Holography and the (Exact) Renormalization Group

Rob Leigh

University of Illinois

Kolymbari: September 2014

Introduction

- An appealing aspect of holography is its interpretation in terms of the renormalization group of quantum field theories — the ‘radial coordinate’ is a geometrization of the renormalization scale.
- Its simplest incarnation is for CFTs

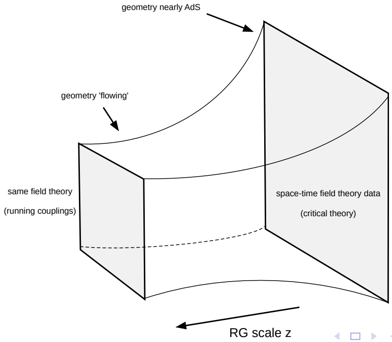
$$\begin{array}{rcl} AdS_{d+1} & \leftrightarrow & CFT_d \\ \textit{isometries} & \leftrightarrow & \textit{global symmetries} \\ \textit{scale isometry} & \leftrightarrow & \textit{RG invariance} \end{array}$$

- Usually, the correspondence is in terms of

$$\begin{array}{rcl} \textit{weakly coupled gravity} & \leftrightarrow & \textit{strongly coupled QFT} \\ \hbar & \leftrightarrow & \frac{1}{N} \sim \left(\frac{\ell_{Pl}}{\ell_{AdS}}\right)^4 \end{array}$$

Introduction

- We regard gravity as a small sector of a much bigger theory, such as a string theory (although most CMT applications ignore this...).
- More generally, RG flows (couplings and correlators changing as we coarse-grain) correspond to specific geometries that have scale isometry only asymptotically.



The Holographic Dictionary

- In a field theory, we have operators. We can talk about adding them to the action, with a corresponding coupling, and we can talk about their expectation values.
- In a CFT, operators have well-defined scaling properties

$$\hat{\mathcal{O}}_z(x) \to \lambda^{\Delta} \hat{\mathcal{O}}_{\lambda z}(\lambda x)$$

- In holography, for each such operator, there is a field propagating in the geometry (satisfies classical equation of motion).
- e.g., for a scalar field, $\Phi(z; x)$, EOM is second-order PDE, and asymptotically (i.e., near the (conformal) boundary, corresponding to near criticality)

$$\Phi(z; x) \sim z^{\Delta_-} \varphi^{(-)}(x) + z^{\Delta_+} \varphi^{(+)}(x)$$

with $\Delta_\pm$ determined by mass of field

The Holographic Dictionary

- Given

$$\Phi(z;x) \sim z^{\Delta_-}\varphi^{(-)}(x) + z^{\Delta_+}\varphi^{(+)}(x)$$

- The correspondence is:

$$\begin{aligned} \varphi^{(-)}(x) \quad & \to \textit{source} \qquad \langle \ldots e^{-\int_x \varphi^{(-)}(x)\hat{\mathcal{O}}(x)} \rangle \\ \varphi^{(+)}(x) \quad & \to \textit{expectation value} \qquad \langle \hat{\mathcal{O}}(x) \rangle \\ \Delta_+ \quad & \to \textit{operator scaling dimension} \end{aligned}$$

- This applies to all types of fields

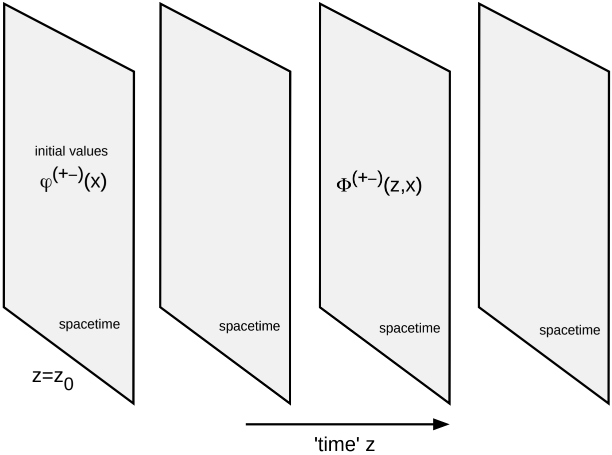
$$\begin{aligned} \textit{gauge field } A_\mu(z;x) \quad & \to \textit{conserved charge current } \hat{j}^\mu(x) \\ \textit{graviton } g_{\mu\nu}(z;x) \quad & \to \textit{conserved en} - \textit{mom tensor } \hat{T}^{\mu\nu}(x) \end{aligned}$$

