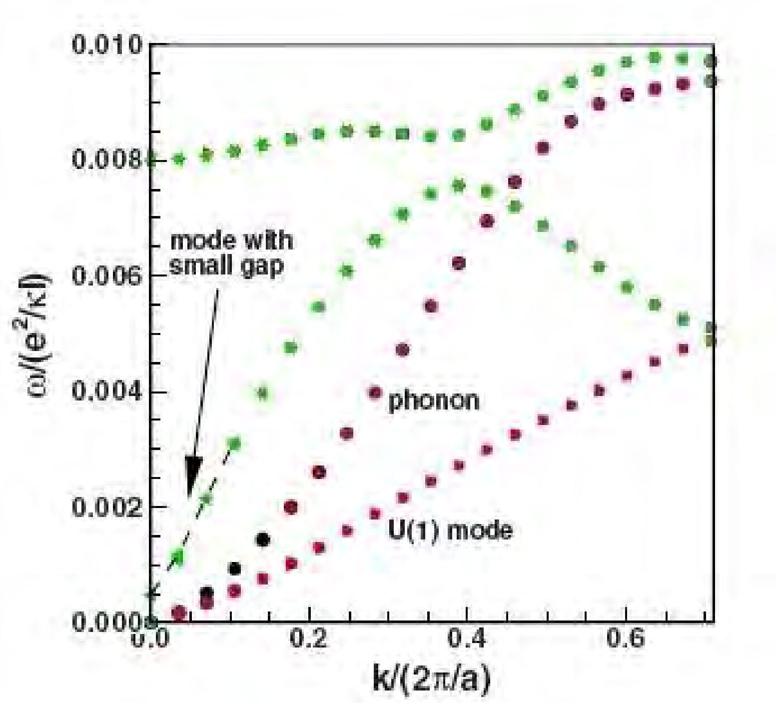
The case for entangled textures (II)

Bilayer with charge imbalance



Ezawa, Tsitsishvili,
Phys. Rev. B **70**, 125304, (2004)

Collective mode spectrum



Côté et al.,
Phys. Rev. B **76**, 125320, (2007)

$CP^{(d-1)}$ model for exchange energy

d-component spinor field $|\psi(r)\rangle$ parametrizes a Slater determinant at $\nu = 1$.
Assume $SU(d)$ *global* symmetry and *local* gauge symmetry:
$|\psi(r)\rangle \to e^{i\phi(r)}|\psi(r)\rangle$.

$$\mathcal{E}_{ex} = \int d^{(2)}r \left(\frac{\langle \nabla\psi|\nabla\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\nabla\psi|\psi\rangle\langle\psi|\nabla\psi\rangle}{\langle\psi|\psi\rangle^2} \right)$$

Berry connection: $\mathcal{A} = \frac{1}{i}\langle\psi|\nabla\psi\rangle$
Topological charge: $\oint \mathcal{A}.d\mathbf{r} = 2\pi N_{\rm top}$

$$\mathcal{E} \geq \pi |N_{\rm top}|$$

Lower bound is reached when $|\psi(r)\rangle$ is holomorphic or anti-holomorphic: leading to a huge degeneracy.

Spectrum of the Hessian matrix (I)

Consider small deviations $|\psi\rangle \to |\psi\rangle + \sqrt{\langle\psi|\psi\rangle}|\phi\rangle$ away from analytic spinor $|\psi\rangle$.

$$\mathcal{E} = \pi|N_{\rm top}| + 2\langle\phi|M^{+}PM|\phi\rangle + ...$$

$$M|\phi\rangle = |\partial_{\bar{z}}\phi\rangle + \frac{1}{2}\frac{\langle\partial_{\bar{z}}\psi|\psi\rangle}{\langle\psi|\psi\rangle}|\phi\rangle$$

$$P|\phi\rangle = |\phi\rangle - \frac{|\psi(z)\rangle\langle\psi(z)|}{\langle\psi(z)|\psi(z)\rangle}|\phi\rangle$$

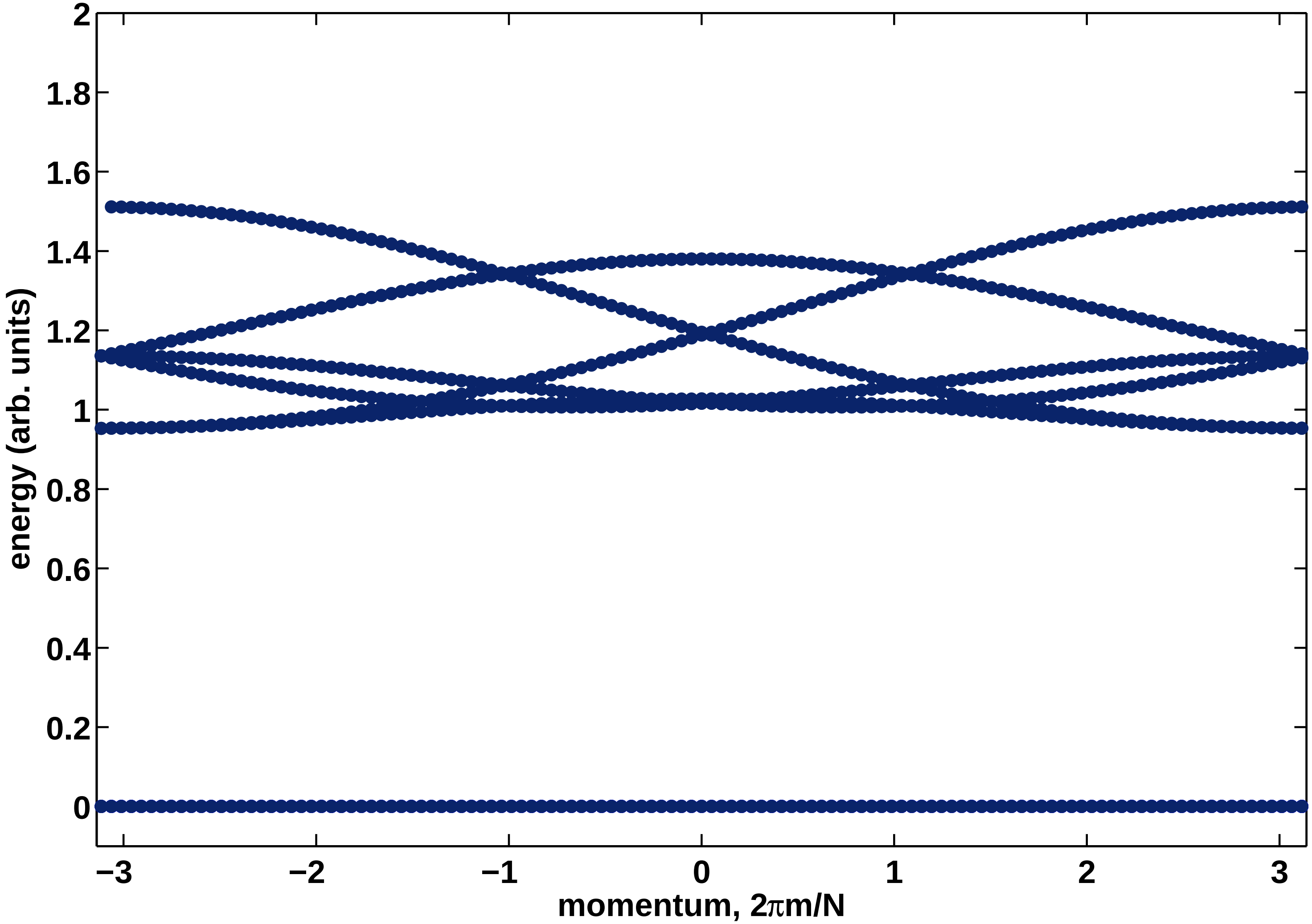
Key property:

