# Scalar instantons and the Hartle-Hawking-Maldacena proposal for dS/CFT

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(based on 1406.6148 with Sebastian de Haro)

# Outline

# Introduction and Motivation

- The HHM proposal: consider scalar fields $\phi$ on a fixed $dS_4$ background i.e.

$$ds^2 = \frac{\ell^2}{\eta^2}(-d\eta^2 + d\vec{x}^2)\,, \quad \eta \in (-\infty, 0)\,, \ \phi(\eta, \vec{x}) \sim \eta\varphi_{\vec{k}}(\eta)e^{i\vec{k}\vec{x}}$$

- The path integral of configurations born out of the Bunch-Davies vacuum $|E\rangle$ (i.e. having positive frequencies at $\mathcal{I}^-$) and ending into real configurations $\varphi$ at late times (i.e. at $\mathcal{I}^+$) yields the probability for the late time configuration $\varphi$

$$\Psi_E[\eta_0, \varphi] \sim e^{iS(\eta_0,\varphi)} \quad , \ \mathcal{P}_\varphi = |\Psi_E[\eta_0, \varphi]|^2\,, \ \ \eta_0 \to 0$$

- The latter can be used to calculate e.g. late time correlation functions in $dS_4$

$$\lim_{\eta,\eta'\to 0}\langle E|\phi(\eta,\vec{x})\phi(\eta',\vec{x}')|E\rangle = \int(\mathcal{D}\varphi)\mathcal{P}_\varphi\varphi(\vec{x})\phi(\vec{x}')$$

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- According to Hartle and Hawking, the previous wave functional can be calculated by a path integral of the form

$$\Psi_{\text{HH}}[\chi] = \int_{\Phi|_{\partial M}\equiv\chi} \mathcal{D}\Phi \; e^{-S_E[\Phi]} \; ,$$

$S_E[\Phi]$ is the *Euclidean* action of all bulk fields, including the metric.

- $\partial M$ is a 3-dimensional spacelike hypersurface near the future infinity (i.e. of an asymptotically de Sitter space-time with radius $\ell_{\text{dS}}$.)
- This is a generalisation of the usual calculation for the ground state in QM using path integral configurations that vanish under Wick rotation in the Euclidean past - "no boundary condition".

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- Then, Maldacena's proposal entails that the HH state can be obtained if one calculates the corresponding renormalized on-shell action on Euclidean $\mathrm{AdS}_4$ ($\mathrm{EAdS}_4$) with radius $\ell_{\mathrm{AdS}}$, and performs the analytic continuation $\ell_{\mathrm{AdS}} \to i\ell_{\mathrm{dS}}$.
- Since the $\mathrm{EAdS}_4$ on-shell action is reasonably well defined—it gives the partition function of a Euclidean 3-dimensional CFT—Maldacena's proposal gives a way to make sense of and calculate the HH state from AdS/CFT.

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- The HHM proposal appears to give a boost to dS/CFT, however there is still no completely satisfactory string theory description for gravity with a positive cosmological constant.
- An adventurous set up for a concrete realisation of dS/CFT was proposed by *[Anninos et. al (11)]* based on Vassiliev's higher-spin (HS) theory which provides the only known consistent classical description of interacting higher-spin gauge fields in a de Sitter background.
- The holographic dual is the Euclidean Sp($N$) vector model with anti-commuting scalars. It is a free CFT$_3$ when the higher-spin symmetry is unbroken.
- An analytic continuation is also at work here, as $N \mapsto -N$, where $N \sim \ell_{\rm dS}^2/G_{\rm N}$ ($G_{\rm N}$ fixed), maps the Sp($N$) anti-commuting vector model to the usual (commuting) O($N$) vector model which is believed to be the holographic dual of HS theory on AdS$_4$.