- local symmetry in bulk $\to$ conserved quantity in field theory

Hamilton-Jacobi Interpretation

- Of course, second order equations can be written as a pair of first order equations
- Thus, there is a Hamiltonian formalism, but with radial coordinate z playing the role of time. (physical time remains one of the field theory coordinates)
- Source $\varphi^{(-)}(x)$ and expectation value $\varphi^{(+)}(x)$ are (boundary values of) canonically conjugate pairs.
- This fits well with Hamilton-Jacobi theory, which can be thought of as a Dirichlet problem – specify initial values — determine time-dependence of canonical variables.

Hamilton-Jacobi Interpretation



Hamilton-Jacobi Interpretation

- In this picture, the 'Hamilton equations' ought to correspond to RG equations — how things change as we change scale, or coarse-grain.
- [de Boer, Verlinde^2 '99]
- [Skenderis '02, Heemskerk & Polchinski '10, Faulkner, Liu & Rangamani '10 ...]
- If the bulk dynamics $\leftrightarrow$ Hamilton-Jacobi, what is the 'Hamiltonian'?

$$\frac{\partial}{\partial z} S_{HJ} = -H$$

- This should encode the entire set of RG equations.
- But can this be formulated in strong coupling?

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]

- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]

- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]

- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '90, '99, '02] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
  - take vector model with O(N) global symmetry in 2+1
  - have $O(N)$-singlet conserved currents $\phi^m \partial_{\mu_1} \ldots \partial_{\mu_s} \phi^m$
  - proposed holographic dual: higher spin gravity theory (Vasiliev)
  - contains an infinite set of higher spin gauge fields propagating on AdS spacetime: $W_\mu^{a_1 \ldots}$ for $s = 0, 2, 4, \ldots$
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
  - take vector model with O(N) global symmetry in 2+1
  - have $O(N)$-singlet conserved currents $\psi^m \partial_{\mu_1} \ldots \partial_{\mu_s} \psi^m$
  - proposed holographic dual: higher spin gravity theory (Vasiliev)
  - contains an infinite set of higher spin gauge fields propagating on AdS spacetime, $W_\mu^{a_1 \cdots a_s}$ for $s = 0, 2, 4, \ldots$
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
  - take vector model with O(N) global symmetry in 2+1
  - have $O(N)$-singlet conserved currents $\psi^m \partial_{\mu_1} \ldots \partial_{\mu_s} \psi^m$
  - proposed holographic dual: higher spin gravity theory (Vasiliev)
  - contains an infinite set of higher spin gauge fields propagating on AdS spacetime, $W_\mu^{a_1 \cdots a_s}$ for $s = 0, 2, 4, \ldots$
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
  - take vector model with O(N) global symmetry in 2+1
  - have $O(N)$-singlet conserved currents $\psi^m \partial_{\mu_1} ... \partial_{\mu_s} \psi^m$
  - proposed holographic dual: higher spin gravity theory (Vasiliev)
  - contains an infinite set of higher spin gauge fields propagating on AdS spacetime, $W_\mu^{a_1 \cdots a_s}$ for $s = 0, 2, 4, ...$
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
  - take vector model with O(N) global symmetry in 2+1
  - have $O(N)$-singlet conserved currents $\psi^m \partial_{\mu_1} ... \partial_{\mu_s} \psi^m$
  - proposed holographic dual: higher spin gravity theory (Vasiliev)
  - contains an infinite set of higher spin gauge fields propagating on AdS spacetime, $W_\mu^{a_1 \cdots a_s}$ for $s = 0, 2, 4, ...$
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
  - take vector model with O(N) global symmetry in 2+1
  - have $O(N)$-singlet conserved currents $\psi^m \partial_{\mu_1} ... \partial_{\mu_s} \psi^m$
  - proposed holographic dual: higher spin gravity theory (Vasiliev)
  - contains an infinite set of higher spin gauge fields propagating on AdS spacetime, $W_\mu^{a_1 \cdots a_s}$ for $s = 0, 2, 4, ...$
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