$[M, M^{+}] = \frac{1}{2}\mathcal{B}(r) = \pi Q(r)$

If $\mathcal{B}(r)$ constant, the spectrum of $M^{+}M$ is $\{\frac{\mathcal{B}}{2}n, n = 0, 1, 2, ...\}$.
At large d, we may expect that the effect of P is small.

Most likely, Hessian of $CP^{(d-1)}$ model is gapped, with an energy gap of order $\frac{e^2}{4\pi\epsilon l}nl^2$. ($l = \sqrt{\hbar/eB}$, $\overline{Q(r)} = n$).

Spectrum of the Hessian matrix (II)



Variational evaluation of the hessian spectrum for $d = 3$

Variational approach for lattice of textures

$\mathcal{E} = \mathcal{E}_{ex} + \mathcal{E}_{el}, \quad \mathcal{E}_{el} = \frac{1}{2}\int d^{(2)}r_1 \int d^{(2)}r_2 Q(r_1)u(r_1 - r_2)Q(r_2)$
$u(r) = \frac{e^2}{4\pi\epsilon|r|}$

Assume an average charge density $\overline{Q(r)} = n$, then $\mathcal{E}_{el}/\mathcal{E}_{ex} = ln^{1/2}$, where $l = \sqrt{\hbar/eB}$. In the *dilute limit*, $\mathcal{E}_{ex} \gg \mathcal{E}_{el}$.

Main approximation: Minimize $\mathcal{E}$ among the configurations that minimize $\mathcal{E}_{ex}$. That is, we look for holomorphic d-component spinor configurations $|\Psi(r)\rangle$ with given $\overline{Q(r)} = n$, such that $\mathcal{E}_{el}$ is minimum.

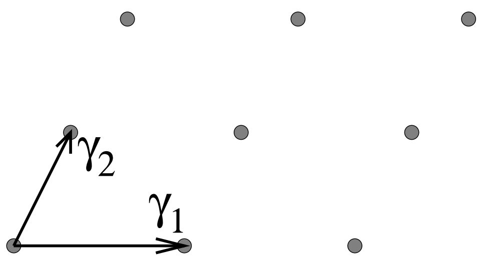
Physical intuition: One should make $Q(r)$ as homogeneous as possible. In particular, it is natural to consider first periodic patterns.

Construction of periodic textures

Problem: construct periodic holomorphic maps from torus to projective space

Answer: use Theta functions

$$\gamma_1 = \pi\sqrt{d}$$
$$\gamma_2 = \pi\sqrt{d}\tau$$



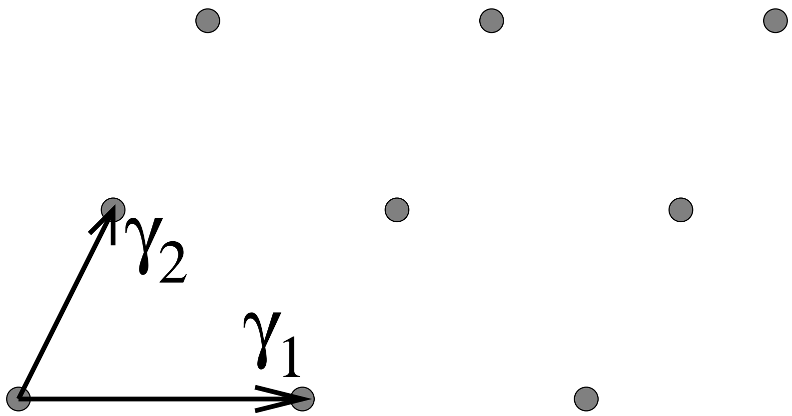
$$\theta(z+\gamma) = e^{a_\gamma z + b_\gamma}\theta(z)$$

$$\gamma = n_1\gamma_1 + n_2\gamma_2$$

n_1 and n_2 integers

Fixing the topological charge d

$$\frac{1}{i}\int_{\mathcal{C}(\gamma_1,\gamma_2)} \frac{\theta'(z)}{\theta(z)} = \frac{1}{i}\left(a_{\gamma_1}\gamma_2 - a_{\gamma_2}\gamma_1\right) = 2\pi d$$



Theta functions of a fixed type carrying topological charge d on the elementary (γ_1, γ_2) parallelogram form a complex vector space of dimension d (Riemann Roch theorem on torus).

Lattice of allowed translations

$$\mathcal{T}_w \theta(z) = e^{\mu(w) z} \theta(z - w)$$

$$\frac{\mathcal{T}_w \theta(z + \gamma)}{\mathcal{T}_w \theta(z)} = e^{a_\gamma z + b_\gamma} e^{\mu(w)\gamma - a_\gamma w}$$

Type conservation:

$$\mu(w)\gamma - a_\gamma w \in 2\pi\mathbb{Z}$$

for any lattice vector γ.

Quantized translations:

$$\begin{aligned} w &= \frac{1}{d}(m_1\gamma_1 + m_2\gamma_2) \\ \mu(w) &= \frac{1}{d}(m_1 a_{\gamma_1} + m_2 a_{\gamma_2}) \end{aligned}$$

$$\mathcal{T}_w \mathcal{T}_{w'} = e^{i\frac{2\pi}{d}(m_1 m_2' - m_2 m_1')} \mathcal{T}_{w'} \mathcal{T}_w$$

$(m_1 m_2' - m_2 m_1')/d =$ topological charge inside parallelogram delimited by w and w'.