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- In *[Maldacena (11)]* the on-shell action on Euclidean $dS_4$ (the four-sphere) was shown to analytically continue to *twice* the finite part of the volume of $EAdS_4$, after dropping counterterms which correspond to imaginary phases and do not contribute to the measure $|\Psi_{HH}|^2$
- Also, generating functionals for 2-pt functions in $EAdS_4$ and $dS_4$ where shown to be related by suitable analytic continuations i.e. *[Harlow et. al. (11)]*
- The HHM proposal has not been tested for cosmological theories with matter fields and generic potentials. This would entail a discussion of exact non-trivial solutions and the analytic continuation of their moduli spaces.
- Also, it is important to know how boundary deformations (such as multitrace deformations) of the CFT dual to $EAdS_4$ carry over to $dS_4$, i.e. complexification of the deformation parameters.

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# Scalar Instantons in Euclidean $AdS_4$ and $dS_4$

- In AdS/CFT the properly renormalized bulk on-shell action is usually evaluated as a functional of the boundary conditions for a suitably regular solutions of the bulk e.o.m. Depending on the choice of boundary conditions it yields either a generating functional for quantum correlation functions or an effective action for a putative CFT living on the boundary.
- However, given a *particular* solution of the bulk equations of motion, namely one where the bulk fields assume fixed boundary values, the renormalized bulk on-shell action evaluates the *free energy F* of the boundary theory and therefore the partition function as:

$$Z = e^{-S_{\text{on-shell}}} = e^{-F} .$$

- For example, using the Poincaré patch of $EAdS_4$ the renormalized EH action gives zero, while using a suitable global parametrization yields the non-zero free energy of a CFT on $S^3$.

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- Conformally coupled scalars in $EAdS_4$, in the Poincaré patch

$$\phi(z,\vec{x}) \to z\,\phi_{(0)}(\vec{x}) + z^2\,\phi_{(1)}(\vec{x})\,, \quad z \to 0$$

- When $\phi_{(0)}(\vec{x})$ takes a *fixed* form - instanton solutions

$$S_{\text{on-shell}}[\phi_{(0)}] = -\ln \tilde{Z}_0$$

  gives the partition function of the *dual* boundary CFT, that has in its spectrum the operator $\mathcal{O}_1$ , $\Delta = 1$ and $\langle \mathcal{O}_1 \rangle \sim \phi_{(0)}$.
- By a Legendre transformation we can obtain the partition function of the CFT having an operator $\mathcal{O}_2$ of dimension $\Delta = 2$ and $\langle \mathcal{O}_2 \rangle \sim \phi_{(1)}$.
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$$S = \frac{1}{2\kappa^2}\int_{M_\epsilon} \mathrm{d}^4x\,\sqrt{g}\,(-R+2\Lambda)$$

$$+\int_{M_\epsilon} \mathrm{d}^4x\,\sqrt{g}\left(\frac{1}{2}\,(\partial_\mu\phi)^2+\frac{R}{12}\,\phi^2+\frac{\lambda}{4!}\,\phi^4\right)$$

$$-\frac{1}{2\kappa^2}\int_{\partial M_\epsilon} \mathrm{d}^3x\,\sqrt{\gamma}\,2K\left(1-\frac{\kappa^2}{6}\,\phi^2\right),$$

where $\kappa^2 = 8\pi G_{\mathrm{N}}$ and $\Lambda = \mp\frac{3}{\ell^2}$.

- We add the usual counter terms in the conformal boundary $\partial M_\epsilon$

$$S_{\mathrm{ct}}^{\mathrm{EAdS}} = \frac{1}{\kappa^2}\int_{\partial M_\epsilon} \mathrm{d}^3x\,\sqrt{\gamma}\left(\frac{4}{\ell}+\ell\,R[\gamma]\right).$$

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# Scalar Instantons in Euclidean $AdS_4$ and $dS_4$

- The instanton solutions are constructed using the Weyl invariance of the matter part of the action. We will use global coordinates conformal to $I \times \partial M$, where $I$ is a (finite or infinite) interval.
- Here $\partial M = S^3$. In $EAdS_4$ we will use conformal cylinder coordinates:

$$ds^2_{\mathrm{EAdS}} = \frac{1}{\sinh^2 \frac{\tau}{\ell}} \left( d\tau^2 + \ell^2_{\mathrm{AdS}} \, d\Omega_3^2 \right) , \quad \tau \in (0, \infty) \, .$$