The Wilson-Polchinski Exact Renormalization Group

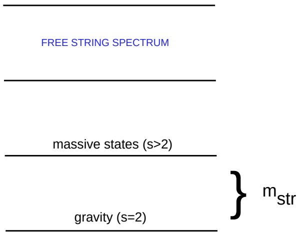
- with Onkar Parrikar & Alex Weiss, 1402.1430, 1407.4574
- Idea:
  - apply ERG to weakly coupled field theories
  - interpret ERG equations as Hamilton-Jacobi flow in one higher dimension
  - deduce geometric structure, emergence of AdS, etc.
- Weak coupling in field theory is not weak gravity!
- So what is it?
- A conjecture of [Klebanov-Polyakov '02]
  - take vector model with O(N) global symmetry in 2+1
  - have $O(N)$-singlet conserved currents $\psi^m \partial_{\mu_1} ... \partial_{\mu_s} \psi^m$
  - proposed holographic dual: higher spin gravity theory (Vasiliev)
  - contains an infinite set of higher spin gauge fields propagating on AdS spacetime, $W_\mu^{a_1 \cdots a_s}$ for $s = 0, 2, 4, ...$
- see also [Sezgin & Sundell '02, Leigh & Petkou '03] [Vasiliev '96, '99, '12] [de Mello Koch, et al '11]

Punch Lines

- We will study free field theories perturbed by arbitrary bi-local 'single-trace' operators ($\rightarrow$ still 'free', but the partition function generates all correlation functions).
- We identify a formulation in which the operator sources correspond (amongst other things) manifestly to a connection on a *really big* principal bundle — related to 'higher spin gauge theories'
- The 'gauge group' can be understood directly in terms of field redefinitions in the path integral, and consequently there are exact Ward identities that correspond to ERG equations.
- This can be formulated conveniently in terms of a jet bundle.
- The space-time structure extends in a natural way (governed by ERG) to a geometric structure over a spacetime of one higher dimension, and *AdS* emerges as a geometry corresponding to the free fixed point.

Relation to Standard Holography?

- it's often conjectured that the higher spin theory is some sort of tensionless limit of a string theory
- not clear that this can make any sense
- however, one does expect that interactions give anomalous dimensions to almost all of the higher spin currents
- in the bulk, the higher spin symmetries are Higgsed, and the higher spin gauge fields become massive



- Dream: derive geometry of weakly coupled field theory, turn on interactions and follow to strong coupling
- Not clear what the analogue of this might be in terms of string theory (rather than higher spin theory).

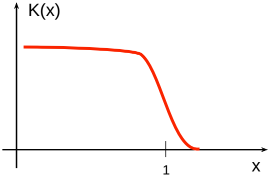
Majorana Fermions in $d = 2 + 1$

- To be specific, it turns out to be convenient to first consider the free Majorana fixed point in $2 + 1$. This can be described by the regulated action

$$S_0 = \int_x \widetilde{\psi}^m(x)\gamma^\mu P_{F;\mu}\psi^m(x) = \int_{x,y} \widetilde{\psi}^m(x)\gamma^\mu P_{F;\mu}(x,y)\psi^m(y)$$

- Here $P_{F;\mu}$ is a regulated derivative operator [Polchinski '84]

$$P_{F;\mu}(x,y) = K_F^{-1}(-\Box/M^2)\partial_\mu^{(x)}\delta(x,y)$$



Majorana Fermions in $d = 2 + 1$

- In 2+1, a complete basis of 'single-trace' operators consists of

$$\hat{\Pi}(x,y) = \widetilde{\psi}^m(x)\psi^m(y), \quad \hat{\Pi}^\mu(x,y) = \widetilde{\psi}^m(x)\gamma^\mu\psi^m(y)$$

- We introduce bi-local sources for these operators in the action

$$S_{int} = \frac{1}{2}\int_{x,y} \widetilde{\psi}^m(x)\Big(A(x,y) + \gamma^\mu W_\mu(x,y)\Big)\psi^m(y)$$

- One can think of these as collecting together infinite sets of local operators, obtained by expanding near $x \to y$. This quasi-local expansion can be expressed through an expansion of the sources

$$A(x,y) = \sum_{s=0}^{\infty} A^{a_1\cdots a_s}(x)\partial^{(x)}_{a_1}\cdots\partial^{(x)}_{a_s}\delta(x-y)$$

(similarly for W_μ). The coefficients are sources for higher spin local operators.