Useful set of theta functions

$$\theta_p(z) = \sum_n e^{i\left(\pi\tau d(n-p/d)(n-1-p/d)+2\sqrt{d}(n-p/d)z\right)}$$

Pattern of zeros (d=4)

$$\begin{aligned}\mathcal{T}_{\frac{\gamma_1}{d}}\theta_p &= e^{i\frac{2\pi p}{d}}\theta_p \\ \mathcal{T}_{\frac{\gamma_2}{d}}\theta_p &= \lambda\theta_{p+1}\end{aligned}$$

$$\lambda = \exp\left(-i\pi\tau(d+1/d)\right)$$

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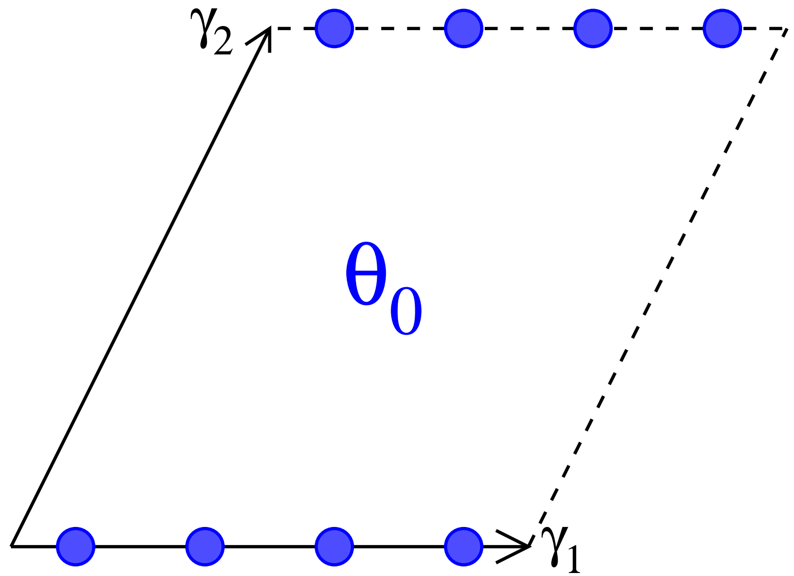

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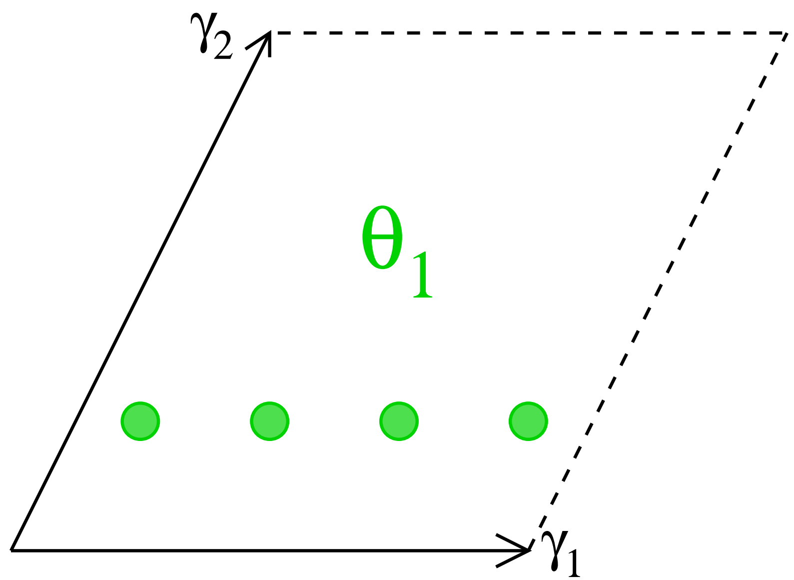

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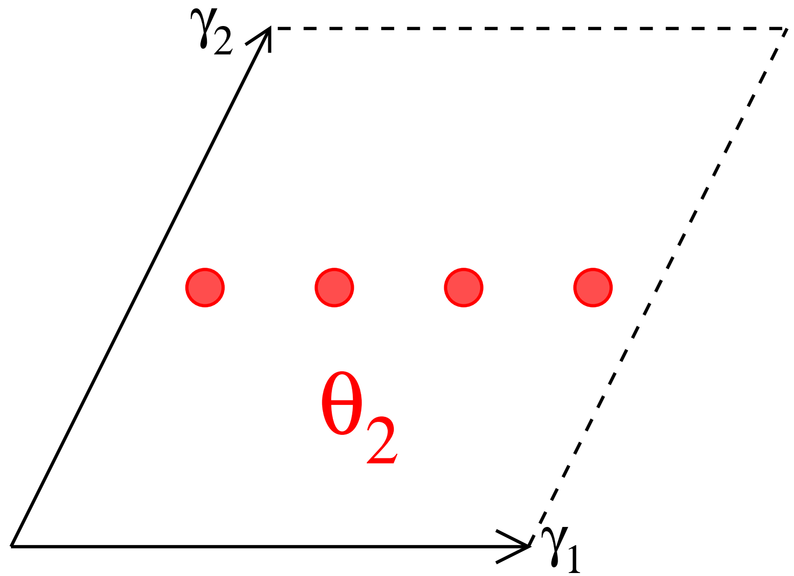

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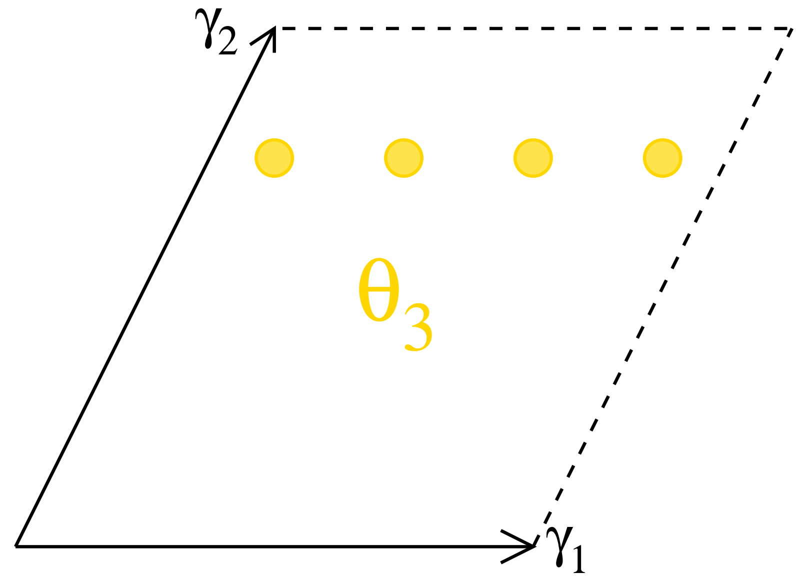
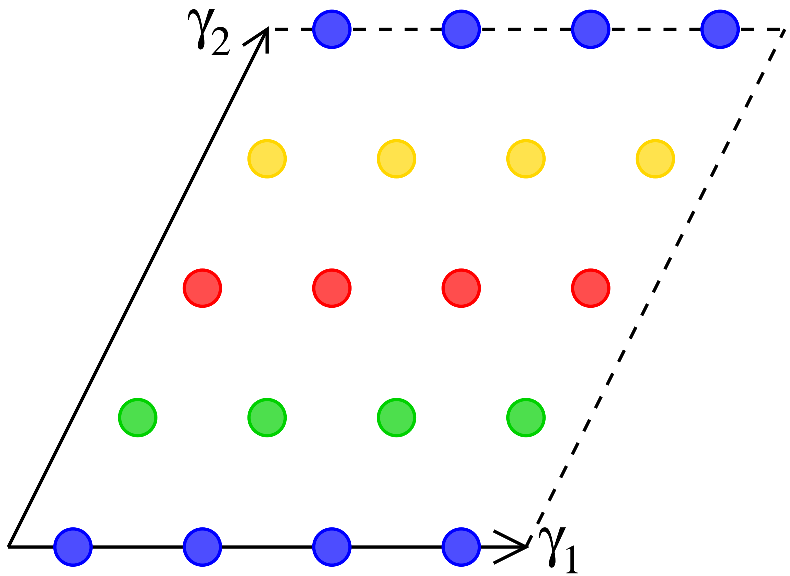