- For practical reasons (to avoid the 2 mentioned above) we will work with half Euclidean $dS_4$ i.e. the southern hemisphere of $S^4$.

$$ds^2_{S^4_-} = \frac{1}{\cosh^2 \frac{r}{\ell_{\mathrm{dS}}}} \left( dr^2 + \ell^2_{\mathrm{dS}} \, d\Omega_3^2 \right) , \quad r \in (-\infty, 0] \, .$$

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- Here $\partial M = S^3$. In EAdS$_4$ we will use conformal cylinder coordinates:

$$ds^2_{\rm EAdS} = \frac{1}{\sinh^2 \frac{\tau}{\ell}} \left( d\tau^2 + \ell^2_{\rm AdS}\, d\Omega_3^2 \right) , \quad \tau \in (0, \infty) \, .$$

- For practical reasons (to avoid the 2 mentioned above) we will work with half Euclidean dS$_4$ i.e. the southern hemisphere of $S^4$.

$$ds^2_{S^4_-} = \frac{1}{\cosh^2 \frac{r}{\ell_{\rm dS}}} \left( dr^2 + \ell^2_{\rm dS}\, d\Omega_3^2 \right) , \quad r \in (-\infty, 0] \, .$$

- Lorentzian de Sitter is obtained by Wick rotating back $r \to -i\ell_{\rm dS}\,\tau$.

# Scalar Instantons in Euclidean AdS$_4$ and dS$_4$

- The relevant solutions of the scalar sector are obtained by solving the Klein-Gordon equation

$$\Box\phi - \frac{2\Lambda}{3}\,\phi - \frac{\lambda}{6}\,\phi^3 = 0$$

together with the requirement that the stress-energy tensor vanishes. The latter requirement turns out to give

$$\left(\nabla_\mu\nabla_\nu - \frac{1}{4}\,g_{\mu\nu}\Box\right)\phi^{-1} = 0\,.$$

- The solutions on EAdS$_4$ and $S^4$ are then given by

$$\phi^{\varepsilon,b_I}_{\mathrm{EAdS}_4}(\tau,\Omega_3) = \frac{\varepsilon\sinh\frac{\tau}{\ell}}{b_0\cosh\frac{\tau}{\ell} + b_5\sinh\frac{\tau}{\ell} + b_i\Omega_i}$$

$$\phi^{\varepsilon,a_I}_{S^4}(r,\Omega_3) = \frac{\varepsilon\cosh\frac{r}{\ell}}{a_0\sinh\frac{r}{\ell} + a_5\cosh\frac{r}{\ell} + a_i\Omega_i}\,,$$

$i = 1,\ldots,4$, $\Omega_i$ is a unit vector normal to $S^3$ and $\varepsilon = \pm 1$.

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# Scalar Instantons in Euclidean $AdS_4$ and $dS_4$

- The moduli spaces are non-trivial,

$$\text{on EAdS}_4: \quad -b_0^2 + b_5^2 + b_i^2 = \frac{\lambda_{\text{AdS}}}{12}\,\ell_{\text{AdS}}^2\,, \quad i = (1,\ldots,4)$$

$$\text{on } S^4: \qquad a_0^2 - a_5^2 + a_i^2 = \frac{\lambda_{S^4}}{12}\,\ell_{\text{dS}}^2\,.$$

- Namely, moduli spaces of instantons on $EAdS_4$ and $dS_4$ are themselves $EAdS_4$ or $dS_4$ depending on the sign and values of the quartic couplings $\lambda_{\text{Ads}}$ and $\lambda_{\text{dS}}$
- For example, the moduli space of the solutions is $EAdS_4$ if $\frac{\lambda_{\text{AdS}}}{12}\,\ell_{\text{AdS}}^2 < b_5^2$, which is the condition required for regularity of the bulk solution. At the critical value $\frac{\lambda_{\text{AdS}}}{12}\,\ell_{\text{AdS}}^2 = b_5^2$ the effective potential of the dual field theory becomes unbounded from below, which was interpreted as an instability against marginal deformations *[de Haro (06)]*.