The $O(L_2)$ symmetry

- the full action takes the form

$$S = \frac{1}{2}\int_{x,y} \tilde{\psi}^m(x)\left[\gamma^\mu(P_{F;\mu} + W_\mu)(x,y) + A(x,y)\right]\psi^m(y)$$
$$\equiv \tilde{\psi}^m \cdot \left[\gamma^\mu(P_{F;\mu} + W_\mu) + A\right]\cdot\psi^m$$

- Now we consider the following map of elementary fields

$$\psi^m(x) \mapsto \int_y \mathcal{L}(x,y)\psi^m(y)$$

- The ψ^m are just integration variables in the path integral, and so this is just a trivial change of integration variable. I'm using here the same logic that might be familiar in the Fujikawa method for the study of anomalies.
- So, we ask, what does this do to the partition function?

The $O(L_2)$ symmetry

- We look at the action

$$\begin{aligned} S &= \widetilde{\psi}^m \cdot \left[\gamma^\mu (P_{F;\mu} + W_\mu) + A\right] \cdot \psi^m \\ &\to \widetilde{\psi}^m \cdot \mathcal{L}^T \left[\gamma^\mu (P_{F;\mu} + W_\mu) + A\right] \cdot \mathcal{L} \cdot \psi^m && (1) \\ &= \widetilde{\psi}^m \cdot \gamma^\mu \mathcal{L}^T \cdot \mathcal{L} \cdot P_{F;\mu} \cdot \psi^m && (2) \\ &\quad + \widetilde{\psi}^m \cdot \left[\gamma^\mu (\mathcal{L}^T \cdot \left[P_{F;\mu}, \mathcal{L}\right] + \mathcal{L}^T \cdot W_\mu \cdot \mathcal{L}) + \mathcal{L}^T \cdot A \cdot \mathcal{L}\right] \cdot \psi^m \end{aligned}$$

- Thus, if we take $\mathcal{L}$ to be orthogonal, $\mathcal{L}^T \cdot \mathcal{L}(x,y) = \int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \delta(x,y)$, the kinetic term is invariant, while the sources transform as

$O(L_2)$ gauge symmetry

$$\begin{aligned} W_\mu &\mapsto \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot \left[P_{F;\mu}, \mathcal{L}\right] \\ A &\mapsto \mathcal{L}^{-1} \cdot A \cdot \mathcal{L} \end{aligned}$$

The $O(L_2)$ Ward Identity

- But this was a trivial operation from the path integral point of view, and so we conclude that there is an exact Ward identity

$$Z[M, g_{(0)}, W_\mu, A] = Z[M, g_{(0)}, \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot P_{F;\mu} \cdot \mathcal{L}, \mathcal{L}^{-1} \cdot A \cdot \mathcal{L}]$$

- this is the usual notion of a background symmetry: a transformation of the elementary fields is compensated by a change in background
- more generally, we can turn on sources for arbitrary multi-local multi-trace operators — the sources will generally transform tensorially under $O(L_2)$
  (see later, perhaps)

The $O(L_2)$ symmetry

- Note what is happening here: the $O(L_2)$ symmetry leaves invariant the (regulated) free fixed point action. W_μ is interpreted as a gauge field (connection) for this symmetry, while A transforms tensorily. $D_\mu = P_{F;\mu} + W_\mu$ plays the role of covariant derivative.
- More precisely, the free fixed point corresponds to any configuration

$$(A, W_\mu) = (0, W_\mu^{(0)})$$

where $W^{(0)}$ is any flat connection

$$dW^{(0)} + W^{(0)} \wedge W^{(0)} = 0$$

The $O(L_2)$ Symmetry

- In fact, the subgroup of $O(L_2)$ leaving $W^{(0)}$ invariant is $O(2,d)$, the conformal group of the boundary theory
- Thus the quasi-local expansion that we previously wrote

$$A(x,y) = \sum_{s=0}^{\infty} A^{a_1\cdots a_s}(x)\partial_{a_1}^{(x)}\cdots\partial_{a_s}^{(x)}\delta(x-y)$$

  should best be reformulated as a sum over conformal modules (the representation of $O(L_2(\mathbb{R}^d))$ being reducible as a direct sum of $O(2,d)$ irreps)
- soon, $W^{(0)}$ will be extended to a corresponding (Cartan) connection in the bulk, and we will identify it with that corresponding to *AdS* geometry

The $CO(L_2)$ symmetry

- We can generalize the $O(L_2)$ condition to include scale transformations

$$\int_z \mathcal{L}(z,x)\mathcal{L}(z,y) = \lambda(x)^{2\Delta_\psi}\delta(x-y)$$