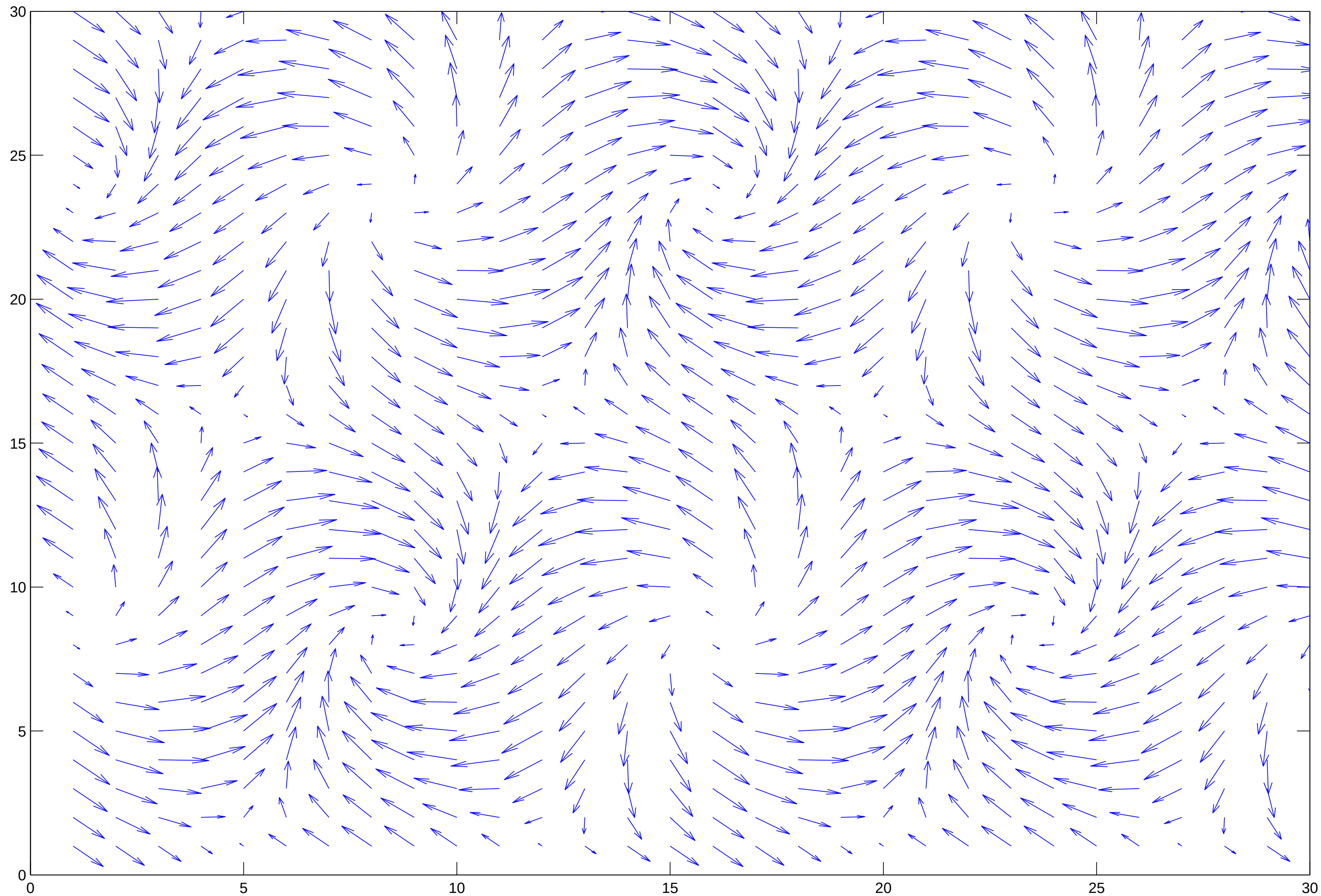
Periodic textures with lowest energy

$$|\Psi(z)\rangle = \begin{pmatrix} \theta_0(z) \\ \theta_1(z) \\ \cdot \\ \cdot \\ \cdot \\ \theta_{d-1}(z) \end{pmatrix}$$

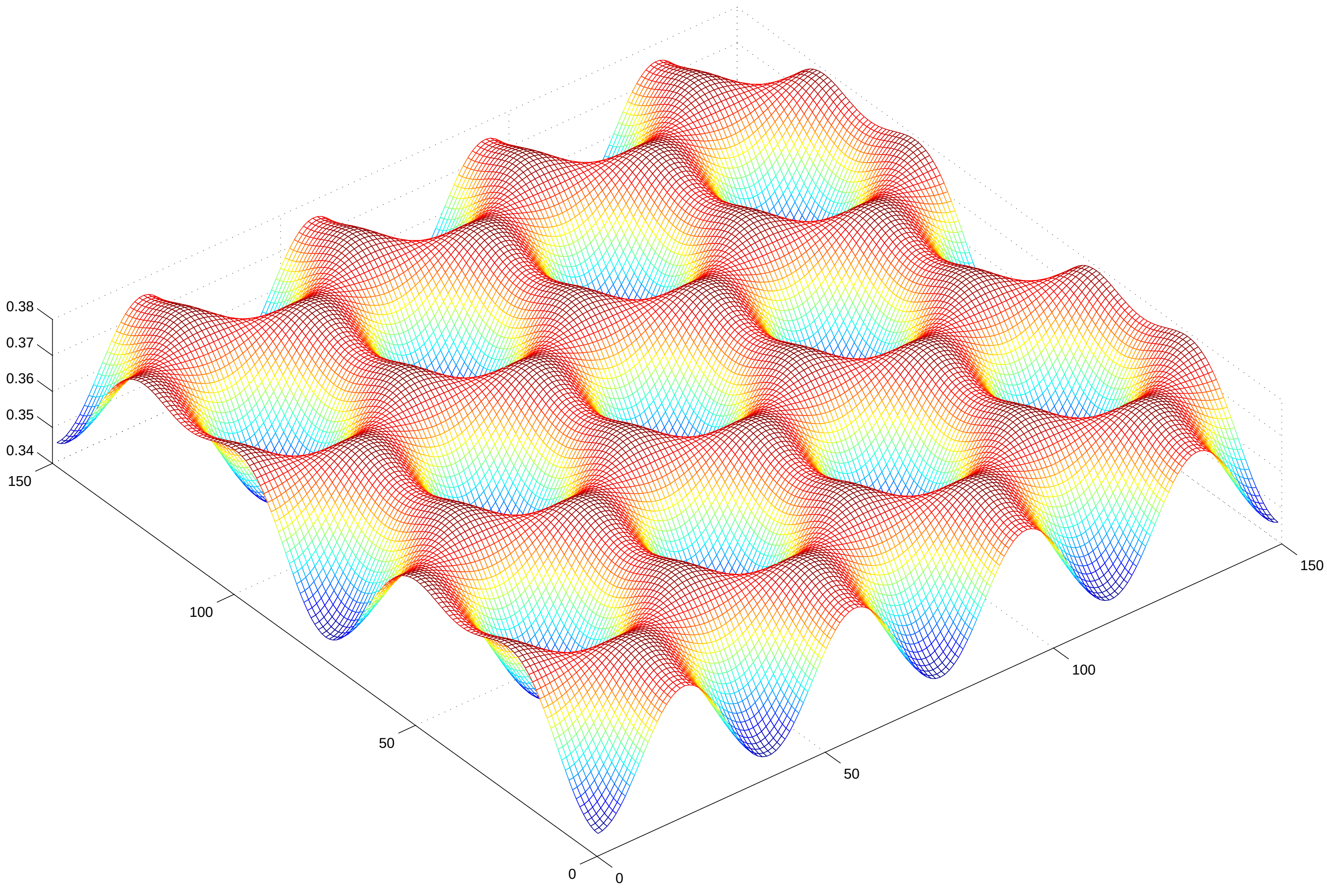
Pattern of zeros
(d=4)