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# Scalar Instantons in Euclidean $AdS_4$ and $dS_4$

- Not all of the parameters $a_I, b_I$ ($I = 0, \ldots, 5$) are moduli. Two of them are boundary conditions relating the leading and subleading modes of the scalar:

$$\phi_{S^4_-}(-\epsilon\,\ell,\Omega) = \frac{\varepsilon}{a_5 + a_i\Omega_i} + \frac{\epsilon\,\varepsilon\,a_0}{(a_5 + a_i\Omega_i)^2} + \mathcal{O}(\epsilon^2) = \Phi^{a_5,a_i}_{(0)} + \epsilon\,\Phi^{a_5,a_i}_{(1)} + ..$$

$$\phi_{\rm EAdS}(\epsilon\,\ell,\Omega) = \frac{\epsilon\,\varepsilon}{b_0 + b_i\Omega_i} - \frac{\epsilon^2\,\varepsilon\,b_5}{(b_0 + b_i\Omega_i)^2} + \mathcal{O}(\epsilon^3) = \epsilon\,\Phi^{b_0,b_i}_{(0)} + \epsilon^2\,\Phi^{b_0,b_i}_{(1)} + ..$$

- The leading and subleading terms in the expansion of the field are related by:

$$\Phi^{a,a_i}_{(0)}(\Omega) \equiv \frac{\varepsilon}{a + a_i\Omega_i}$$

$$\Phi^{a,a_i}_{(1)}(\Omega) = \pm\,\varepsilon\,\alpha\left(\Phi^{a,a_i}_{(0)}(\Omega)\right)^2\,,\quad \mathrm{EAdS}/S^4_-\,,$$

where $\alpha = b_5$ for $\mathrm{EAdS}_4$ and $\alpha = a_0$ for $S^4_-$. Thus, $b_5, a_0$ parametrize marginal triple trace deformations that change the boundary conditions from Dirichlet to mixed.

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# Scalar Instantons in Euclidean $AdS_4$ and $dS_4$

- The solutions are exact: they have zero stress-energy tensor hence the EAdS/dS background stays unmodified. Thus we can compute the on-shell effective action including its finite part. For simplicity we set the spherical modes $a_i = b_i = 0$ and get:

$$S_{\text{EAdS}}^{\text{on-shell}} = \frac{4\pi^2\ell_{\text{AdS}}^2}{\kappa^2} - \frac{\lambda_{\text{AdS}}\,\pi^2\ell_{\text{AdS}}^4}{12}\frac{2b_0+b_5}{3b_0^3(b_0+b_5)^2} + \pi^2\ell_{\text{AdS}}^2\frac{b_5}{b_0^3} + \mathcal{O}(\epsilon)$$
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- In the special case of a Neumann boundary condition $b_5 = 0$, which requires negative coupling $\lambda_{\text{AdS}} < 0$, we get

$$S_{\text{on-shell, EAdS}}^{b_5=b_i=0} = \frac{4\pi^2\ell_{\text{AdS}}^2}{\kappa^2} - \frac{8\pi^2}{\lambda_{\text{AdS}}} > 0.$$

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$$-\log\Psi_{\rm HH}=S^{\text{on-shell}}_{S^4_-}=-\frac{4\pi^2\ell^2_{S^4}}{\kappa^2}-\frac{\lambda_{\rm dS}\,\pi^2\ell^4_{S^4}}{12}\frac{2a_5-a_0}{3a_5^3(a_5-a_0)^2}-\pi^2\ell^2_{S^4}\,\frac{a_0}{a_5^3}$$
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After the analytic continuation $\ell_{S^4}\mapsto -i\ell_{\rm AdS}$, $\lambda_{S^4}\mapsto\lambda_{\rm AdS}$ this agrees with the $EAdS_4$ result.