- This is a symmetry (in the previous sense) provided we also transform the metric, the cutoff and the sources

$$g_{(0)} \mapsto \lambda^2 g_{(0)}, \quad M \mapsto \lambda^{-1} M$$

$$A \mapsto \mathcal{L}^{-1} \cdot A \cdot \mathcal{L}$$

$$W_\mu \mapsto \mathcal{L}^{-1} \cdot W_\mu \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot \left[P_{F;\mu}, \mathcal{L}\right].$$

- A convenient way to keep track of the scale is to introduce the conformal factor $g_{(0)} = \frac{1}{z^2}\eta$. Then $z \mapsto \lambda^{-1} z$. This z should be thought of as the renormalization scale.

The Renormalization group

- To study RG systematically, we proceed in two steps:

Step 1: Lower the cutoff $M \mapsto \lambda M$, by integrating out the "fast modes"

$$Z[M, z, A, W] = Z[\lambda M, z, \widetilde{A}, \widetilde{W}] \qquad (\textit{Polchinski})$$

Step 2: Perform a $CO(L_2)$ transformation to bring the cutoff back to M, but in the process changing $z \mapsto \lambda^{-1} z$

$$Z[\lambda M, z, \widetilde{A}, \widetilde{W}] = Z[M, \lambda^{-1} z, \mathcal{L}^{-1} \cdot \widetilde{A} \cdot \mathcal{L}, \mathcal{L}^{-1} \cdot \widetilde{W} \cdot \mathcal{L} + \mathcal{L}^{-1} \cdot [P_F, \mathcal{L}]]$$

- We can now compare the sources at the same cutoff, but different z. Thus, z becomes the natural flow parameter, and we can think of the sources as being z-dependent.
  - Thus we have the Polchinski formalism extended to include both a cutoff and an RG scale — required for a holographic interpretation.

The RG Equations

- These equations have the form

$$\partial_z \mathcal{A} + [\mathcal{W}_z, \mathcal{A}] = \beta^{(\mathcal{A})}$$

$$\partial_z \mathcal{W}_\mu - [P_{F;\mu}, \mathcal{W}_z] + [\mathcal{W}_z, \mathcal{W}_\mu] = \beta_\mu^{(\mathcal{W})}$$

- get 'gauge theory' in $d + 1$ dimensions
- fixed point (zero of β-fns $\leftrightarrow$ flat connection)
- flat connection $\leftrightarrow$ AdS geometry
- gauge group $\leftrightarrow$ higher spin symmetry

Hamilton-Jacobi Structure

- Indeed, if we identify $Z = e^{iS_{HJ}}$, then a fundamental relation in H-J theory is

$$\frac{\partial}{\partial z} S_{HJ} = -\mathcal{H}$$

- We can thus read off the Hamiltonian of the theory, for which the RG equations are the Hamilton equations

$$\begin{aligned} \mathcal{H} &= -\mathrm{Tr}\Big\{ \Big(\Big[\mathcal{A}, \mathcal{W}_{\underline{e}_z^{(0)}} \Big] + \beta^{(\mathcal{A})} \Big) \cdot \mathcal{P} \Big\} \\ &\quad -\mathrm{Tr}\Big\{ \Big(\Big[P_{F;\mu} + \mathcal{W}_\mu, \mathcal{W}_{\underline{e}_z^{(0)}} \Big] + \beta_\mu^{(\mathcal{W})} \Big) \cdot \mathcal{P}^\mu \Big\} \\ &\quad -\frac{N}{2}\mathrm{Tr}\Big\{ \Big(\Delta^\mu \cdot \widehat{\mathcal{W}}_\mu + \Delta^z \cdot \widehat{\mathcal{W}}_{\underline{e}_z^{(0)}} \Big) \Big\} \end{aligned} \tag{3}$$

- Encodes all information (concerning $O(N)$ singlet operators) in the field theory.

Remarks

- We have seen how the rich symmetry structure of the free-fixed point allows us to geometrize RG.
- The resulting structure has striking similarities with Vasiliev higher spin theory, and (at least in some cases) might be equivalent.
- The β-functions encode the information about *n*-point functions, which correspond to interactions in the bulk.
- There are many generalizations of this scheme (e.g., to fixed points with different symmetries/properties) that give rise to higher spin theories with no Vasiliev analogue. [hep-th:1407.4574]
  - e.g., the $z = 2$ free field theory has a holographic dual that is a new higher spin theory on the Schrödinger geometry (rather than AdS).