Periodic texture $d = 2$



Periodic texture $d = 4$



Spatial variations of topological charge

$Q(r)$ is always γ_1/d and γ_2/d periodic.

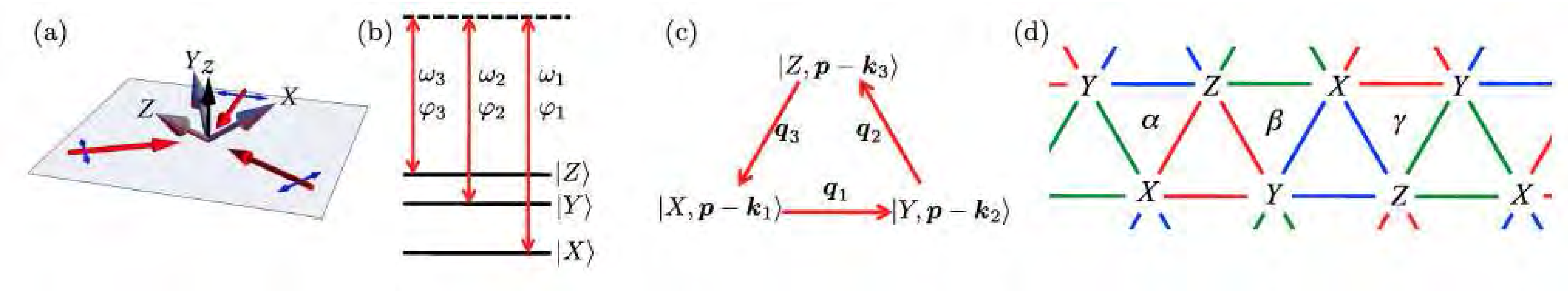
At large d the modulation contains mostly the lowest harmonic, and its amplitude decays exponentially with d.

Large d behavior for a square lattice:

$$Q(x,y) \simeq \frac{2}{\pi} - 4de^{-\pi d/2}[\cos(2\sqrt{d}x) - 2e^{-\pi d/2}\cos^2(4\sqrt{d}x) + (x \leftrightarrow y)] + \ldots$$

Only the triangular lattice seems to yield a true local energy minimum. This is most directly seen by computing eigenfrequencies of small deformation modes.

Applications of a flat topological charge profile



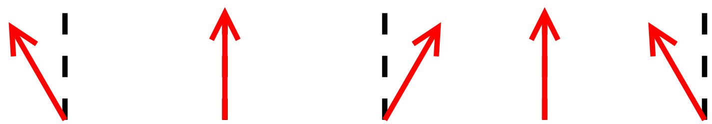
N. Cooper and J. Dalibard, PRL **110**, 185301 (2013); N. Cooper and R. Moessner, PRL **109**, 215302 (2012)

Tight binding model in momentum space with a non-zero average flux (*à la* Hofstadter) corresponds, in the large N limit to a periodic texture in real space $r \to |\psi(r)\rangle$ with very flat Berry curvature. After adding kinetic energy of atoms, this generates a very flat effective orbital magnetic field.

For $N = 3$, $\Omega = 3E_{\rm R}$, get Landau level with a bandwidth

$W = 0.015E_{\rm R}$.

Collective mode spectrum (I)

Analogy with spin-wave theory: 

$$\psi_a(r) = (\delta_{ab} + U_{ab}(r))\theta_b(r)$$

$U_{ab}(r)$ gives d^2 degrees of freedom for each *pseudo-momentum*, so there are d^2 branches (positive frequencies) in the excitation spectrum: the situation is reminiscent of a non-collinear antiferromagnet.

Consequences of $U(d)$ symmetry

Zero-momentum sector $(m_1, m_2) = (0, 0)$
Hamiltonian system with $N = d^2$ degrees of freedom.

If $g \in U(d)$, the transformation
$U \to gU, \ (gU)_{ac}(m_1, m_2) \equiv g_{ab}U_{bc}(m_1, m_2)$ preserves equations of motion.

This gives $f = d^2$ flat directions, tangent to the ground-state manifold at the periodic texture configuration.

Finite momentum sector

Get one *magnetophonon* with $\omega \simeq k^{1+\alpha/2}$ if $u(r) \simeq r^{-\alpha}$, and $d^2 - 1$ *spin-waves* with linear dispersion.