- This is negative for $\lambda_{S^4}<-2\kappa^2/\ell^2_{S^4}$ and vanishes for $\lambda_{S^4}=-2\kappa^2/\ell^2_{S^4}$. The latter is the critical value arising in the consistent truncation of $\mathcal{N}=8$ sugra, after the analytic continuation of the $dS_4$ radius.

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- The matching under analytic continuation of the gravity part is already nontrivial i.e. the $EAdS_4$ calculation needs to be regularized and renormalized, whereas the $S^4_-$ calculation of the HH wave function is completely finite.
- In order to match the matter contributions, however, in the second term we need to analytically continue the couplings as well. This analytic continuation from $EAdS_4$ to $S^4_-$ is an invertible map $\gamma$:

$$\gamma(\ell_{\rm AdS}) = i\,\ell_{\rm dS}\,,\gamma(b_0) = i\,a_5\,,\gamma(b_5) = -i\,a_0\,,\gamma(b_i) = ia_i\ .$$

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# Scalar Instantons in Euclidean AdS$_4$ and dS$_4$

- The moduli have to be analytically continued since they are dimensionful quantities, to be measured in units of the radius. Defining dimensionless moduli $y_I = a_I/\ell_{\rm dS}$, $z_I = b_i/\ell_{\rm AdS}$, $I = 0, \ldots, 5$, we find

$$-z_0^2 + z_5^2 + z_i^2 = \frac{\lambda_{\rm AdS}}{12}\ ,\ y_0^2 - y_5^2 + y_i^2 = \frac{\lambda_{\rm dS}}{12}\ ,\ i = 1, \ldots, 4.$$

- These are all real quantities on both sides. The moduli space is O(1,5) invariant:

$$\eta^{IJ} y_I y_J = \frac{\lambda_{\rm dS}}{12}\ ,\quad I, J = 0, \ldots, 5\ ,$$

with $\eta^{IJ}$ the O(1,5) Minkowski metric. The analytic continuation is then simply an SO(1,5) map of the moduli space onto itself:

$$z_I = \epsilon_I{}^J y_J\ ;\quad \epsilon = \delta \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & \mathbf{1}_{4\times 4} \end{pmatrix},\quad \delta^2 = 1.$$

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- We note here that in order to enforce the conformal boundary conditions to the solutions of the bulk equations of motion, one needs to add a further "triple trace deformation" term to the action

$$S_{\text{bdy def}} = -\frac{b_5 \ell_{\text{AdS}}^2}{3} \int d\Omega_3 \, \Phi_{(0)}^3(\Omega) = -\frac{2\pi^2 \ell_{\text{AdS}}^2 b_5}{3 b_0^3} \, .$$

This of course agrees, after the analytic continuation with the term one gets in the $dS_4$ case, $-\frac{2\pi^2 \ell_{\text{dS}}^2 \, a_0}{3 a_5^3}$. These terms simply add up to the solutions above.

# Geometric Interpretation

- The 3-sphere partition function $Z_{S^3}$ of a three-dimensional CFT is a measure of its degrees of freedom. It has been argued that for unitary CFTs the corresponding free energy is given (in suitably chosen units) by

$$F = -\log |Z_{S^3}| \,,$$

  which is positive and satisfies an $F$-theorem, namely it decreases along RG flows from the UV to the IR *[Jafferis (10)]*.

- Holographically, the partition function is usually calculated using the bulk gravitational action on $EAdS_4$ with matter fields set to zero. This is the first term of the results above. It is proportional to the dimensionless ratio $\ell^2_{\rm AdS}/\kappa^2$.
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# Geometric Interpretation

- Consider the conformally related metrics

$$g_{\mu\nu} = \Omega^{-2}\, h_{\mu\nu}$$

- One can then show that

$$-I_h := \frac{1}{2\kappa^2}\int d^4x\,\sqrt{h}\,(R[h] - 2\Lambda_h)$$

$$= \int d^4x\,\sqrt{g}\left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{R[g]}{12}\phi^2 + \frac{\lambda}{4!}\phi^4\right) - \frac{1}{2}\int d^4x\,\partial_\mu\left(\sqrt{g}g^{\mu\nu}\phi\partial_\nu\phi\right)$$

$$=: I_{(g,\phi)}$$

where we have defined

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- Then, the on-shell value of the instanton action $I_{(g,\phi)}$ gives