Geometry: The Infinite Jet bundle

- we can put the non-local transformation $\psi(x) \mapsto \int_y \mathcal{L}(x,y)\psi(y)$ in more familiar terms by introducing the notion of a jet bundle
- The simple idea is that we can think of a differential operator $\mathcal{L}(x,y)$ as a matrix by "prolongating" the field

$$\psi^m(x) \mapsto \left(\psi^m(x), \frac{\partial \psi^m}{\partial x^\mu}(x), \frac{\partial^2 \psi^m}{\partial x^\mu \partial x^\nu}(x) \cdots \right) \quad \text{"jet"}$$

- Then, differential operators, such as $P_\mu(x,y) = \partial_\mu^{(x)}\delta(x-y)$ are interpreted as matrices $\mathbb{P}_\mu$ that act on these vectors
- The bi-local transformations can be thought of as local gauge transformations of the jet bundle.
- The gauge field W is a connection 1-form on the jet bundle, while A is a section of its endomorphism bundle.

Bosonic Relativistic Free Fixed Point

- Another example consists of N complex scalar fields. In this case, we formulate the single-trace deformations in terms of the $CU(L_2)$ connection.

$$S = \int \phi_m^* \cdot \Big(\big[D_{F;\mu} + W_\mu \big]^2 + B \Big) \cdot \phi^m$$

- The ERG equations give rise to an 'A-model' in any dimension.
- Here though there is an extra background symmetry

$$Z[M, z, B, W_\mu^{(0)}, \widehat{W}_\mu + \Lambda_\mu] = Z[M, z, B + \{\Lambda^\mu, D_\mu\} + \Lambda_\mu \cdot \Lambda^\mu, W_\mu^{(0)}, \widehat{W}_\mu]$$

- this background symmetry allows for fixing $W_\mu \to W_\mu^{(0)}$, and the corresponding transformed B sources all single-trace currents.

[This was the starting point of Douglas, et al, and so geometry was not manifest.]

The Bulk Action and Correlation Functions

- For the bosonic theory, the bulk phase space action is

$$I = \int dz \, \mathrm{Tr} \left\{ \mathcal{P}^I \cdot \left(\mathcal{D}_I \mathfrak{B} - \beta_I^{(\mathfrak{B})} \right) + \mathcal{P}^{IJ} \cdot \mathcal{F}_{IJ}^{(0)} + N \, \Delta_B \cdot \mathfrak{B} \right\}$$

- Here Δ_B is a derivative with respect to M of the cutoff function.
- As in any holographic theory, we solve the bulk equations of motion in terms of boundary data, and obtain the on-shell action, which encodes the correlation functions of the field theory.
- It is straightforward to carry this out exactly for the free fixed point.
- Here we have

$$I_{o.s.} = N \int \Delta_B \cdot \mathfrak{B}$$

where now $\mathfrak{B}$ is the bulk solution

The Bulk Action and Correlation Functions

- The RG equation

$$\left[\mathcal{D}_z^{(0)}, \mathfrak{B}\right] = \beta_z^{(\mathfrak{B})} = \mathfrak{B} \cdot \Delta_B \cdot \mathfrak{B}$$

can be solved iteratively

$$\mathfrak{B} = \alpha \mathfrak{B}_{(1)} + \alpha^2 \mathfrak{B}_{(2)} + ...,$$

$$\begin{aligned}
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(1)}\right] &= 0 \\
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(2)}\right] &= \mathfrak{B}_{(1)} \cdot \Delta_B \cdot \mathfrak{B}_{(1)} \\
\left[\mathcal{D}_z^{(0)}, \mathfrak{B}_{(3)}\right] &= \mathfrak{B}_{(2)} \cdot \Delta_B \cdot \mathfrak{B}_{(1)} + \mathfrak{B}_{(1)} \cdot \Delta_B \cdot \mathfrak{B}_{(2)} \\
&\vdots
\end{aligned}$$

The Bulk Action and Correlation Functions

- The first equation is homogeneous and has the solution

$$\mathfrak{B}_{(1)}(z;x,y) = \int_{x',y'} K^{-1}(z;x,x') b_{(0)}(x',y') K(z;y',y)$$

where we have defined the boundary-to-bulk Wilson line

$$K(z) = P.\exp \int_{\epsilon}^{z} dz'\, \mathcal{W}_z^{(0)}(z')$$

with the boundary being placed at $z=\epsilon$. (UV cutoff $\sim M/\epsilon$)