Time-dependent Hartree-Fock equation

Impose that $|\Psi(t)\rangle$ is a Slater determinant. For a texture, this is completely determined by the spinor configuration $|\psi(r,t)\rangle$.
Dynamics is obtained from $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ with :

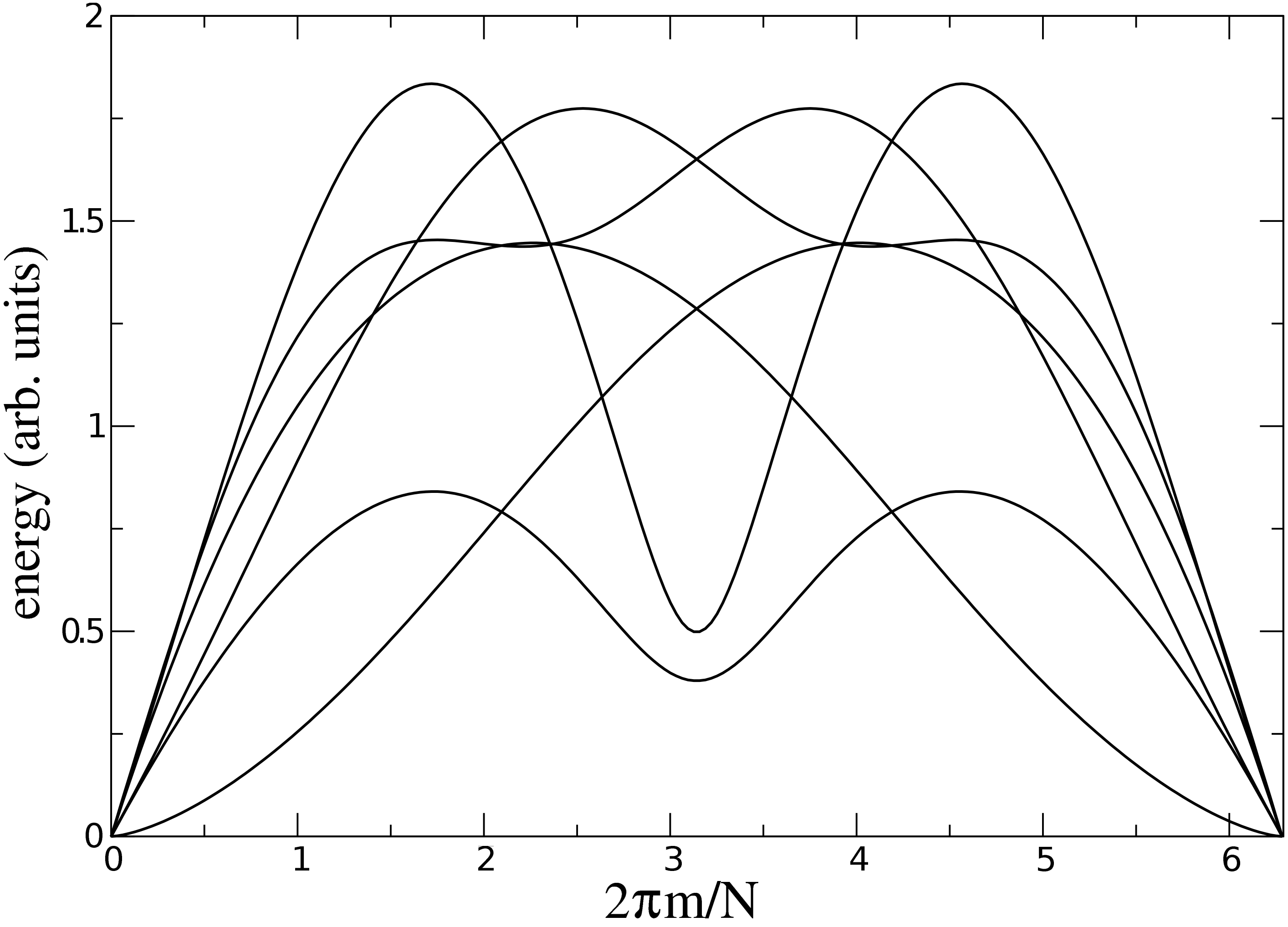
$$\mathcal{S}_1 = i \int_{t_1}^{t_2} dt \int d^{(2)}r \frac{\langle\psi(r,t)|\dot{\psi}(r,t)\rangle}{\langle\psi(r,t)|\psi(r,t)\rangle} dt$$

$$\mathcal{S}_2 = -\frac{1}{2} \int_{t_1}^{t_2} dt \int d^{(2)}r_1 \int d^{(2)}r_2 Q(r_1) u(r_1 - r_2) Q(r_2)$$

The variation of $\mathcal{S}$ has to be taken within the subspace of analytic spinors.
Equations of motion have a similar structure as in Bogoliubov theory of superfluids in the presence of a vortex lattice, see Matveenko and Shlyapnikov, PRA, **83**, 033604, (2011). Because of high symmetry of the $Q(r)$ profile, matrix structure breaks into small blocks of size 2 by 2 !

Collective mode spectrum (II)

Numerical spectrum for $d = 3$ and Coulomb interactions



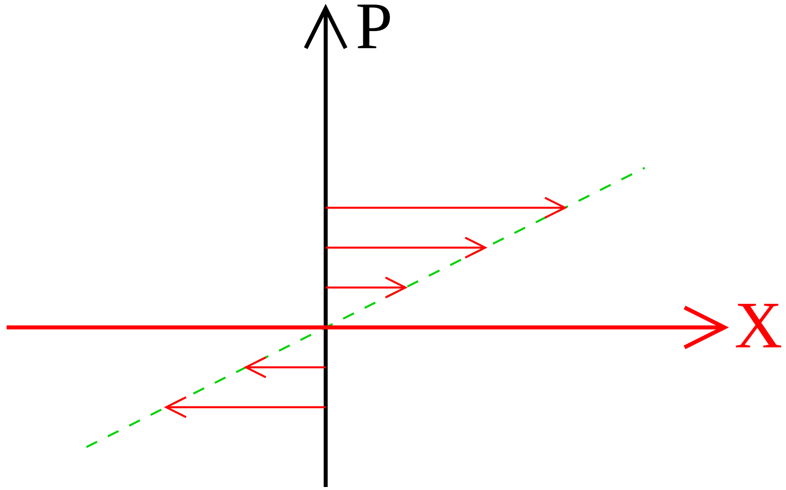
D. Kovrizhin, B. D. and R. Moessner, Phys. Rev. Lett. **110**, 186802, (2013)

Quadratic Hamiltonians with flat directions (I)

$$N = 1, \, f = 1$$

$$H = \frac{1}{2} P^2$$

$$\begin{pmatrix} \dot{X} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ P \end{pmatrix}$$



flat direction: X axis

Moving away by ϵ along the P axis generates drift motion parallel to the flat direction with velocity ϵ.

Quadratic Hamiltonians with flat directions (IIa)

$$N = 2,\ f = 2$$

Assume flat subspace is isotropic, generated by X_1, X_2 directions.

$$H = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2$$

$$\begin{pmatrix} \dot{X}_1 \\ \dot{P}_1 \\ \dot{X}_1 \\ \dot{P}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ P_1 \\ X_2 \\ P_2 \end{pmatrix}$$

One Jordan block for each flat direction.

Generating functions of drift motions, P_1 and P_2 commute everywhere, and in particular on the ground-state subspace.

Quadratic Hamiltonians with flat directions (IIb)

$$N = 2,\ f = 2$$

Assume flat subspace is *not* isotropic, generated by X_1, P_1 directions.

$$H = \frac{\omega}{2}\left(X_2^2 + P_2^2\right)$$

$$\begin{pmatrix} \dot{X}_1 \\ \dot{P}_1 \\ \dot{X}_1 \\ \dot{P}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ P_1 \\ X_2 \\ P_2 \end{pmatrix}$$

Only one zero eigenvector for each flat direction (no Jordan block). There is a finite frequency oscillator.