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- The crucial observation now is that the instanton solutions $\phi_{\text{inst}}$ on either $S^4_-$ and EAdS$_4$, with the moduli set to specific values, correspond exactly to the conformal factor relating the two metrics.
- Hence, the on-shell action of instantons on EAdS$_4$ corresponds to the volume of (half) $S^4$ and conversely, the on-shell action of instantons on the half $S^4$ corresponds to the volume of EAdS$_4$,

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- The instanton deformation parameter $b$ (which corresponds to $a_0$ in the previous section) regulates the volume of $\text{EAdS}_4$.
- Let us see how this arises. On the half $S^4$ with curvature radius $\ell$ and metric

$$ds^2 = \frac{4}{\left(1+\frac{\rho^2}{\ell^2}\right)^2}\left(d\rho^2 + \rho^2 d\Omega_3^2\right)$$

the instanton solutions are given by

$$\phi_{\text{inst}}(\rho) = \pm\sqrt{\frac{12}{\lambda_{S^4}}\frac{1}{b}\frac{1+\frac{\rho^2}{\ell^2}}{1-\frac{\rho^2}{b^2}}}$$

where $\lambda_{S^4} > 0$.
- It is important to note the presence of the instanton modulus $b$ which is in principle unrelated to $\ell$. In particular, since the range of the radial coordinate is $\rho \in [0, \ell)$, if we consider $b > \ell$ the solution is everywhere regular.

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# Geometric Interpretation

- Next we notice that half $S^4$ and $\mathrm{EAdS}_4$ are conformally related metrics. In particular

$$
\mathrm{d}s^2 = \frac{4}{\left(1+\frac{\rho^2}{\ell^2}\right)^2}\left(\mathrm{d}\rho^2+\rho^2\mathrm{d}\Omega_3^2\right) =
$$

$$
= \left(\frac{1-\frac{\rho^2}{b^2}}{1+\frac{\rho^2}{\ell^2}}\right)^2 \frac{4}{\left(1-\frac{\rho^2}{b^2}\right)^2}\left(\mathrm{d}\rho^2+\rho^2\mathrm{d}\Omega_3^2\right) ,
$$

where on the right $\mathrm{EAdS}_4$ has a generally different radius $b$ which is set equal to the instanton modulus. Hence, the calculation of the on-shell action for instantons on half $S^4$ boils down to the calculation of the *regularized* volume of $\mathrm{EAdS}_4$.

# Geometric Interpretation

- Notice that it is the presence of the instanton modulus $b > \ell$ that gives rise to a particular regularization of the volume of the EAdS$_4$ space with radius $\ell$.
- Explicitly, we obtain ($\alpha = \frac{\ell}{b} < 1$)

$$I^{\text{on-shell}}_{(g,\phi)}\left(S^4_-\right) = \frac{\Lambda_h}{\kappa^2}\int d^4x\sqrt{h} = \frac{8\pi^2 b^2}{\kappa^2}\frac{\alpha^4(\alpha^2-3)}{(1-\alpha^2)^3} = \frac{16\pi^2}{\lambda_{S^4}}\frac{\alpha^4(\alpha^2-3)}{(1-\alpha^2)^3}$$

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- Next we consider the instanton solutions on $\mathrm{EAdS}_4$ with radius $\ell$. These have the form

$$\phi_{\text{inst}}(\rho) = \pm\sqrt{\frac{12}{-\lambda_{\text{AdS}}}}\frac{1}{b}\frac{1-\frac{\rho^2}{\ell^2}}{1+\frac{\rho^2}{b^2}}\,.$$

with $\lambda_{\text{AdS}} < 0$ and $b$ the modulus which we take here again to be $b > \ell$. The on-shell action would give part of the volume of $S^4$ with radius $b$. We find

$$I_h^{\text{on-shell}}(\mathrm{EAdS}_4) = \frac{8\pi^2 b^2}{\kappa^2}\frac{\alpha^4(\alpha^2+3)}{(\alpha^2+1)^3} = -\frac{16\pi^2}{\lambda_{\text{AdS}}}\frac{\alpha^4(\alpha^2+3)}{(\alpha^2+1)^3}\,,$$

where here $\lambda_{\text{AdS}} < 0$.