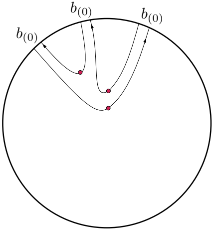
- $b_{(0)}$ has the interpretation of a boundary source
- this can then be inserted into the second order equation and the whole system solved iteratively

The Bulk Action and Correlation Functions

- At k^{th} order, one finds a contribution to the on-shell action

$$
\begin{aligned}
I^{(k)}_{o.s.} &= N \int_\epsilon^\infty dz_1 \int_\epsilon^{z_1} dz_2 ... \int_\epsilon^{z_{k-1}} dz_k \\
&\quad \times \mathrm{Tr}\, H(z_1) \cdot b_{(0)} \cdot H(z_2) \cdot b_{(0)} \cdot ... \cdot H(z_k) \cdot b_{(0)} \\
&\quad + permutations
\end{aligned}
$$

where $H(z) \equiv K^{-1}(z) \cdot \Delta_B(z) \cdot K(z) = \partial_z g(z)$



The Witten diagram for the bulk on-shell action at third order.

The Bulk Action and Correlation Functions

- The z-integrals can be performed trivially, resulting in

$$I^{(k)}_{o.s.} = \frac{N}{k} \mathrm{Tr} \left(g_{(0)} \cdot b_{(0)} \right)^k$$

  where $g_{(0)} = g(\infty)$ is the boundary free scalar propagator
- These can be resummed, resulting in

$$Z[b_{(0)}] = \det{}^{-N} \left(1 - g_{(0)} b_{(0)} \right)$$

  which is the exact generating functional for the free fixed point.
- Thus, this holographic theory does everything that it can for us.

Remarks

- What of standard gravitational holography?
- A standard piece of higher spin lore is expected to kick in here — when interactions in the field theory are included, the higher spin symmetry of the bulk breaks spontaneously (the operators get anomalous dimensions, corresponding to masses in the bulk). Presumably, if we follow the theory to strong coupling, all that is left behind is gravity.
- It is an interesting challenge to show that precisely this happens generically (!!).
- Perturbatively nearby fixed points (e.g., large N saddle points) are accessible, and will have an operator spectrum whose anomalous dimensions scale as $1/N^x$.
  - N is insignificant prior to the introduction of field theory interactions ($1/N$ does not (yet) act as the bulk $\hbar$)

Interactions

- the free fixed point can always be thought of as a vectorial theory, but interactions determine how to think of the field content (depending on what the interactions do to the global symmetries)
- the simplest possibility is to turn on all $O(N)$- (or $U(N)$)-invariant multi-trace interactions

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int B_k(x_1, y_1; ...; x_k, y_k) \phi^*_{m_1}(x_1) \phi^{m_1}(y_1) ... \phi^*_{m_k}(x_k) \phi^{m_k}(y_k)$$

- the sources B_k are paired with vevs

$$\Pi_k(x_1, y_1; ...; x_k, y_k) \equiv \langle \phi^*_{m_1}(x_1) \phi^{m_1}(y_1) ... \phi^*_{m_k}(x_k) \phi^{m_k}(y_k) \rangle$$

- these objects are $U(L_2)$ tensors, corresponding to an infinite set of canonically conjugate pairs in the bulk

Interactions

- the ERG equations couple all of these together – there is generically no ‘consistent truncation’ (other than restricting to single-trace ops, as before)
- a solution to this system of equations corresponds to an RG fixed point
- indeed there are such solutions visible at large N, specified by a choice of boundary values for the $\{B_k\}$ — bare couplings
- large N factorization here corresponds to a collapse of the phase space to

$$\Pi_k(x_1, y_1; ...; x_k, y_k) = \Pi_1(x_1, y_1)...\Pi_1(x_k, y_k)$$

- e.g., turn on the double-trace coupling: expect that essentially B_1 and Π_1 swap roles (Legendre transform). This is the interacting critical point.

[RGL+OP, to appear]

Open Questions

- (interacting) matrix theories?
- Gauge interactions (various)?
- Geometry of global symmetries?
- Emergence of just gravity?
- Entanglement? MERA?
- Other spacetime topologies?
- Other states (e.g., finite temperature), corresponding geometries?