Generating functions of drift motions, X_1 and P_1 do not commute on the ground-state subspace.

Two spins with antiferromagnetic couplings

$$N = 2,\ f = 2$$

$$H = \vec{S}_1 \cdot \vec{S}_2, \qquad ||\vec{S}_1||^2 = s_1,\ ||\vec{S}_2||^2 = s_2$$

Ground-state manifold: $\vec{S}_1 = s_1\vec{n}$, $\vec{S}_2 = -s_2\vec{n}$, $||\vec{n}|| = 1$

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$$\frac{d\vec{S}_i}{dt} = (\vec{S}_1 + \vec{S}_2) \wedge \vec{S}_i$$

Eigenvalue spectrum: $\{0, 0, s_1 - s_2, s_2 - s_1\}$

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Ground-state manifold: $\vec{S}_1 = s_1 \vec{n}$, $\vec{S}_2 = -s_2 \vec{n}$, $||\vec{n}|| = 1$

If $s_1 \neq s_2$: non-isotropic, one finite frequency oscillator.
$\vec{S}_1 + \vec{S}_2 \neq 0$ on ground-state manifold.

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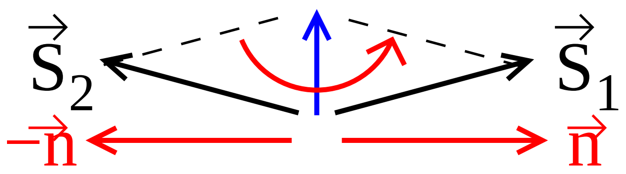
Ground-state manifold: $\vec{S}_1 = s_1\vec{n}$, $\vec{S}_2 = -s_2\vec{n}$, $||\vec{n}|| = 1$

If $s_1 = s_2$: isotropic, two Jordan blocks.

$\vec{S}_1 + \vec{S}_2 = 0$ on ground-state manifold.



Phase-space as cotangent bundle, spanned by $\vec{n}$ and $\dot{\vec{n}}$. Gives rise to a non-linear σ-model.

$$\mathcal{S} = g \int dt \int d^{(2)}r \left[(\partial_t \vec{n})^2 - (\partial_x \vec{n})^2 - (\partial_y \vec{n})^2\right]$$

An $U(d)$ σ-model for collective dynamics? (I)

For zero momentum:

A Jordan block is associated to each flat direction.

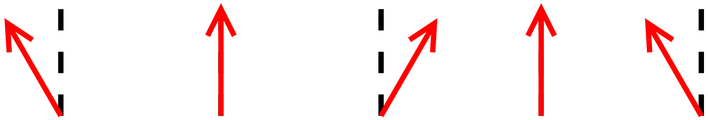
$U(d)$-orbit of the periodic texture configuration is isotropic.

Small deviations from periodic texture: ($U_{ab}(m_1, m_2)$ small)

$$\psi_a(r) = \theta_a(r) + \sum_{b,m_1,m_2} U_{ab}(m_1, m_2)\chi_b(m_1, m_2)(r)$$

An $U(d)$ σ-model for collective dynamics? (II)

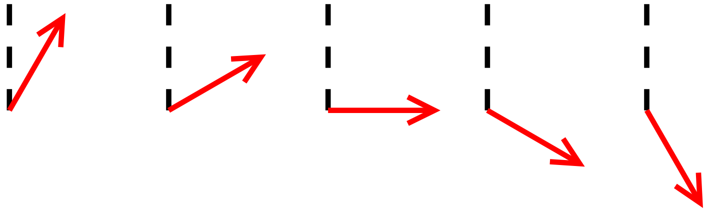
Linear spin-waves



$\psi_a(r) = (\delta_{ab} + U_{ab}(r))\theta_b(r)$

$U_{ab}(r) = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \tilde{U}_{ab}(\vec{k})$

Sigma model (gradient expansion)



$\psi_a(r) = g_{ab}(r)\theta_b(r)$, $g_{ab}(r)$ unitary

$\mathcal{S}$ local functional of derivatives of g_{ab}.

An $U(d)$ σ-model for collective dynamics ? (III)

Projection on a space of holomorphic functions not compatible with unitarity condition $\sum_b g_{ba}(r)\overline{g_{bc}(r)} = \delta_{ac}$.

Our "spin-wave theory" has the following structure:

$\psi_a(r) = \left[(\delta_{ab} + \hat{U}_{ab})\theta_b\right](r)$ with $\hat{U}_{ab}(r) = \mathcal{P}_{hol}\left(\sum_{\vec{k}} U_{ab}(\vec{k})e^{i\vec{k}\cdot\vec{r}}\right)$

Suggests to construct gradient expansion using $\mathcal{P}_{hol}$:

$\psi_a(r) = \mathcal{P}_{hol}\left(g_{ab}(r)\theta_b\right)(r)$?

Note: $\mathcal{P}_{hol} f \mathcal{P}_{hol} g\theta = \mathcal{P}_{hol}(f \star g)\theta$

But is there an optimal choice of $\mathcal{P}_{hol}$?

$\mathcal{S}$ non-local functional of derivatives of g_{ab}. Can we approximate it by a local one in the long wave-length limit ?

Open questions

- Derivation of our effective $CP^{(d-1)}$ model from microscopic model ?
- Is there a degeneracy lifting effect from zero point motion energy of finite frequency modes of the Hessian ?
- Are the collective degrees of freedom described by an emerging $U(d)$ σ-model ?
- Role of non-commutativity of physical plane ?
- Role of quantum fluctuations $\rightarrow$ quantum melting of Skyrmion crystal?
- Effect of non-infinite stiffness in $CP^{(d-1)}$ model $\rightarrow$ admixture of non-analytic components.
- Extension to higher integer filing factors $\rightarrow$ $CP^{(d-1)}$ replaced by Grassmanian manifolds.

Effect of valley anisotropy (I)

AlAs quantum wells: three valleys with mass anisotropy,
$\lambda = (m_x/m_y)^{1/2} \simeq 2$
Nematic anisotropy term: $H_N = 2\Delta_0\kappa \sum_{i\neq j} |\psi_i|^2|\psi_j|^2$
Generalizes (Abanin, Parameswaran, Kivelson, Sondhi, PRB **82**, 035428 (2010)).
For AlAs at $n_{el} = 2.5 \times 10^{11}\ \mathrm{cm}^{-1}$, $\Delta_0\kappa = 2.5\ \mathrm{K}$, $\rho_s = 5.2\ \mathrm{K}$.