# Geometric Interpretation

- The $\mathrm{EAdS}_4$ and $\mathrm{dS}_4$ results are mapped to each other by the transformation $b \mapsto i\,b$. This is natural, if we recall that $b$ is the radius of both the $S^4$ and $\mathrm{EAdS}_4$ of the associated EH actions.
- Thus, we have shown that the on-shell instanton action on $\mathrm{EAdS}_4$ can be viewed as a partition function on a 3-sphere of radius $b$, arising from a bulk gravitational action with Newton's constant $\kappa$, if we identify the coupling $\lambda \sim \kappa^2/b^2$.
- The instantons in this part are related to the previously discussed ones by

$$b^2 = \ell_{\mathrm{dS}}^2 \left(1 - \frac{a_5}{a_0}\right),$$

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- The limit $\alpha \to 1$ of the above action, in which the instanton deformation parameter approaches the curvature radius, corresponds to the limit $a_5^2 \to 0^-$ and $a_0^2 \to \frac{\lambda_{S^4}}{12}\ell_{\rm dS}^2{}^+$ which is the critical value at which the moduli space shrinks to zero curvature radius. In this limit, the instanton part of the action diverges as $1/a_5^3$, which is precisely the divergence of the $\mathrm{EAdS}_4$ volume.
- Thus, the instanton is computing the $\mathrm{EAdS}_4$ volume, and the boundary deformation parameter $a_0$ regulates this volume.
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# Discussion

- In this note we have tested the HHM proposal in the case of scalar instantons. We have calculated the on-shell action for instantons on half-$S^4$, which yields the late-time HH state, and compared it with the on-shell action for instantons on $\mathrm{EAdS}_4$. The results match under the HHM prescription of analytically continuing the curvature radii.
- Additionally, we have found that it is also necessary to analytically continue the boundary condition, which corresponds to a marginal triple trace deformation. This provides new evidence that the HHM proposal works for exact, but non-trivial configurations.
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- On the EAdS side, the instanton modulus $b_5$ corresponds to the coefficient of a marginal triple trace deformation for an operator of dimension 1. Under the analytic continuation to dS space this is again a triple trace deformation. The partition function as a function of the boundary deformation parameter (i.e. imaginary magnetic field) may have zeroes (i.e. Lee-Yang zeroes) which usually signal an instability/phase transition of the theory. The presence of instabilities seems to be connected to the fact that our free energy result may be negative, hence it corresponds to a non-unitary $\mathrm{CFT}_3$.
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- It would also be interesting to work out the relationship of this result with holographic stochastic quantisation.

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- We also found an interesting geometric realization of the same computation, in which the $S^4$ instantons (for particular values of the moduli) are seen to compute the regularized volume of $\mathrm{EAdS}_4$, and the $\mathrm{EAdS}_4$ instantons are seen to compute the volume of the four-sphere. The regulator of the $\mathrm{EAdS}_4$ volume is $a_0$, with the divergence appearing precisely for the critical value $a_0^2 = \frac{\ell_{S^4}}{12}\,\ell_{\mathrm{dS}}^2$. This might imply that a sector of the $\mathrm{Sp}(N)$ model with this marginal deformation is dual to a pure gravitational theory with no scalars, and hence signal a duality between $\mathrm{Sp}(N)$ models with different values of the deformation parameter.

# Discussion

- Our instanton solutions are intimately related to the SO(4) and SO(3, 1) invariant solutions of 4-dimensional HS theory found by Sezgin and Sundell, which are also related to a consistent truncation of $\mathcal{N} = 8$ gauge supergravity down to a single scalar of the SO(8) group. In that sense, our results should also provide the partition functions of both the above theories, at the scalar instanton vacua.
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