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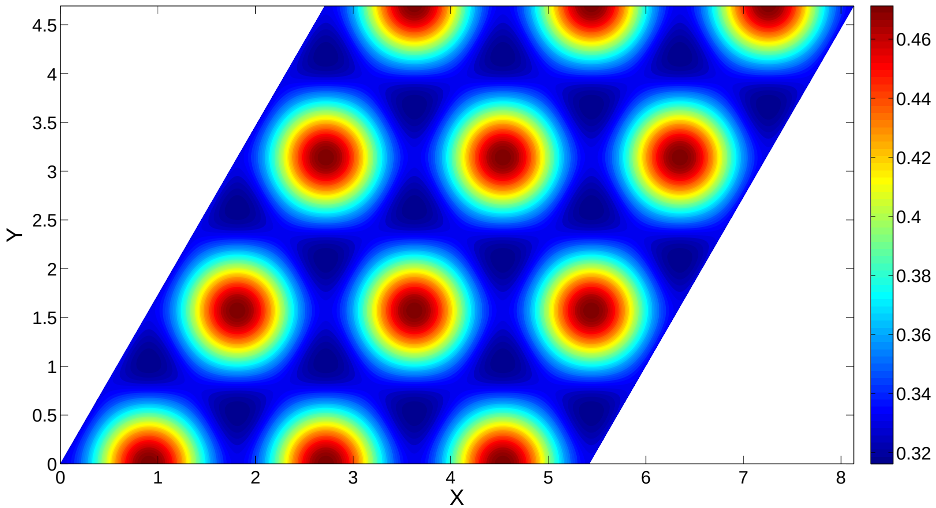

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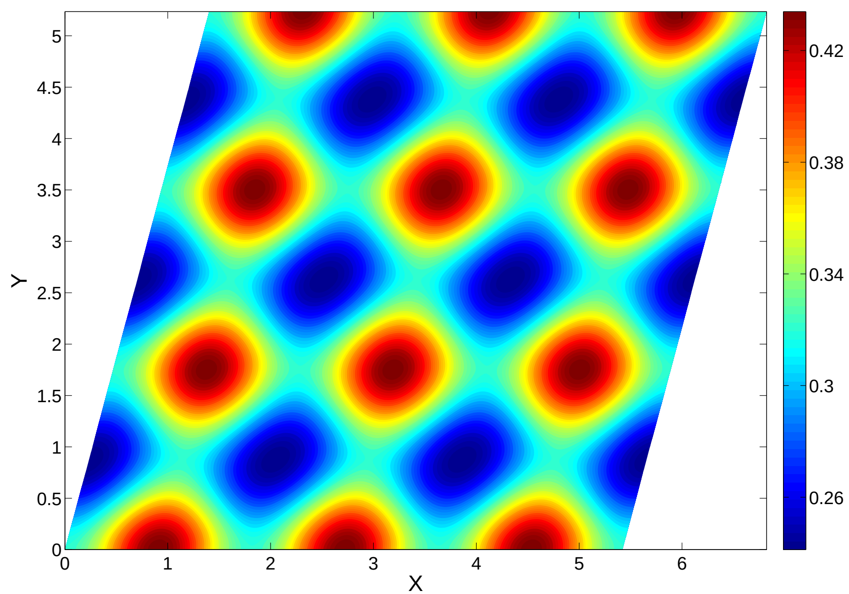

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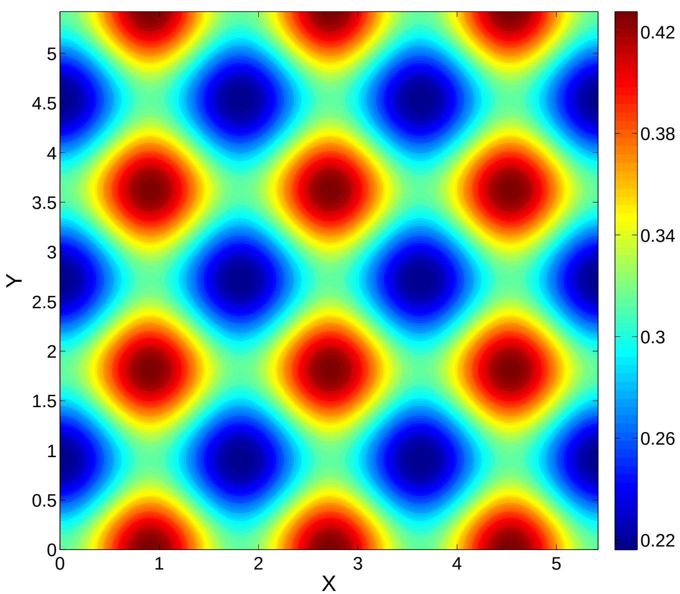
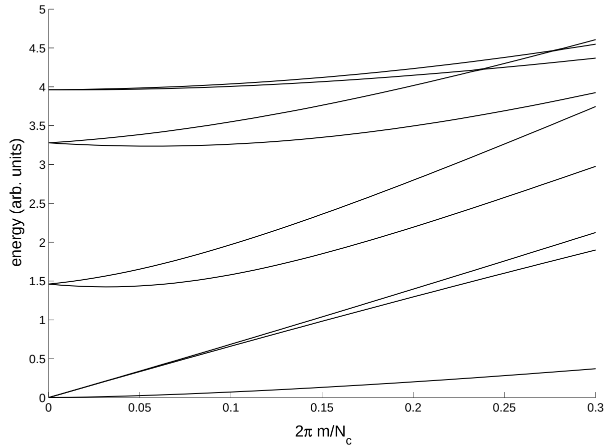
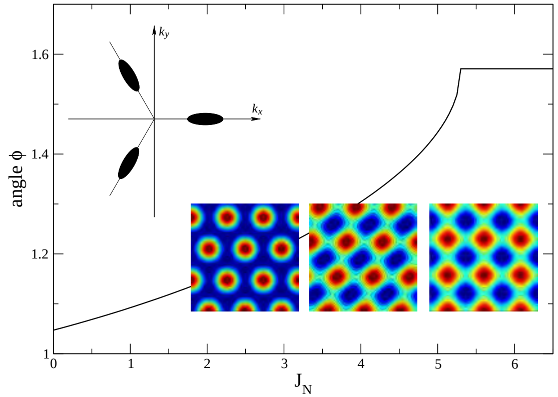
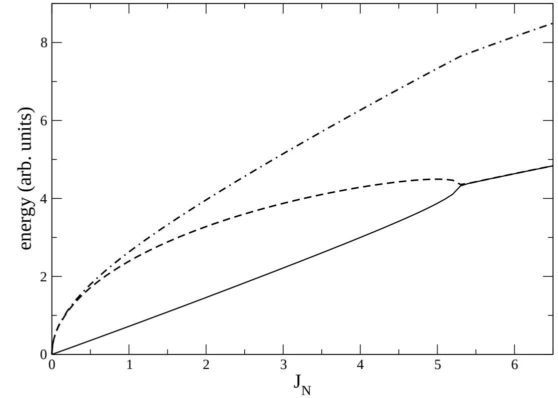


Effect of valley anisotropy (II)







In presence of anisotropy, the three lowest states remain gapless: the *magnetophonon* and two *spin-waves* associated to a Cartan subalgebra of the $su(3)$ Lie algebra.
A gap develops for the six remaining states which come in pairs.